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Lipschitz Denjoy domains, Brownian motion and positive harmonic functions

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Summary

Let $\Omega=\mathbb{R}^{d}\setminus E$ be a domain in $\mathbb{R}^{d}$ where E is a compact subset of a Lipschitz graph Σ. It is known that there is a partition $E=E_{1}\cup E_{2}$ of E where E_{1} is the set of *simple points* (w.r. to the Martin compactification of Ω) in E and E_{2} is the set of *double* points in E [2]. Also the harmonic measure (with respect to Ω) and $H_{d-1|E_{2}}$ are mutually absolutely continuous in E_{2} and $H_{d-1}(E_{1})=0$ [4]. The main result is that for almost every point ζ of E_{1} (w.r. to the harmonic measure in Ω) every open "cone" C in Σ with vertex at ζ is minimally unthin (w.r. to Ω) at ζ (or, in other words, a Brownian motion conditioned to exit from Ω at ζ hits with probability one $C\cap\Omega=C\setminus E$ during its lifetime). This extends to Lipschitz Denjoy domains one of the results of C. J. Bishop in [7] and in the same time gives a different method of proof. Using a similar method, another natural asymptotic result is also obtained.

1 Introduction and preliminaries.

In this paper, we deal with questions related to some of the results of C. J. Bishop in [7] (see **1.7** and **1.8** below) about the behavior of the Brownian motion in a Denjoy domain in $\mathbb{R}^{d}$, $d\geq 2$, that is a domain in $\mathbb{R}^{d}$ whose complement is contained in the hyperplane $x_{d}=0$.

We consider here a *Lipschitz Denjoy* domain Ω in $\mathbb{R}^d$: more precisely $\Omega = \mathbb{R}^d \setminus E$, with E compact in $\mathbb{R}^d$ and contained in the graph Σ_f of a function $f : \mathbb{R}^{d-1} \longrightarrow \mathbb{R}$ which is assumed to be k-Lipschitz for some constant $k \geq 1$ (fixed in all what follows). The main results (Theorems 2.1, 3.2 and 4.1) are first described in **1.9**.

Denote $\infty_{\mathbb{R}^d}$ the point at infinity in the Alexandrov compactification of $\mathbb{R}^d$ and set $\widehat{E} = E \cup \{\infty_{\mathbb{R}^d}\}$. If B is an open subset of $\widehat{E}$ and if h is harmonic in Ω, we say that h vanishes on B if (i) for each $\zeta \in B$, h is bounded in the trace on Ω of some neighborhood of ζ, and (ii) the set $B_h = \{\zeta \in B\,;\ \limsup\limits_{\Omega \ni x \to \zeta} |h(x)| > 0\}$ is polar. Recall (or take as a convention) that $\{\infty_{\mathbb{R}^d}\}$ is polar iff $d = 2$. If $h = 0$ on B, then h has the limit 0 at every Dirichlet-regular boundary point for Ω lying in B (including $\infty_{\mathbb{R}^d}$ if $d \geq 3$ and $B \ni \infty_{\mathbb{R}^d}$).

We now state some known basic properties of positive harmonic functions in such a domain Ω. Let $\mathcal{H}_+(\Omega)$ denote the set of all nonnegative harmonic functions in Ω.

1.1. For each point $P \in E$ the dimension of the cone $\mathcal{H}_P^+ = \{u \in \mathcal{H}_+(\Omega)\,;\ u$ vanishes in $\widehat{E} \setminus \{P\}\}$ is one or two, $\mathcal{H}_P^+$ being generated by one or two minimal harmonic functions in Ω (See [1], [6] for E contained in a hyperplane -or even a $C^{1,1}$ hypersurface [1]-, and [2] for the general case). Moreover P is a unique pole (see [17] for a definition) on $\widehat{E}$ for these minimal functions. Accordingly P is said to be a *simple boundary point* of Ω if $\mathcal{H}_P^+$ is one dimensional, and a *double boundary point* otherwise ([4]).

1.2. The set of double points is a Borel subset of E (in fact a F_σ-subset -see remark 1.3 below) and given a double point $P \in E$, one of the minimal attached to P is the limit -in the Martin topology- of every sequence $\{x_n\}$ converging nontangentially to P in $\Sigma_f^+ = \{(x', x_d)\,;\ x_d > f(x')\}$ (see [11]). This minimal is denoted h_P^+. The other minimal, denoted h_P^-, is similarly related to P and the strict subgraph Σ_f^- .

1.3. The property in **1.1** above is obtained in [2] (see p. 254) by establishing a boundary Harnack principle. Let $\zeta = (\zeta', \zeta_d) \in \Sigma_f$ and $r > 0$. Define $T_\zeta(r) := B_{d-1}(\zeta', r) \times (\zeta_d - 10\,k\,r, \zeta_d + 10\,k\,r)$ (where B_{d-1} means an open ball in $\mathbb{R}^{d-1}$). In the sequel it will be convenient to say that $T_\zeta(r)$ is a Σ_f adapted cylinder, that ζ is its center and r its radius. The point $A^+_{T_\zeta(r)} = (\zeta', \zeta_d + 5k\,r)$, (resp. $A^-_{T_\zeta(r)} = (\zeta', \zeta_d - 5k\,r)$) is the upper (resp. lower) reference point of $T_\zeta(r)$. A form of the above mentioned boundary Harnack principle says that if $f,\ g,\ h \in \mathcal{H}_+(\Omega \setminus T_\zeta(r/4))$ and $f = g = h = 0$ in $\widehat{E} \setminus T_\zeta(r/2)$, then with $A^\pm = A^\pm_{T_\zeta(r)}$ we have

$$h(x) \leq C\,\{\,\frac{h(A^+)}{f(A^+)}\,f(x) + \frac{h(A^-)}{g(A^-)}\,g(x)\,\} \quad \text{for } x \in \Omega \setminus T_\zeta(r) \tag{1.1}$$

Here C is a constant depending only on d and the bound k for the Lipschitz constant of f.

1.4. In the flat case, i.e. when $f = 0$, M. Benedicks [6] uses a different method and gives also a criterion of simplicity of $P \in E$. For a general Lipschitz-Denjoy domain there is no similar criterion (in contrast with the flat case, the multiplicity of $\zeta \in E$ is not in general an increasing function of the set E [5]).

1.5. Let $\omega = \omega^\Omega_Q$ denote the harmonic measure in Ω of some point $Q \in \Omega$ (viewed as a measure in E). Let $\omega = \omega_a + \omega_s$ be the Lebesgue decomposition of ω with respect to H_{d-1}, where ω_a is absolutely continuous with respect to H_{d-1} and ω_s is singular with respect to H_{d-1} (H_{d-1} will mean here the natural Riemannian measure on the graph Σ_f).

