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# Boundary Harnack principle and the quasihyperbolic boundary condition — Expository and correction —

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## 1. Introduction

This is an expository and correction of [2]. We observe that the global boundary Harnack principle holds for domains satisfying some conditions related to the quasihyperbolic metric. Let $D$ be a bounded domain in $\mathbb{R}^n$ with $n \geq 2$ and let $\delta_D(x) = \operatorname{dist}(x, \partial D)$. We use the following notation: $B(x,r)$ (resp. $S(x,r)$) stands for the open ball (resp. the sphere) center at $x$ and radius $r$. By $A$ we denote a positive constant which may change from one occurrence and the next. We write $f \approx g$ if $A^{-1} \leq f/g \leq A$.

Consider a pair $(V, K)$ of a bounded open set $V \subset \mathbb{R}^n$ and a compact set $K \subset \mathbb{R}^n$ such that

$$K \subset V,\ K \cap D \neq \emptyset \text{ and } K \cap \partial D \neq \emptyset. \tag{1}$$

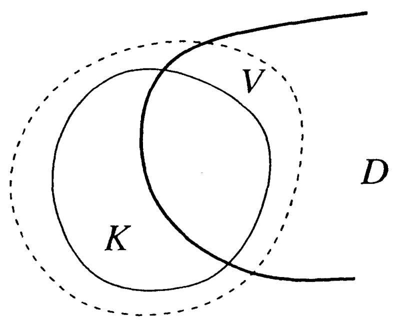



Figure 1. A pair $(V, K)$ with (1).

**Definition 1** (Global boundary Harnack principle). We say that $D$ *enjoys* the *global boundary Harnack principle* if for all $(V, K)$ with (1), there exists $A_1$ depending only on $D$, $V$ and $K$ with the following property: If

(i) $u$ and $v$ are *positive superharmonic functions on* $D$,
(ii) $u$ and $v$ are bounded, positive and harmonic in $V \cap D$,
(iii) $u$ and $v$ vanish on $V \cap \partial D$ except for a polar set,

then

$$\frac{u(x)/u(y)}{v(x)/v(y)} \leq A_1 \quad \text{for } x, y \in K \cap D.$$

*Remark* 1. We have the following remarks.

(i) In general, $K \cap D$ *may be disconnected*, so $u$ and $v$ need to be defined on the whole $D$ and positive and superharmonic there.
(ii) Jerison-Kenig [12] and Bass-Burdzy-Bañuelos [8] and [9] assume that $u$ and $v$ are positive and *harmonic over the whole D.*
(iii) Our formulation of the boundary Harnack principle is slightly stronger.

For a Lipschitz domain the global boundary Harnack principle was proved by Ancona [4], Dahlberg [11] and Wu [14] independently. Caffarelli-Fabes-Mortola-Salsa [10] and Jerison-Kenig [12] gave significant extensions. From the probabilistic point of view, Bass-Burdzy-Bañuelos [8] and [9] proved the global boundary Harnack principle for a *Hölder domain*, a domain whose boundary is locally given by the graph of a Hölder continuous function in $\mathbb{R}^{n-1}$.

Define the *quasihyperbolic metric* $k_D(x, y)$ by

$$k_D(x, y) = \inf_{\gamma} \int_{\gamma} \frac{ds(z)}{\delta_D(z)},$$

where the infimum is taken over all rectifiable curves $\gamma$ connecting $x$ to $y$ in $D$ and $ds(z)$ stands for the line element on $\gamma$. Smith-Stegenga [13] said that $D$ is a "Hölder domain" if

$$k_D(x, x_0) \leq A \log \frac{\delta_D(x_0)}{\delta_D(x)} + A' \quad \text{for all } x \in D \tag{2}$$

with some positive constants $A$ and $A'$. Bañuelos [7] said that such a domain is a Hölder domain of order 0. To avoid the confusion, we say that $D$ satisfies the *quasihyperbolic boundary condition (of order 0)* if (2) holds. One of the most significant properties of domains satisfying the quasihyperbolic boundary condition is the exponential integrability of the quasihyperbolic metric: there exists a positive constant $\varepsilon$ such that

$$\int_D \exp(\varepsilon k_D(x, x_0))dx < \infty$$

(Smith-Stegenga [13]).

