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# Some recent results on James and von Neumann-Jordan constants

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Among various geometric constants of a Banach space $X$ the von Neumann-Jordan constant $C_{NJ}(X)$ and James constant $J(X)$ have been treated most widely. Since Kato, Maligranda and Takahashi [3] showed a relation (inequalities) between these constants, several authors have improved their result. Recently covering all the previous results, the present authors showed that the quite simple inequality $C_{NJ}(X)\leq J(X)$ holds for all Banach spaces $X$. In this expository short note we shall present a concise description of these developments.

Let $X$ be a real Banach space with $\dim X\geq 2$. The closed unit ball and unit sphere of $X$ are denoted by $B_X$ and $S_X$, respectively. The *James (non-square) constant* of $X$ is

$$J(X)=\sup\{\min(\|x+y\|,\|x-y\|):\ x,y\in S_X\},\tag{1}$$

and the *von Neumann-Jordan constant* of $X$ is

$$C_{NJ}(X)=\sup\left\{\frac{\|x+y\|^2+\|x-y\|^2}{2(\|x\|^2+\|y\|^2)}:\ x\in S_X,\ y\in B_X\right\}.\tag{2}$$

The modified von Neumann-Jordan constant $C'_{NJ}(X)$ is defined by taking the supremum over all $x,y\in S_X$ in (2). Recall that a Banach space $X$ is *uniformly non-square* provided $J(X)<2$ or equivalently $C_{NJ}(X)<2$.

**1.** The relation between $J(X)$ and $C_{NJ}(X)$ was first investigated by Kato-Maligranda-Takahashi [3] who proved that

$$\frac{J(X)^2}{2} \leq C_{NJ}(X) \leq \frac{J(X)^2}{1+(J(X)-1)^2}, \tag{3}$$

where all the terms coincide if $J(X)=2$, i.e., $X$ is not uniformly non-square. The first inequality attains equality with many uniformly non-square spaces, while the second is strict for all uniformly non-square spaces.

**2.** In 2004 Nikolova-Persson-Zachariades [5] improved the second inequality as

$$C_{NJ}(X) \leq \frac{J(X)^2}{4} + 1 + \frac{J(X)}{4}\left[\sqrt{J(X)^2 - 4J(X) + 8} - 2\right] \tag{4}$$

(see also Maligranda et al. [4], Takahashi [7]). Maligranda formulated the following conjecture: For all Banach space $X$

$$C_{NJ}(X) \leq \frac{J(X)^2}{4} + 1. \tag{5}$$

**3.** In 2008 Alonso et al. [1] proved that

$$C_{NJ}(X) \leq 2\left[1 + J(X) - \sqrt{2J(X)}\right], \tag{6}$$

which was obtained by combining the inequalities

$$C'_{NJ}(X) \leq J(X) \tag{7}$$

and

$$C_{NJ}(X) \leq 2\left[1 + C'_{NJ}(X) - \sqrt{2C'_{NJ}(X)}\right]. \tag{8}$$

Note that

$$2\left[1 + J(X) - \sqrt{2J(X)})\right] \leq J(X)^2/4 + 1, \tag{9}$$

where equality holds only when $J(X)=2$. This indicates that the inequality (6) is sharper than (5), a fortiori, (4). Thus they answered Maligranda's conjecture (5) affirmatively.

**4.** A slight improvement of the inequality (6) was obtained by Wang-Pang [10] in 2009:

$$C_{NJ}(X) \leq J(X) + \sqrt{J(X)-1}\left[\sqrt{1+(1-\sqrt{J(X)-1})^2}-1\right]. \tag{10}$$

To prove (10) they used the inequalities

$$\rho_X(1) \leq \sqrt{J(X)-1} \tag{11}$$

and

$$C'_{NJ}(X) \leq J(X), \tag{12}$$

where $\rho_X(\tau)$ is the modulus of smoothness of $X$.

**5.** Covering all the previous results, we proved in [9] that the inequality

$$C_{NJ}(X) \leq J(X) \tag{13}$$

holds true for all Banach spaces $X$, where equality holds only when $X$ is not uniformly non-square. This answered affirmatively the question mentioned in Alonso et al. [1, Question 1]. More precisely, we first improved the inequality (11) by Wang-Pang [10] as

$$\rho_X(1) \leq 2\left\{1-\frac{1}{J(X)}\right\} \leq \sqrt{J(X)-1}. \tag{14}$$

The first inequality of (14) is equivalent to

$$\frac{2}{2-\rho_X(1)} \leq J(X). \tag{15}$$

Secondly we showed that

$$C_{NJ}(X) \leq 1+\rho_X(1)\left[\sqrt{\{1-\rho_X(1)\}^2+1}-\{1-\rho_X(1)\}\right]. \tag{16}$$

It is easy to see that

$$1+\rho_X(1)\left[\sqrt{\{1-\rho_X(1)\}^2+1}-\{1-\rho_X(1)\}\right] \leq \frac{2}{2-\rho_X(1)}.$$

Therefore by combining (16) and (15), we obtained that $C_{NJ}(X) \leq J(X)$.

**6.** Recently we considered the inequality $C_{NJ}(X) \leq J(X)$ by another approach with the modified von Neumann-Jordan constant $C'_{NJ}(X)$ and obtained another proof of the above inequality and some other results, which will appear elsewhere.

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