Then, H_{d-1}–almost all point $\zeta \in E$ is a double point (for Ω) and ω_s–almost all point in E is simple. Moreover $\omega_a \sim H_{d-1}$ on E (that is: the restrictions to E of ω_a and H_{d-1} are mutually absolutely continuous). ([4], see also [11])

A proof of these facts follows for the reader's convenience.

1.6. *Proof.* Recall $\Sigma_f^+ = \{(x,t) \in \mathbb{R}^{d-1} \times \mathbb{R}\,;\, t > f(x)\}$ denotes the open epigraph of f. By Naïm's results [17], if ω^+ is the harmonic measure in Σ_f^+ (evaluated at some point $x_0 \in \Sigma_f^+$), then $\omega^+_{|E}$ a.e. $P \in E$ is a pole of a minimal h_P in Ω for which Σ_f^- is thin. But ω^+ is equivalent to H_{d-1} (Dahlberg's theorem) and repeating the argument with Σ_f^- we find that H_{d-1}–a.e. point $P \in E$ is a double point for which moreover each "half-space" Σ_f^+, Σ_f^-, is minimally thin with respect to one of the minimals at P (h_P^- and h_P^+ respectively).

Thus there is a Borel set A of full H_{d-1}–measure in E and such that every $\zeta \in A$ is a double point for Ω. Since the harmonic measure ω^Ω in Ω is larger than $\omega^+_{|E}$, we may also assume that ω^Ω and H_{d-1} are mutually absolutely continuous on A.

It remains to see that ω^Ω–a.e. point in $E \setminus A$ is simple. If not, there exists a compact subset L of $E \setminus A$, consisting of double points and such that $\omega^\Omega(L) > 0$. Consider then the projection map $\pi : \Delta_1 \longrightarrow \widehat{E}$ on the minimal Martin boundary Δ_1 of Ω and the decomposition of $D = \pi^{-1}(L)$ into the disjoint sets $D_+ = \{h_P^+\,;\, P \in L\}$ and $D_- = \{h_P^-\,;\, P \in L\}$. These are G_δ subsets of Δ_1 (see remark 1.4 below) and hence also Polish spaces. By a standard regularity result we may find compact subsets K and K' of D_+ and D_- respectively such that $\omega^\Omega(\pi(K)) \geq \frac{3}{4}\omega^\Omega(L)$ and $\omega^\Omega(\pi(K')) \geq \frac{3}{4}\omega^\Omega(L)$ (note that $\alpha \mapsto \omega^\Omega(\pi(\alpha))$ defines a Borel measure on D_+ or on D_- –the map $\pi^{-1} : L \longrightarrow D_+$ being Borel as follows from a simple approximation of π^{-1} by a sequence of continuous maps, or from [8] p. 135). A slightly different argument can also be obtained using the capacitability theorem [8]. Passing to $L_0 = \pi(K) \cap \pi(K')$, we get a compact subset of L with positive harmonic measure and such that $\pi^{-1}(L_0)$ is a disjoint union $K_0^+ \cup K_0^-$ of two compact subsets of D_+ and D_- respectively. Moreover $L_0 = \pi(K_0^+) = \pi(K_0^-)$.

Since the harmonic measure $\tilde{\omega}^\Omega$ in Ω with respect to the Martin boundary projects onto the harmonic measure in Ω with respect to $\widehat{E}$, one of the sets

K_0^+ and K_0^- in the minimal Martin boundary of Ω has positive harmonic measure in Δ_1.

Assume that $\tilde{\omega}^\Omega(K_0^+) > 0$. Fixing some closed neighborhood V of K_0^- in the Martin compactification of Ω with $\overline{V} \cap K_0^+ = \emptyset$ we may attach to each point $P \in L_0$ a closed "downward" cone $\mathcal{C}_P^-$ of fixed small aperture with vertex at P and such that P is not in the closure in $\mathbb{R}^d$ of $(\mathcal{C}_P^- \setminus \{P\}) \setminus V$. Say $\mathcal{C}_P^- = P + \mathcal{C}^-$, $\mathcal{C}^- = \{(x', x_d)\,;\, x_d \leq -10\, k_f\, |x'|\,\}$ †.

Let F denote the union of these cones $\mathcal{C}_P^-$. For every $\zeta^+ \in K_0^+$, ζ^+ is not in the closure of $F \setminus L_0$, thus the set $F \setminus L_0$ is thin at ζ^+. Using again standard general properties of the Martin compactification [17], it follows that L_0 should be of positive harmonic measure in $\Omega \setminus F$. But $\mathbb{R}^d \setminus F$ is a Lipschitz domain and $H_{d-1}(L_0) = 0$, so this contradicts Dahlberg's theorem [12]. $\square$

Remark 1.1 In particular if $H_{d-1}(E) = 0$ then ω^Ω-a.e. point $\zeta \in E$ is simple.

Remark 1.2 The proof also shows that for ω a.e. double point $P \in E$, $\Sigma_f \setminus E$ is minimally thin at $h_P^\pm$.

Remark 1.3 Denote K_x the Martin kernel in Ω with pole at $x \in \Omega$ normalized at some fixed point $Q \in \Omega$ with say $d(Q, \Sigma_f) \geq 10$. Let $F_n :=$ $\{P \in E\,;\, \|K_x - K_{x'}\|_{\infty, B(Q,1)} \geq \frac{1}{n}$ for $x \in \mathcal{C}_P^+$, $x' \in \mathcal{C}_P^-$ and $|x - x'| \leq \frac{1}{n}\}$. The set F_n is closed in E and the set E_2 of double points in E is $\cup_{n \geq 1} F_n$. Thus E_2 is an F_σ subset of E.

Remark 1.4 For $h \geq 0$ in Ω and $A \subset \Omega$ recall the notation ${}^\Omega R_h^A$ (or R_h^A) for the function $w = \inf\{s\,;\, s \geq 0$ and superharmonic in Ω, $s \geq h$ in $A\}$ [9], [10]. Set $\mathcal{C}^n = \{(x', x_d) \in \mathcal{C}^+\,;\, \frac{1}{n} \leq x_d \leq 1\,\}$, $\mathcal{C}_P^n = P + \mathcal{C}_n$ and $\mathcal{C}_P = \cup_{n \geq 1} \mathcal{C}_P^n$. For $Q \in \Omega$, the map $h \mapsto R_h^{\mathcal{C}^n_{\pi(h)}}(Q)$ is continuous on Δ_1,

† Later we also use the notation $\mathcal{C}^+ = -\mathcal{C}^-$, $\mathcal{C}_P^+ = P + \mathcal{C}^+$.

so $h \mapsto R_h^{C_{\pi(h)}} = \sup_{n\geq 1} R_h^{C^n_{\pi(h)}}(Q)$ is l.s.c. It follows since $D^+ = \{h \in \Delta_1\,;\ \pi(h) \in L,\ R_h^{C_{\pi(h)}}(Q) = h(Q)\,\}$ (if $d(Q, \Sigma_f) \geq 10$) that D_+ is a G_δ set in Δ_1.