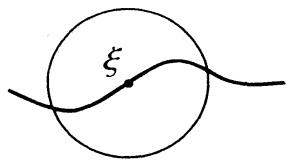
Extending (2), we consider the following condition:

$$k_D(x, x_0) \le A\Big(\frac{\delta_D(x_0)}{\delta_D(x)}\Big)^{\alpha} + A' \quad \text{for all } x \in D \tag{3}$$

with some positive constants $A$ and $A'$. Let us say that $D$ satisfies the *quasihyperbolic boundary condition of order $\alpha$* if (3) holds. The above condition and (2) are *interior* conditions. Let us consider an *exterior* condition.

**Definition 2.** By Cap we denote the logarithmic capacity if $n = 2$, and the Newtonian capacity if $n \geq 3$. We say that the *capacity density condition* holds if there exist constants $A > 1$ and $r_0 > 0$ such that

$$\operatorname{Cap}(B(\xi, r) \setminus D) \geq \begin{cases} Ar & \text{if } n = 2, \\ Ar^{n-2} & \text{if } n \geq 3, \end{cases}$$




whenever $\xi \in \partial D$ and $0 < r < r_0$. See Armitage-Gardiner [6] for the capacity, which illustrates the inhomogeneity between $n = 2$ and $n \geq 3$.

Bañuelos [7] said that $D$ is a *uniformly Hölder domain of order $\alpha$* if (3) and the capacity density condition hold. It seems that the capacity density condition is needed for $\alpha > 0$ because of the lack of the exponential integrability of the quasihyperbolic metric.

*Remark* 2. We have the following remarks:

(i) Bañuelos [7] showed the intrinsic ultra-contractivity for a domain satisfying the quasihyperbolic boundary condition of order $\alpha$, $0 \leq \alpha < 2$, and the capacity density condition.

(ii) Bass-Burdzy-Bañuelos [8] and [9] proved the boundary Harnack principle for a Hölder domain. The main tool was the so-called *box argument.*

(iii) They also claimed that the boundary Harnack principle may hold for a domain satisfying the quasihyperbolic boundary condition of order $\alpha$, $0 < \alpha < 1$, and the capacity density condition. However, no proof has not been provided.
(iv) Bass-Burdzy-Bañuelos [9] and [8] were very probabilistic.
(v) Aikawa [1] proved the local (or scale-invariant) boundary Harnack principle for a uniform domain by an elementary and analytic argument.

Aikawa [3] gave a completely different approach to the boundary Harnack principle: the *Domar method* and the equivalence between the boundary Harnack principle and the Carleson estimate.

**Definition 3** (Global Carleson estimate). We say that $D$ *enjoys* the *global Carleson estimate* if for all $(V, K)$ with (1), and a point $x_0 \in K \cap D$, there exists $A_2$ depending only on $D$, $V$, $K$ and $x_0$ with the following property: If

(i) $u$ is *positive superharmonic on* $D$,
(ii) $u$ is bounded, positive and harmonic in $V \cap D$,
(iii) $u$ vanishes on $V \cap \partial D$ except for a polar set,

then

$$u(x) \leq A_2 u(x_0) \quad \text{for } x \in K \cap D. \tag{4}$$

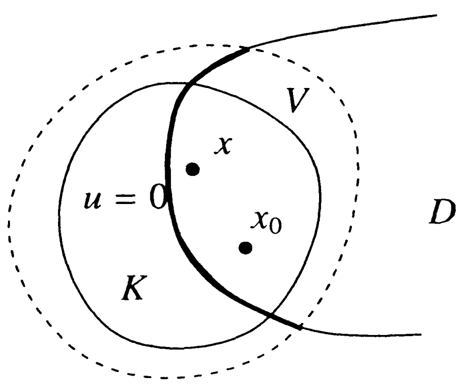



FIGURE 2. The global Carleson estimate.