1.7. In the case $d = 2$ and for $f = 0$ (so Ω is a standard Denjoy domain), C. Bishop answered some questions of K. Burdzy about the behavior of the Brownian motion in Ω (ref. [7]).

One of the main results in [7] can be rephrased as follows in theorem 1.5 below. Note that in the flat case ($f = 0$) at hand, it is easy to see (by a symmetry argument) that, given $P \in E$, the set $\Sigma_f \setminus E$ is minimally thin at every minimal in $\mathcal{H}_+(\Omega)$ with pole at $P \in E$ if and only if P is a "double" point for Ω. Recall also the probabilistic meaning (given a minimal function $h \in \mathcal{H}_+(\Omega)$) of "$A \subset \Omega$ is minimaly thin at h": the last exit time from A of the h-Brownian motion in Ω is a.s. strictly smaller than his lifetime.

Theorem 1.5 (see Theorem 1 in [7]) *Suppose $d = 2$ and E is flat (more precisely $f = 0$). At ω-almost all simple point $P \in E$ the graph Σ_f (or rather $\Sigma_f \setminus E$) is minimally unthin on the "right" and on the "left" of P.*

Here we say that $\Sigma_f \setminus E$ is thin (resp. unthin) on the right at $P = (a, f(a)) \in E$ if $(\Sigma_f \setminus E) \cap \{(x, y)\,;\ x > a\,\}$ is minimally thin (resp. minimally unthin) at each minimal Martin boundary point above P. Using the results in the previous paragraphs, theorem 1.5 is easily translated into the following property of the Brownian motion (in short BM) -actually, the original formulation by Bishop, see Theorem 1 in [7].

Given $\varepsilon > 0$, for almost all $x \in E \subset \mathbb{R}$ (with respect to the harmonic measure in $\Omega = \mathbb{C} \setminus E$), a Brownian motion in Ω conditioned to exit at x will hit the interval $(x, x + \varepsilon)$ with probability one iff it hits the interval $(x - \varepsilon, x)$ with probability one.

Theorem 1.5 also means that if S is the lifetime of the BM in Ω, then almost surely the BM hits $\Sigma_f \setminus E$ on the right of B_S infinitely often as $t \uparrow S$ (i.e. there is a sequence of random times $t_n < S$ with $\lim t_n = S$ and $B_{t_n} \in \Sigma_f$, $(B_{t_n})_1 \geq (B_S)_1$) iff it hits $\Sigma_f \setminus E$ i.o. as $t \uparrow S$. Equivalently for almost every *simple* point $P \in E$- with respect to harmonic measure- the BM conditioned to exit from Ω at P a.s. hits Σ_f on the left of P and on the right of P. In particular if $H_1(E) = 0$ then the BM hits before time S the set Σ_f on the left of B_S almost surely.

1.8. A generalization of theorem 1.5 to dimensions $d \geq 3$ (for $f = 0$) is also obtained in [7] (see Theorem 5 there). Let us note another result from [7] which will not be considered here: if $d = 2$ and dim(E) < 1, then with probability one the planar BM stopped at time S separates its exit point from the rest of E. A conjecture of Lyons and Burdzy says that this should be true for every compact set E of dimension < 1 in the plane (for the BM stopped on E).

1.9. In what follows we show that theorem 1.5 extends to general Lipschitz functions f (the case $d \geq 3$ is considered in section 3). This also gives a new proof of theorem 1.5. Its proof in [7] relies among other things on a criterion (close to Benedicks criterion) characterizing left minimal thinness of $\Sigma_f \setminus E$ at $P \in E$ for $f = 0$ (Theorem 2 in [7]). As mentioned above a similar characterization is not available when f is only assumed to be k-Lipschitz.

1.10. From now on f is assumed to be a general Lipschitz function. As before, let S denote the lifetime of the BM stopped on E. We will show that almost surely

(i) after some time $t_1 < S$, B_t stays on one side of Σ_f if B_S is double.

(ii) If B_S is simple, then the BM $\{B_t\}_{t<S}$, hits "infinitely often" each of the upper and lower cones $C^+_{B_S}$, $C^-_{B_S}$ as $t \uparrow S$. (If Ω is a "flat" Denjoy domain, i.e. f is constant, for *every* simple point $\zeta \in E$ this is true -and

easy to see- for the BM conditioned to hit E at ζ. But this is not true for a general Lipschitz-Denjoy domain Ω [11].)

(iii) If $d = 2$, on the event $\{B_S$ is simple $\}$, the BM hits a.s. Σ_f on the right and on the left of $P = B_S$. For $d \geq 3$, conditionaly on $\{B_S$ is a simple point in $E\}$, the BM hits $\Gamma \cap \Sigma_f$ for every cone $\Gamma = B_S + C \times \mathbb{R}$ of vertex B_S (where C is a cone with vertex 0 and nonempty interior in $\mathbb{R}^{d-1}$). In other words, Theorem 1 and Theorem 5 in [7] can be extended to Lipschitz Denjoy domains.

In particular, (ii) and (iii) holds a.s. if $H_{d-1}(E) = 0$.

2 Extension of Theorem 1.5

In this section, we assume that $d = 2$. Recall that E is a compact subset of the Lipschitz graph (curve) Σ_f. Let $\omega = \omega^\Omega$ denote the harmonic measure with respect to Ω.

Theorem 2.1 *For ω^Ω almost every simple point $P \in E$ the graph Σ_f is minimally unthin on the right and on the left of P (w.r. to the unique minimal with pole at P). On the other hand, at ω^Ω–almost every double point $\zeta \in E$ the graph Σ_f (or rather $\Sigma_f \setminus E$) is minimally thin at both h_ζ^+ and h_ζ^-.*

The last claim has essentially been noticed in **1.6** above (first two paragraphs). It follows from the fact that the harmonic measures ω^+, ω^- (of some fixed points $Q^+ \in \Sigma_f^+$, $Q^- \in \Sigma_f^-$ respectively) in the Lipschitz domains Σ_f^+ and Σ_f^- are such that $H_1 \sim \omega^+ \sim \omega^-$ on E (and even on Σ_f). From this and Martin boundary theory [17], it follows that for H_1 a.e. point $\zeta \in E$, the subgraph Σ_f^- is minimally thin at h_ζ^+ (w.r. to Ω) (the minimal for the epigraph is then proportionnal to $h_\zeta^+ - {}^\Omega R_{h_\zeta^+}^{\Sigma_f^-}$). Whence the claim since the set of double points has full H_1 measure in E and $H_1 \sim \omega$ on this set.