We have the following theorems.

**Theorem 1.** *For arbitrary domains*

$$\textit{Global Boundary Harnack principle} \iff \textit{Global Carleson estimate.}$$

**Theorem 2.** *The global boundary Harnack principle holds for a domain satisfying the quasihyperbolic boundary condition (of order 0).*

**Theorem 3.** *Let $D$ be a bounded domain satisfying the capacity density condition and the quasihyperbolic boundary condition of order $\alpha$ for some $0 < \alpha < 1$. Then the global boundary Harnack principle holds for $D$.*

*Remark* 3. We do not know whether the restriction $\alpha < 1$ is sharp or not. Bañuelos [7] showed the intrinsic ultra-contractivity for a domain satisfying the quasihyperbolic boundary condition of order $\alpha$, $0 \le \alpha < 2$, and the capacity density condition. It is interesting to study the case $1 \le \alpha < 2$.

## 2. $\beta$-John domains

Let $0 < \beta \le 1$. We say that $D$ is a *$\beta$-John domain*, if there is a point $x_0 \in D$ and every point $x \in D$ can be connected to $x_0$ by a rectifiable curve $\gamma$ with

$$\delta_D(y)^\beta \ge A\ell(\gamma(x,y)) \quad \text{for all } y \in \gamma,$$

where $\ell(\gamma(x,y))$ is the arc length of the subcurve $\gamma(x,y)$ connecting $x$ and $y$ along $\gamma$. See Ancona [5, 9.2].

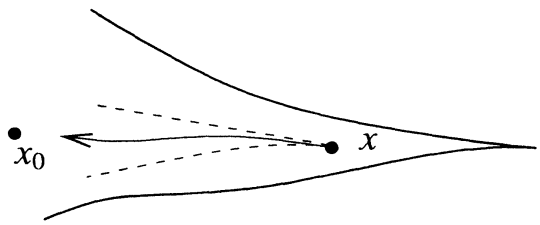



FIGURE 3. A $\beta$-John domain.

*Remark* 4. We have the following remarks.

(i) If $\beta = 1$, then a $\beta$-John domain is a classical John domain and the quasihyperbolic boundary condition of order 0 is satisfied.

(ii) In general, a $\beta$-John domain satisfies the quasihyperbolic boundary condition of order $1 - \beta$.

Hence, we obtain the following corollary to Theorem 3.

**Corollary 1.** *Let $0 < \beta \leq 1$. Then a $\beta$-John domain with the capacity density condition enjoys the global boundary Harnack principle.*

*Remark* 5. In view of Theorem 2, the capacity density condition is superfluous in case $\beta = 1$.

A typical example of a $\beta$-John domain is a $\beta$-Hölder domain, whose boundary is given locally by the graph of a $\beta$-Hölder continuous function in $\mathbb{R}^{n-1}$. A 1-Hölder domain is a Lipschitz domain, so that the boundary Harnack principle is classically established. A $\beta$-John domain need not satisfy the capacity density condition if $0 < \beta < 1$. However, Bass-Burdzy [9] showed that the global boundary Harnack principle holds for a $\beta$-Hölder domain without the capacity density condition for $\frac{1}{2} < \beta < 1$, and then Bass-Burdzy-Bañuelos [9] for $0 < \beta < 1$.

**Theorem 4.** *Let $0 < \beta < 1$. A $\beta$-Hölder domain enjoys the global boundary Harnack principle.*

Their proof is very probabilistic. We shall give an elementary analytic proof for $\frac{1}{2} < \beta < 1$.

*Remark* 6. There were mistakes in [2]. Here a corrected proof will be given. Unfortunately, the $0 < \beta \leq \frac{1}{2}$ cannot be covered. Its analytic simple proof remains open.