To get the first claim it suffices to show the following: If K is compact $\subset E$, $H_1(K) = 0$, and $\omega(K) > 0$ there exists a simple point $\zeta \in K$ such that $\Sigma_f \setminus E$ is minimally unthin on the right of ζ in Ω.

Equivalently, it suffices to show that the following assumption

(H) : " $\exists K$ compact $\subset E$ such that $\omega(K) > 0$, every point of K is simple (w.r. to Ω) and the graph Σ_f is minimally thin (in Ω) on the right at each point of K"

leads to a contradiction. Let us transform (H). Passing to $E_0 = K$, $\Omega_0 = \mathbb{C} \setminus K$ we see that $H_1(E_0) = 0$, $\text{cap}(E_0) > 0$ and there exists a Borel set $A \subset E_0$ of positive harmonic measure in Ω_0 consisting of points $\zeta \in E_0$ that are simple points for Ω_0 and such that Σ_f is minimally thin on the right at ζ (w. r. to Ω_0). This is because the set of double points in K w.r. to Ω_0 has zero harmonic measure in Ω_0 (**1.5**, section 1) and because, by [17], at ω^Ω almost every point P in K, $\Omega_0 \setminus \Omega$ is minimally thin at P (w.r. to Ω_0) and minimal thinness w.r. to Ω implies minimal thinness w.r. to Ω_0.

Thus, there is a compact subset $K_0 \subset E_0$ of simple points for Ω_0 with $\omega_0(K_0) > 0$ and such that the graph $\Sigma_f \setminus E_0$ is thin on the right at each $\zeta \in K_0$.

We now distinguish two cases:

A. First case. Assume that the initial compact subset K of E has a lower capacitary density larger than some real $c_0 > 0$: by this we mean that for each point $x \in K$ and each $r \in (0, 1]$ the harmonic measure $\omega_{r,x,K}(y)$ of $B(x, r/4) \cap K$ in $B(x, r)$ is larger than c_0 at every point $y \in \partial B(x, r/2)$. We use later the notation $\underline{\delta}_K(x) := \inf\{\omega_{r,x,K}(y)\,;\, 0 < r \leq 1, y \in \partial B(x, r/2)\,\}$.

For $\zeta \in E_0$ simple with respect to Ω_0, denote by h_ζ the minimal harmonic function in Ω_0 with pole at ζ and normalized at some fixed point ξ_0 in Ω_0. Denote ω_0 the harmonic measure of ξ_0 in Ω_0. Consider the function

$$s(x) = \int_{E_0 \setminus K_0} h_\zeta(x)\, d\omega_0(\zeta) + \int_{K_0} \widehat{R}_{h_\zeta}^{\Sigma_f^{r,\zeta}}(x)\, d\omega_0(\zeta), \quad x \in \Omega_0$$

(recall that ω_0–a.e. point of E_0 is simple for Ω_0). Here the "réduite" is performed in Ω_0 and we have set $\Sigma_f^{r,\zeta} = (\Sigma_f \setminus E_0) \cap \{(x_1, x_2) \in \mathbb{R}^2\, ;\, \zeta_1 < x_1 \leq \zeta_1 + \ell\}$ where ℓ is fixed and such that $\ell \geq \operatorname{diam}(E_0)$.

It is easily checked that s is a positive superharmonic function in Ω_0 (using Fatou's lemma and Fubini theorem). Since for $A \subset E_0$, $\omega_\xi^{\Omega_0}(A) = \int_A h_\zeta(\xi)\, d\omega_0(\zeta)$ and since $\widehat{R}_{h_\zeta}^{\Sigma_f^{r,\zeta}}(x) = h_\zeta(x)$ in $\Sigma_f^{r,\zeta}$ we see that if $\Gamma = \widehat{\alpha\beta}$ is an open sub-arc of Σ_f contiguous to E_0, $s(x) \geq \omega_0(T_\alpha)$ in Γ where T_α is that part T_α of E_0 standing on the left of α. Using the capacitary density assumption on $K = E_0$ (at α) and Harnack inequality, it follows that for some positive constant c depending only on c_0 and k (and also the diameter of E_0 if larger than 1), we have $s(x) \geq c$ in $\Gamma' = \widehat{\alpha\beta'}$, the subarc of Γ such that $\beta_1' - \alpha_1 = \frac{2}{3}(\beta_1 - \alpha_1)$. In particular, $s(x) \geq c$ in $\Gamma'' := \widehat{\alpha'\beta'}$ if $\alpha' \in \Gamma'$ is such that $\alpha_1' - \alpha_1 = \frac{1}{3}(\beta_1 - \alpha_1)$. We note $\Gamma'' = \frac{1}{3}\Gamma$.

Thus $s \geq c$ in the "centered third" $\frac{1}{3}\Gamma_j$ of every bounded connected component Γ_j of $\Sigma_f \setminus E_0$. It follows then (see the next lemmas) that $s \geq c' > 0$ on all intervals contiguous to E_0 (where c' is another positive constant). Since $H_1(E_0) = 0$ the union of these intervals has full harmonic measure in Ω_0. Thus $s \geq c'$ in $\mathbb{C} \setminus E_0$, i.e. $s(x) \geq c' \int_{E_0} h_\zeta(x)\, d\omega_0(\zeta)$ for $x \in \Omega_0$.

Passing to the greatest harmonic minorant of s we get (since each $\widehat{R}_{h_\zeta}^{\Sigma_f^{r,\zeta}}$ for $\zeta \in K_0$ is a potential in Ω_0 by the minimal thinness at ζ of $\Sigma_f^{r,\zeta}$) that

$$\int_{E_0 \setminus K_0} h_\zeta(x)\, d\omega(\zeta) \geq c' \int_{K_0} h_\zeta(x)\, d\omega(\zeta)$$

for $x \in \Omega_0$. This is in contradiction with the fact that the Martin representation map $\mathcal{M}(\,\Delta_1(\Omega_0)\,) \to \mathcal{H}_+(\Omega_0)$ is a cone isomorphism. So theorem

2.1 is proven in the case where the lower capacitary density of K is $\geq c > 0$ in K. $\square$

Let us now complete this part of the proof by showing that $s \geq c'$ on each subarc I of Σ_f contiguous to E_0.

Lemma 2.2 *Let L be a compact subset of Σ_f such that $H_1(L) = 0$, denote $\tilde{\omega}$ the harmonic measure in $\mathbb{R}^2 \setminus L$, $\mathcal{I}$ the set of all bounded component of $\Sigma_f \setminus L$ and set $J_I = \frac{1}{3}I$ for $I \in \mathcal{I}$, $F := \cup_{I \in \mathcal{I}} J_I$. If T is a Σ_f-adapted cylinder such that $T \cap \Sigma_f = \overset{\frown}{PQ}$ with P, $Q \in L$ and if $A = A_T^+$, we have*

$$\tilde{\omega}_A(F \cap T) \geq c$$

for some constant $c > 0$ depending only on k.

This is well-known and can be deduced from the doubling property of $\tilde{\omega}$ (see [14]). A more direct agument is obtained as follows. For $I \in \mathcal{I}$, let T_I be the Σ_f adapted cylinder such that $T_I \cap \Sigma_f = I$ and $A_I^{\pm}$ the reference points of T_I. It is clear that $\tilde{\omega}_{A_I^{\pm}}(I) \leq 1 \leq C\,\tilde{\omega}_{A_I^{\pm}}(J_I)$ for some constant $C \geq 1$. So by the boundary Harnack principle $\tilde{\omega}_A(I) \leq cC\tilde{\omega}_A(J_I)$ (with another constant c). Summing over all $I \in \mathcal{I}$, $I \subset \overset{\frown}{PQ}$

$$cC\,\tilde{\omega}_A(F) \geq \tilde{\omega}_A((\Sigma_f \setminus L) \cap \overset{\frown}{PQ}) \geq u_A(\overset{\frown}{PQ})$$

if u is the harmonic measure with respect to $T \cap \Sigma_f^+$. We have used the fact that $u_A(L \cap \overset{\frown}{PQ}) = 0$ since $H_1(L) = 0$. Now $u_A(\overset{\frown}{PQ}) \geq c'$ for some constant $c' > 0$ depending only on c_0 and the lemma follows. $\square$

Lemma 2.3 *We retain the hypothesis and the notations of lemma 2.2. Then,*

$$\tilde{\omega}_x(F) \geq c_2$$

for every $x \in I$, $I \in \mathcal{I}$, and a constant c_2 depending only on k.

Proof. Let $I = \overset{\frown}{ab}$ be an "interval" in $\mathcal{I}$ and let $x \in I$, say in the left half of I (i.e. $x_1 \leq (a+b)/2$). By Harnack inequality we may assume that $x_1 - a_1 \leq \frac{1}{10}(b-a)$.

Let y be the leftmost point in L such that $a_1 - y_1 \leq \frac{3}{2}\,(x_1 - a_1)$. Then by lemma 2.2 applied to $\widehat{ya}$, one gets (if $y \neq a$) that $\omega_A(F) \geq c$ if $A = A_T^+$, T being the Σ_f–adapted cylinder such that $T \cap \Sigma_f = \widehat{ya}$.

If $a_1 - y_1 \geq \frac{1}{2}\,(x_1 - a_1)$, Harnack's inequalities imply that $\omega_x(F) \geq c''\omega_A(F) \geq c''\,c$, with $c'' = c''(k)$, which gives the desired conclusion.

Suppose now that $a_1 - y_1 < \frac{1}{2}\,(x_1 - a_1)$. Let z be the rightmost point in L with $z_1 < y_1$ if any. Denote $\tilde{T}$ the Σ_f–adapted cylinder of center a and radius $r = \min\{b_1 - a_1, a_1 - z_1\}$ (where $a_1 - z_1 = +\infty$ if $y_1 = \inf\{u_1\,;\,u \in L\}$).

Since $F \cap \tilde{T}$ contains an arc of length $\geq \frac{1}{6}r$ we have $\omega_{A_{\tilde{T}}}(F) \geq c_1 = c_1(k)$ and $\omega_\xi(F) \geq c'c_1$, $c' = c'(k)$, for ξ on the boundary of the shrinked cylinder $\frac{9}{10}\tilde{T}$. Since $\omega_\bullet(F) \geq c''\,c$, $c'' = c''(k)$, on the boundary of $\frac{11}{10}\,T$, it follows by the maximum principle that $\omega_x(F) \geq \min\{c'c_1, c''\,c\}$ (note that the case $y = a$ is trivial). $\square$

Remark 2.4 If one adds the assumption that L has a positive lower capacitary density, a slightly simpler argument gives a constant $c_2 = c_2(L) > 0$. This would suffice to complete the proof of theorem 2.1.

B. Second case. We are now left with the case where the lower capacitary density $\underline{\delta}_K(x)$ of K is zero for almost every $x \in K$ with respect to the harmonic measure class in Ω. Otherwise replacing K by a suitable subcompact $K' \subset K$ we are again in case A. For the notation $\underline{\delta}_K$ see the beginning of part A of the proof.

By restricting K we may even assume that $\underline{\delta}_K$ is zero in K. Moreover substituting $E_0 = K$ to E and choosing suitably $K_0 \subset K$ we may assume that : $\underline{\delta}_{E_0} = 0$ in E_0, $H_1(E_0) = 0$ (this is implied by $\underline{\delta}_{E_0} = 0$), $\omega(K_0) > 0$, every point ζ of K_0 is simple and such that the set $\Sigma_f \setminus E_0$ is minimally thin on the right of ζ (using once again the results of [17]).

To get a contradiction we will rely on the following lemma.

Lemma 2.5 *Let T be a Σ_f-adapted cylinder and let L be a compact subset of Σ_f. There is an $\varepsilon_1 > 0$ such that whenever the harmonic measure at A_T^+ of $T \cap L$ in T is less than ε_1, then for every function $h \in \mathcal{H}_+(T \setminus L)$*

$$\varepsilon_1\, h(A_T^+) \leq h(A_T^-) \leq \varepsilon_1^{-1}\, h(A_T^+) \tag{2.1}$$

and moreover, if $U := T \setminus L$, if x is the center of T,

$$R_h^{U \cap \Sigma_f^{r,x}}(A_T^+) \geq \varepsilon_1\, h(A_T^+) \tag{2.2}$$

and similarly $R_h^{U \cap \Sigma_f^{r,x}}(A_T^-) \geq \varepsilon_1\, h(A_T^-)$.

The réduites above are taken with respect ot the domain $U = T \setminus L$ and we have set here $\Sigma_f^{r,x} = \{z \in \Sigma_f\,;\, z_1 > x_1\}$.