## 3. Boundary Harnack principle and Carleson estimate in terms of the Green function

The Riesz decomposition theorem says that a positive superharmonic function $u$, which is harmonic in $D \cap V$ and vanishes on $\partial D \cap V$ except for a polar set, can be represented as the Green potential $G\mu$ of a measure $\mu$ on $D \cap \partial V$ in $D \cap V$. One more geometrical observation is relevant. Let $\overline{B}$ be a closed ball including $D$. Then $F = \overline{B} \setminus V$ is a compact set and $D \setminus V = D \cap F$ and $K \cap F = \emptyset$. Consider a pair of disjoint compact sets $K$ and $F$ with

$$K \cap \partial D \neq \emptyset,\ K \cap D \neq \emptyset,\ F \cap \partial D \neq \emptyset \text{ and } F \cap D \neq \emptyset. \tag{5}$$

**Definition 4.** We say that a domain $D$ enjoys the *global Carleson estimate in terms of the Green function* if for each pair of disjoint compact sets $K$ and $F$ with (5) and a point $x_0 \in K \cap D$, there exists a constant $A_2$ depending only on $D$, $K$, $F$ and $x_0$ such that

$$G(x,y) \le A_2 G(x_0,y) \quad \text{for } x \in K \cap D \text{ and } y \in F \cap D.$$

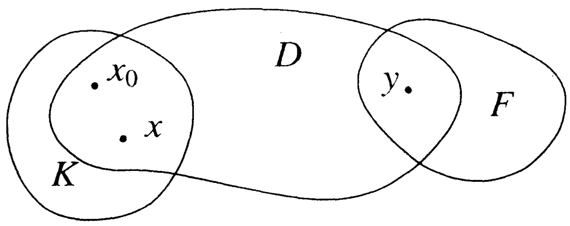



FIGURE 4. The global Carleson estimate in terms of the Green function.

The above discussion gives readily the following theorem.

**Theorem 5.** *The global Carleson estimate and the global Carleson estimate in terms of the Green function are equivalent.*

Using the Riesz decomposition theorem, we similarly obtain the counterpart of the boundary Harnack principle.

**Definition 5.** We say that a domain $D$ enjoys the *global boundary Harnack principle in terms of the Green function* if for each pair of disjoint compact sets $K$ and $F$ with (5), there exists a constant $A_1$ depending only on $D$, $K$ and $F$ such that

$$\frac{G(x,y)/G(x',y)}{G(x,y')/G(x',y')} \le A_1 \quad \text{for } x,\, x' \in K \cap D \text{ and } y,\, y' \in F \cap D.$$

**Theorem 6.** *The global boundary Harnack principle and the global boundary Harnack principle in terms of the Green function are equivalent.*

Combine Theorems 1, 5 and 6.

**Theorem 7.** *Let $D$ be a bounded domain in $\mathbb{R}^n$. Then the following statements are equivalent:*

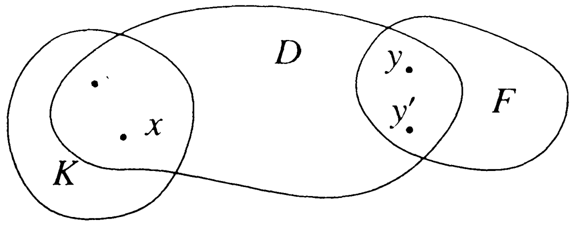



FIGURE 5. The global boundary Harnack principle in terms of the Green function.

(i) *$D$ enjoys the global boundary Harnack principle.*
(ii) *$D$ enjoys the global boundary Harnack principle in terms of the Green function.*
(iii) *$D$ enjoys the global Carleson estimate.*
(iv) *$D$ enjoys the global Carleson estimate in terms of the Green function.*

## 4. LEMMAS

In order to prove the theorems we use the following notions. By $\omega(E, U)$ we denote the harmonic measure over an open set $U$ of $E \subset \partial U$. Let $w_\eta(U)$ be the *capacitary width* defined by

$$w_\eta(U) = \inf\left\{r > 0 : \frac{\mathrm{Cap}(B(x,r) \setminus U)}{\mathrm{Cap}(B(x,r))} \geq \eta \quad \text{for all } x \in U\right\}.$$

Here $0 < \eta < 1$.

**Lemma 1.** *If $D$ satisfies the capacity density condition, then $w_\eta(\{x \in D : \delta_D(x) < r\}) \leq 2r$ for some $\eta$.*

The capacitary width is useful for the estimate of harmonic measure ([1, Lemma 1]).

**Lemma 2.** *There exists $A_3 > 0$ if $x \in U$ and $R > 0$, then*

$$\omega^x(U \cap S(x,R); U \cap B(x,R)) \leq \exp\Big(2 - A_3 \frac{R}{w_\eta(U)}\Big).$$

**Definition 6.** We say that two points $x, y \in D$ are connected by a *Harnack chain* $\{B(x_j, \frac{1}{2}\delta_D(x_j))\}_{j=1}^k$ if $x \in B(x_1, \frac{1}{2}\delta_D(x_1))$, $y \in B(y_k, \frac{1}{2}\delta_D(y_k))$, and $B(x_j, \frac{1}{2}\delta_D(x_j)) \cap B(x_{j+1}, \frac{1}{2}\delta_D(x_{j+1})) \neq \emptyset$ for $j = 1, \ldots, k-1$. The number $k$ is called the *length of the Harnack chain.*

Since the shortest length of the Harnack chain connecting $x$ and $y$ in $D$ is comparable to the quasihyperbolic metric $k_D(x, y)$, we have

**Lemma 3.** *There is a constant $A_4 > 1$ depending only on the dimension $n$ (even independent of $D$) such that*

$$\exp(-A_4(k_D(x,y)+1)) \le \frac{h(x)}{h(y)} \le \exp(A_4(k_D(x,y)+1)) \tag{6}$$

*for every positive harmonic function $h$ on $D$.*

## 5. The boundary Harnack principle for domains with quasihyperbolic boundary condition

We shall show the global Carleson estimate in terms of the Green function. Then Theorem 7 gives the global boundary Harnack principle. Let $D$ be as in Theorem 3, i.e. $D$ satisfies the capacity density condition and the quasihyperbolic boundary condition of order $\alpha$, $0 < \alpha < 1$. Let $K$ and $F$ be a pair of disjoint compact sets satisfying (5) and let $x_0 \in K \cap D$. Set $U(r) = \{x \in \mathbb{R}^n : \operatorname{dist}(x, F) < r\}$ for $r > 0$. Let $2R = \operatorname{dist}(K, F)$ and put $U = U(R)$. Observe $\overline{U} \cap F = \emptyset$. Let $\omega_0 = \omega(D \cap \partial U, D \cap U)$ be the harmonic measure of $D \cap \partial U$ in $D \cap U$.

First let us compare $\omega_0$ and $G(x_0, \cdot)$.

**Lemma 4.** *Let $\omega_0 = \omega(D \cap \partial U, D \cap U)$. Then $\omega_0(y) \le AG(x_0, y)$ for $y \in F \cap D$.*

*Proof.* Let us employ the box argument. The main idea is to slice $U$ according to the distance from $F$ and the value of the Green function. Let $R_0 = R$ and

$$R_j = \Big(1 - \frac{3}{\pi^2}\sum_{k=1}^{j}\frac{1}{k^2}\Big)R \quad \text{for } j \ge 1.$$

Then it is easy to see that $R_{j-1} - R_j = (3R)/(\pi^2 j^2)$, so that

$$\sum_{j=1}^{\infty} R_j = \frac{R}{2}. \tag{7}$$

Since $G(x_0, \cdot)$ is bounded on $U \cap D$, we may assume that $u = G(x_0, \cdot)/A$ is bounded by 1 on $U \cap D$ by a suitable choice of $A > 1$.