Proof. The harmonic measure $\omega(A_T^+; B_T^-, T \setminus L)$ in $T \setminus L$ of the ball $B_T^- := B(A_T^-, r/10k)$ at A_T^+ is larger than $\omega(A_T^+; B_T^-, T) - \varepsilon_1 = c_k - \varepsilon_1 \geq \frac{1}{2}c_k$ if $\varepsilon_1 \leq \frac{1}{2}c_k$. Thus, by Harnack and maximum principle, there is an absolute constant $c' > 0$ such that

$$\begin{aligned} c'\omega(A_T^-; T \cap L, U) &\leq [\inf_{z \in B_T^-} \omega(z; T \cap L, U)]\, \omega(A_T^+; B_T^-, T \setminus L) \\ &\leq \omega(A_T^+; T \cap L, U) \leq \varepsilon_1. \end{aligned}$$

So $\omega(A_T^-; T \cap L, U)$ is also small for ε_1 sufficiently small and as before, for ε_1 small enough,

$$\omega(A_T^-; B_T^+, U) \geq \frac{1}{2}c_k. \tag{2.3}$$

Using again Harnack and maximum principle, it follows that for some constant c'

$$h(A_T^-) \geq \frac{1}{2}c_k\, c'\, h(A_T^+),$$

and similarly $h(A_T^+) \geq \frac{1}{2}c_k\, c'\, h(A_T^-)$, if ε_1 is small enough.

To prove (2.2), introduce the Σ_f-adapted cylinder T' with radius $\rho = r_T/2$ and center x' such that $x'_1 = x_1 + \rho$ (so that $\Sigma_f^{r,x'} \cap T = \Sigma_f^{r,x'} \cap T'$). By Harnack inequalities and standard argument similar to the above, we have for the réduite ${}^U R_h^{\Sigma_f^{r,x} \cap U}$ with respect to U,

$$ {}^U R_h^{\Sigma_f^{r,x} \cap U}(A_{T'}^-) \geq c' h(A_{T'}^+)\, \omega(A_{T'}^-; B_{T'}^+, T' \setminus L) \geq \frac{1}{2} c'\, c_k\, h(A_{T'}^+) \tag{2.4} $$

provided ε_1 is small enough. This follows from the inequality $h(z) \geq c' h(A_{T'}^+)\, \omega(z; B_{T'}^+, T' \setminus L)$ for $z \in U \cap T'$ and the maximum principle (the second inequality is (2.3)). And the lemma follows, using (2.1) and Harnack inequality. □

Remark 2.6 *Let x denote as before the center of T and $\Sigma_f^{r,x}$ the part of Σ_f on the right of x. For $\varepsilon_1 > 0$ small enough, (2.1) in lemma 2.5 holds for h positive harmonic in $T \setminus (\Sigma_f^{r,x} \bigcup L)$*

This remark follows on applying lemma 2.5 to the Σ_f-adapted cylinder T'' with radius $r_{T''} = r_T/2$ and center x'' such that $x''_1 = x_1 - r'$ (using again Harnack's inequality).

We now finish the proof of Theorem 2.1 in case **B**. Let $\varepsilon > 0$ and let $\zeta \in K_0$. We may choose arbitrarily small Σ_f-adapted cylinder T centered at ζ and such that the harmonic measure of E_0 in T is smaller than ε at the references points A_T^+ and A_T^-.

Thus taking $\varepsilon > 0$ small enough, it follows from the previous lemma that for these suitably chosen cylinders T (we let $\Omega = \mathbb{R}^2 \setminus E_0$ and denote h_ζ a minimal harmonic function in Ω with pole at ζ) :

$$ {}^\Omega R_{h_\zeta}^{T \cap \Sigma_f^{r,\zeta}}(A_T^+) \sim {}^\Omega R_{h_\zeta}^{T \cap \Sigma_f^{r,\zeta}}(A_T^-) \sim h_\zeta(A_T^-) \sim h_\zeta(A_T^+) \tag{2.5} $$

(which means that the first three quantities are between two constant times $h_\zeta(A_T^+)$). We have used here the fact that by definition

$$ h_\zeta(A_T^+) \geq {}^\Omega R_{h_\zeta}^{T \cap \Sigma_f^{r,\zeta}}(A_T^+) \geq {}^{T \cap \Omega} R_{h_\zeta}^{T \cap \Sigma_f^{r,\zeta}}(A_T^+) $$

By (2.5) and the boundary Harnack principle (1.1) we have (the réduites being taken w.r.t. to Ω) with another constant c

$$h_\zeta(x) \le c\, R_{h_\zeta}^{T\cap\Sigma_f^{r,\zeta}}(x) \le c\, R_{h_\zeta}^{\Omega\cap\Sigma_f^{r,\zeta}}(x)$$

for $x \in \Omega \setminus 2T$ (if $2T$ is the image of T under the dilation $x \mapsto \zeta + 2(x-\zeta)$).

Since we may choose arbitrarily small cylinders T at ζ, we get that

$$R_{h_\zeta}^{\Sigma_f^{r,\zeta}} \ge ch_\zeta \quad \text{in } \Omega = \mathbb{R}^2 \setminus E_0.$$

This means that $R_{h_\zeta}^{\Sigma_f^{r,\zeta}}$ is not a potential in U (and that in fact $R_{h_\zeta}^{\Sigma_f^{r,\zeta}} = h_\zeta$) and hence that $\Sigma_f^{r,\zeta}$ is not minimally thin at ζ which is a contradiction.

The proof of Theorem 2.1 is complete. □

3 Extension to the higher dimension case (d $\geq$ 3).

3.1. We start with a preliminary lemma asserting the existence of some coverings and allowing the extension of the argument used for the $d = 2$ case. Let C, C' be two nonempty open convex cones in $\mathbb{R}^{d-1}$ with vertex at the origin, rotation-invariant around the axis generated by $u_0 = (1, \ldots, 1)$ in $\mathbb{R}^{d-1}$ and such that

$$\overline{C} \setminus \{0\} \subset C' \subset \mathbb{R}_+^{d-1}. \tag{3.1}$$

Denote $C_1 = \{x \in C\,;\, x_1 + \cdots + x_{d-1} < 1\}$, $C_2 = \{x \in C'\,;\, x_1 + \cdots + x_{d-1} < 1\}$ two corresponding truncated cones, and $T = \{x \in \overline{C_1}\,;\, x_1 + \cdots + x_{d-1} = 1\}$ the base of $\overline{C_1}$.

Choose and fix a positive real ρ_1 so small that $\min\{x_1, \ldots, x_{d-1}\} \ge \rho_1 (x_1 + \cdots + x_{d-1})$ for $x \in C$. Choose then a point $a \in C_1$ on the axis of C_1 close enough to 0 so that $a + C_2 \supset T$ and $|a| < \frac{\rho_1}{2}$. Fix finally a small positive real $\rho < \frac{\rho_1}{2}$ such that $B(a, 2\rho) \subset C_1$ and then $A \ge 1$ such that $0 \in B(a, A\rho)$.