Let $j \geq 1$ and set

$$U_j = \{y \in U(R_j) \cap D : 0 < u(y) < \exp(-2^j)\},$$
$$D_j = \{y \in U(R_j) \cap D : \exp(-2^{j+1}) \leq u(y) < \exp(-2^j)\}.$$

Let

$$q_j = \begin{cases} \sup_{D_j} \omega_0/u, & \text{if } D_j \neq \emptyset, \\ 0 & \text{if } D_j = \emptyset. \end{cases}$$

Since $0 < u < 1$ on $U \cap D$, it follows from (7) that $F \cap \partial D$ is included in the closure of $\bigcup_{j=1}^{\infty} D_j$. Hence it is sufficient to show that $q_j$ is bounded.

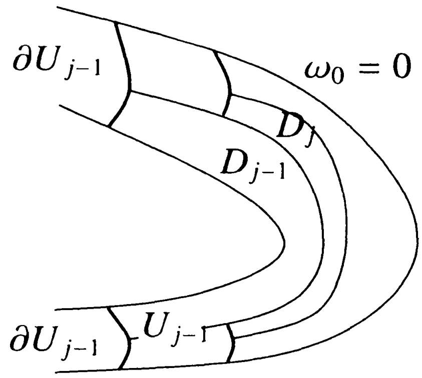



FIGURE 6. Box argument: slice of $U$.

By (3) and (6), we have

$$\exp(-2^j) > u(y) \geq \exp(-A_4(k_D(y, x_0) + 1)) \geq A \exp\Big(-\frac{A}{\delta_D(y)^\alpha}\Big) \quad \text{for } y \in U_j.$$

In other words, $\delta_D(y) \leq A2^{-j/\alpha}$ for $y \in U_j$. Hence, Lemma 1 implies that $w_\eta(U_j) \leq A2^{-j/\alpha}$. Observe that $\operatorname{dist}(D \cap \partial U_{j-1}, U_j) \geq R_{j-1} - R_j$. Applying the maximum principle on $U_{j-1}$, we obtain

$$\omega_0 \leq \omega(D \cap \partial U_{j-1} \setminus \overline{D_{j-1}}, U_{j-1}) + q_{j-1}u \quad \text{on } U_{j-1}.$$

Divide the both sides by $u$ and take the supremum over $D_j$. Then Lemma 2 yields

$$q_j \leq A \exp\big(2^{j+1} - ARj^{-2}2^{j/\alpha}\big) + q_{j-1}.$$

Since $0 < \alpha < 1$, it follows that $\sum_{j=1}^{\infty} \exp(2^{j+1} - ARj^{-2}2^{j/\alpha}) < \infty$, so that $q_j$ is bounded. The proof is complete. □

*Proof of Theorem 3.* It is sufficient to show the global Carleson estimate in terms of the Green function by Theorem 7. Let $K$ and $F$ be disjoint compact sets with (5). Let $2R = \mathrm{dist}(K, F)$ and put $U = U(R)$. Let $\omega_0 = \omega(D \cap \partial U, D \cap U)$. Then Lemma 4 gives $\omega_0(y) \le AG(x_0, y)$ for $y \in F \cap D$. Since the distance between $K$ and $U$ is positive, it is obvious that $G(x, y) \le A$ uniformly for $x \in K \cap D$ and $y \in U \cap D$. Fix $x \in K \cap D$ and apply the maximum principle to $G(x, \cdot)$ and $\omega_0$ to obtain $G(x, y) \le A\omega_0(y)$ for $y \in U \cap D$. Hence Lemma 4 yields

$$G(x, y) \le AG(x_0, y) \quad \text{for } y \in F \cap D,$$

as required. □

## 6. The boundary Harnack principle for Hölder domains

Let $0 < \beta < 1$. We may assume that $D$ is above the graph of a $\beta$-Hölder continuous function $\varphi$ in $\mathbb{R}^{n-1}$. Moreover we may assume that $\|\varphi\|_\infty \le 1$ and $\varphi(x) = 0$ for $|x| \ge 1$. For a point $x \in D$ we define $d(x) = x_n - \varphi(x')$, where $x = (x', x_n)$. Let $x_0$ be *high above* from the boundary. Suppose $x$ is just below $x_0$ and $0 < d(x) < 1$. In general, $\delta_D(x) \approx d(x)$ does not hold. We can assert only that $\delta_D(x)^\beta \ge Ad(x)$.