Lemma 3.1 *Let K be a compact subset of $Q_0 = [0,1]^{d-1}$ such that $0 \in K$. Then there is a family of disjoint balls $B(x^\alpha, r_\alpha) \subset Q_0 \setminus K$, $\alpha \in I$, in $\mathbb{R}^{d-1}$such that $(x^\alpha - \frac{r_\alpha}{\rho} C_2) \cap K \neq \emptyset$ and $\cup_{\alpha \in I} B(x^\alpha, 5Ar_\alpha) \supset (\overline{C} \cap Q_0) \setminus K$. Of course $r_\alpha \leq \frac{1}{2}$.*

Proof. For each point $x \in (\overline{C} \cap Q_0) \setminus K$, $0 \in x - (d-1)\overline{C_1}$ and there is a maximal $R = R_x \in (0, d-1]$ such that $x - R_x(\overline{C}_1 \setminus T) \subset K^c$. Then $(x - R_x T_1) \cap K \neq \emptyset$. Set $y_x = x - R_x a$, $r_x = R_x \rho$ and consider the family of balls $B(y_x, Ar_x)$, $x \in (\overline{C} \cap Q_0) \setminus K$, in $\mathbb{R}^{d-1}$. It covers $(\overline{C} \cap Q_0) \setminus K$ and hence by a well-known lemma from measure theory (see [16] p. 24) there is a subfamily of disjoint balls $B(y_x, Ar_x)$, $x \in I$, such that $\bigcup_{x \in I} B(y_x, 5A\, r_x) \supset (\overline{C} \cap Q_0) \setminus K$.

Also, for $x \in (\overline{C} \cap Q_0) \setminus K$, $B(y_x, r_x) = x - R_x\, B(a, \rho) \subset x - R_x(C_1 \setminus T) \subset K^c$. Since $0 \notin x - R_x(\overline{C}_1 \setminus T)$, we have $x \in \overline{C} \setminus R_x(\overline{C}_1 \setminus T)$ and, by the choice of ρ_1, $\min_k x_k \geq R_x \rho_1$. Thus for $z \in B(y_x, r_x)$ and $1 \leq j \leq d-1$, $z_j = x_j - R_x\, w_j > R\rho_1 - R\rho_1 = 0$ (because $w_j \in \overline{B}(a, \rho) \subset B(0, \rho_1)$) so that $B(y_x, r_x) \subset Q_0$.

Moreover since $y_x - r_x \rho^{-1} C_2 = x - R_x(a + C_2) \supset x - R_x T_1$, we have $(y_x - \frac{r_x}{\rho} C_2) \cap K \neq \emptyset$. So the family $\{B(y_x, r_x)\}_{x \in I}$ fulfills all the requirement of the lemma.□

3.2. We return now to the study of Lipschitz Denjoy domains. We also return to the notations used in section 1 and consider the Lipschitz Denjoy domain $\Omega = \mathbb{R}^d \setminus E$ defined by a compact subset E of the graph Σ_f, where $f : \mathbb{R}^{d-1} \longrightarrow \mathbb{R}$, is k-Lipschitz and $k \geq 1$ fixed .

Fix a nonempty open cone C_0 in $\mathbb{R}^{d-1}$ with vertex at the origin and set $\Gamma = \Gamma_{C_0} = C_0 \times \mathbb{R}$. If $\zeta \in E$ is a simple boundary point w.r. to Ω, we say now that Σ_f is minimally thin (with respect to Ω) at ζ in the direction C_0 (or Γ) if the set $[\Sigma_f \cap (\zeta + \Gamma)] \setminus E$ is minimally thin at ζ (w.r. to Ω).

The next statement generalizes Theorem 5 in [7] to Lipschitz Denjoy domains. As before $\omega = \omega^\Omega$ is the harmonic measure w.r. to the domain Ω.

Theorem 3.2 *For ω^Ω almost every simple point $P \in E$ the graph Σ_f is minimally unthin in the direction C_0 at P. On the other hand, at ω^Ω-almost all double point $\zeta \in E$ the graph Σ_f (or rather $\Sigma_f \setminus E$) is minimally thin at both h_ζ^+ and h_ζ^-.*

Clearly it follows that in fact ω^Ω a.e. simple point $\zeta \in E$ is such that the graph is minimally unthin at ζ in every possible direction Γ.

Proof. A straightforward extension of the argument in the first paragraph after the statement of theorem 2.1 proves the last claim.

To prove the first we may assume, using a rotation, that the cone C_0 in $\mathbb{R}^{d-1}$ contains the half-line $\{(t, \ldots, t)\,;\, t > 0\}$. We then fix two cones C and C' as above in **3.1** with $\overline{C} \setminus \{0\} \subset C' \subset \overline{C}' \setminus \{0\} \subset C_0 \cap (\mathbb{R}_+^*)^{d-1}$. Recall that $C_1 := \{x \in C\,;\, x_1 + \cdots + x_{d-1} = 1\}$, $C_2 := \{x \in C'\,;\, x_1 + \cdots + x_{d-1} = 1\}$.

We then follow the strategy of proof of theorem 2.1 (first claim) and after reductions completely similar to those made in this proof we are left with showing that the following two cases **A** and **B** are impossible.

Case A. The compact E has a lower capacitary density bounded from below on E: $\underline{\delta}_E(x) \geq c_0 > 0$ for $x \in E$, $H_{d-1}(E) = 0$, $K \subset E$ is compact, $\omega(K) > 0$ and Σ_f is minimally thin (w.r. to Ω) in the direction C_0 at each $\zeta \in K$.

We set $\Sigma_f^{\Gamma,\zeta} := (\Sigma_f \setminus E) \cap (\zeta + \Gamma)$ and introduce the following function

$$s(x) = \int_{E \setminus K} h_\zeta(x)\, d\omega_0(\zeta) + \int_K \widehat{R}_{h_\zeta}^{\Sigma_f^{\Gamma,\zeta}}(x)\, d\omega(\zeta) \quad , \; x \in \Omega$$

where h_ζ is a minimal harmonic function in Ω with pole at ζ (normalized at some point in Ω) and $\widehat{R}_{h_\zeta}^{\Sigma_f^{\Gamma,\zeta}} = R_{h_\zeta}^{\Sigma_f^{\Gamma,\zeta}}$ is a réduite with respect to Ω.

Fix $a = (a', a_d) \in E$, $0 < r_0 < 1$, and consider the $d-1$-dimensional cube $a' + r_0 Q_{d-1}$, $Q_{d-1} = [0,1]^{d-1}$. Let E' denote the projection of E in $\mathbb{R}^{d-1}$ and $E'_0 = E' \cap (a' + r_0 Q_{d-1})$.