Considering the line segment connecting $x$ and $x_0$, we obtain the following estimate of the quasihyperbolic metric:

$$k_D(x, x_0) \le A \int_{d(x)}^{d(x_0)} \frac{dt}{t^{1/\beta}} \le Ad(x)^{1-1/\beta} + A. \tag{8}$$

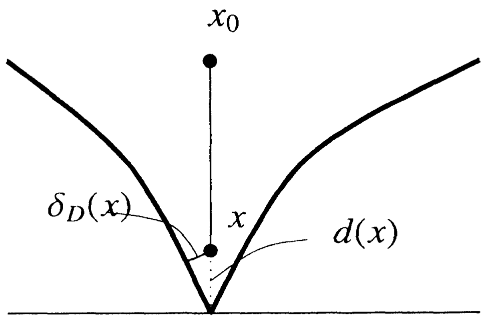



Figure 7. Hölder domain: $\delta_D(x)$ and $d(x)$.

The same estimate holds if $x \in D$ is close to 0, say $|x| < 1$. Now we have the counterpart of Lemma 1.

**Lemma 5.** *Let $D$ be as above. If $0 < r < 1$, then $w_\eta(\{x \in D : d(x) < r\}) \le Ar$ for some $\eta > 0$.*

This lemma can be proved by Fubini's theorem showing the volume density condition. In view of Theorem 7, it is sufficient to show the global Carleson estimate in terms of the Green function. Let $K$ and $F$ be a pair of disjoint compact sets with (5) and let $x_0 \in K \cap D$. Let $U(r) = \{x \in \mathbb{R}^n : \operatorname{dist}(x, F) < r\}$ for $r > 0$. Let $2R = \operatorname{dist}(K, F)$ and put $U = U(R)$. Observe $\overline{U} \cap F = \emptyset$. Let $\omega_0 = \omega(D \cap \partial U, D \cap U)$ be the harmonic measure of $D \cap \partial U$ in $D \cap U$. We can compare $\omega_0$ and $G(x_0, \cdot)$.

**Lemma 6.** *Let $\omega_0 = \omega(D \cap \partial U, D \cap U)$. Then $\omega_0(y) \le AG(x_0, y)$ for $y \in F \cap D$.*

*Proof.* The proof is similar to that of Lemma 4. We take $R_j$, $U_j$, $D_j$ and $q_j$ in the same way as in the proof of Lemma 4. It is sufficient to show that $q_j$ is bounded. By (8) and (6), we have

$$\exp(-2^j) > u(y) \ge \exp(-A_4(k_D(y, x_0) + 1)) \ge A \exp\left(-Ad(y)^{1-1/\beta}\right) \quad \text{for } y \in U_j.$$

In other words, $d(y) \le A2^{-j/(1/\beta-1)}$ for $y \in U_j$. Hence, Lemma 5 implies that $w_\eta(U_j) \le A2^{-j\beta/(1-\beta)}$. Applying the maximum principle on $U_{j-1}$, we obtain

$$\omega_0 \le \omega(D \cap \partial U_{j-1} \setminus \overline{D_{j-1}}, U_{j-1}) + q_{j-1}u \quad \text{on } U_{j-1}.$$

Divide the both sides by $u$ and take the supremum over $D_j$. Then Lemma 2 yields

$$q_j \le A \exp\left(2^{j+1} - ARj^{-2}2^{j\beta/(1-\beta)}\right) + q_{j-1}.$$

Since $\beta/(1-\beta) > 1$ for $\frac{1}{2} < \beta < 1$, it follows that $\sum_{j=1}^{\infty} \exp(2^{j+1} - ARj^{-2}2^{j\beta/(1-\beta)}) < \infty$, so that $q_j$ is bounded. □

*Proof of Theorem 4.* Using Lemma 6, we can prove the theorem in the same way as in the proof of Theorem 3. See [2] for details. □

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