Applying the lemma we obtain a cover of $(a' + r_0(\overline{C} \cap Q_{d-1})) \setminus E'$ by balls $B(x'_i, A' r_i)$ (of $\mathbb{R}^{d-1}$) such that the balls $B(x'_i, r_i)$ are disjoint and contained in $(a' + r_0 Q_{d-1}) \setminus E'$. Moreover there is a point $z'_i \in (x'_i - t_i r_i \overline{C}_2) \cap E'_0$ with $0 < t_i \leq \frac{1}{\rho}$ and we may choose z'_i associated to a *minimal* t_i. So $0 < \theta_0 \leq t_i \leq \frac{1}{\rho}$ (where θ_0 depends only on C_2), $x'_i \in z'_i + \frac{r_i}{\rho} \overline{C}_2$ and for a small enough positive constant β, we have $x'_i \in z' + C_0$ whenever $|z' - z'_i| \leq \beta r_i$ It follows that (if $z_i = (z'_i, f(z'_i))$, $x_i = (x'_i, f(x'_i))$ and the notation B_d refers to balls in $\mathbb{R}^d$)

$$\begin{aligned} s(x_i) &\geq \int_{E \setminus K} h_\zeta(x)\, d\omega_0(\zeta) + \int_{K \cap B_d(z_i, \beta r_i)} \widehat{R}_{h_\zeta}^{\Sigma_f^{\Gamma,\zeta}}(x_i)\, d\omega(\zeta) \\ &\geq \int_{E \cap B_d(z_i, \beta r_i)} h_\zeta(x_i)\, d\omega(\zeta) = \omega(x_i; E \cap B_d(z_i, \beta r_i); \Omega). \quad (3.2) \end{aligned}$$

But the assumption that $\underline{\delta}_E(z_i) \geq c_0$ implies that $\omega(x; E \cap B_d(z_i, \beta r_i); \Omega) \geq c_0$ for $x \in B_d(z_i, \frac{1}{2}\beta r_i)$. Since z'_i belongs to the boundary (in $\mathbb{R}^{d-1}$) of $B_{d-1}(x'_i, r_i) \cup (x'_i - t_i r_i C_2) \cap (a' + r_0 Q_{d-1})$, it follows using Harnack inequalities that $s(x_i) \geq \omega(x_i; E \cap B_d(y_i, \beta r_i); \Omega) \geq c_1$ for some positive constant c_1.

In that way it is now seen that in Σ_f^+ (the open epigraph of f), the function s is larger than a constant times the harmonic measure (in Σ_f^+) of the union of the pseudo-balls $\Sigma_f \cap B_d(x_i, r_i)$.

The doubling property of $\omega(a + r_0 e\,;\, .\,; \Sigma_f^+)$ where $e = (0, \ldots, 0, 1) \in \mathbb{R}^d)$ and the covering property of the balls $B(x_i, A r_i)$ imply that finally $s(a + r_0 e)$ is larger than a constant times the harmonic measure (in Σ_f^+ and at $a + r_0 e$) of $\{x \in \Sigma_f\,;\, x' \in (a' + r_0 Q_{d-1}) \cap (a' + C_1) \setminus E'\}$.

Thus we have shown that there is a positive constant c_3 (depending only on d, k and the positive capacitary condition) such that for each $x \in E$ and $0 < r \leq 1$, we have $s(x + re) \geq c_3$ and similarly $s(x - re) \geq c_3$.

To conclude we may for example use the general Fatou-Doob-Naïm theorem (see [17], [13] or [3],) and the fact that at each boundary simple point $\zeta \in E$ the union $C_\zeta^+ \cup C_\zeta^-$ is not minimally thin at ζ (w.r. to the domain Ω). The extended Fatou theorem says that the fine limit (w.r. to Ω) at $\zeta \in K$ of s is zero for ω-almost all $\zeta \in K$ (since each réduite $\widehat{R}_{h_\zeta}^{\Sigma_f^{r,\zeta}}$ is a potential in Ω, the second expression in the definition of s is a potential, whereas the first is a harmonic function generated by a measure not charging K). So by Harnack we should have for such ζ, $\liminf_{t\downarrow 0}(s(\zeta + te) \wedge s(\zeta - te)) = 0$ This gives a contradiction. $\square$

Case B. The set $E \subset \Sigma_f$ is compact, its lower capacitary density in $\mathbb{R}^d$ is zero at each of its points, and (w.r. to Ω) there is a simple point $x_0 \in E_0$ such that Σ_f is thin at x_0 in the direction C_0.

In fact the argument for the case **B** in the proof of theorem 2.1 does not really use the assumption $d = 2$ as the reader may easily check. It shows that E_0 has only simple points (w.r. to Ω) and that Σ_f is unthin in the direction C_0 at each $\zeta \in E_0$. $\square$

4 Another statistical unthinness result

Recall the notations (section 1) $\mathcal{C}^+ = \{(x', x_d)\,;\, x_d \geq 10\,k\,|x'|\,\} \in \mathbb{R}^{d-1} \times \mathbb{R}$, $\mathcal{C}_P^+ = P + \mathcal{C}^+$, $\mathcal{C}_P^- = P - \mathcal{C}^+$. As before, $\Omega = \mathbb{R}^d \setminus E$ is a Lipschitz Denjoy domain with E a compact subset of $\mathbb{R}^d$ contained in Σ_f.

Theorem 4.1 *For $\omega = \omega^\Omega$ almost every simple point $\zeta \in E$, both upper and lower cones, $\mathcal{C}_\zeta^+$ and $\mathcal{C}_\zeta^-$, are minimally unthin at ζ (w.r. to Ω).*

We note here that this result is well-known when Ω is a flat Denjoy domain and in fact $\mathcal{C}_\zeta^+$ and $\mathcal{C}_\zeta^-$ are in this case unthin at every simple point $\zeta \in E$ (a fact which does not extend to the general Lipschitz Denjoy case).

Proof. We argue again by contradiction and so assume the existence of a compact subset K of E whose elements are all simple and such that :(i) $\omega^\Omega(K) > 0$, (ii) C_ζ^+ is minimally thin at ζ (w.r. to $\Omega = \mathbb{R}^d \setminus E$) for all $\zeta \in K$. This of course implies that $H_{d-1}(K) = 0$.

Again after replacing E by $E_0 = K$, we may assume that ω almost every point of E is simple ($H_{d-1}(E) = 0$).

As in the preceding sections, we distinguish two cases:

A. First case : E has a lower capacitary density $\geq c > 0$ at each of its points.

Denote $C^s = \{(x', x_d)\,;\, x_d \geq 20\,k\,|x'|\,\} \in \mathbb{R}^{d-1} \times \mathbb{R}$, $C_P^s = P + C^s$. Let $\Phi = \overline{B}(0,R) \cap [\bigcup_{\zeta \in K} C_\zeta^s]$ where R is so large that $E \subset B(0, R/2)$ and denote u the harmonic measure of $E \setminus K$ in $\Omega = \mathbb{R}^d \setminus E$. We construct now the function:

$$p(x) = q(x) + \int_K \widehat{R}_{h_\zeta}^{C_\zeta^s}(x)\, d\omega(\zeta) \tag{4.1}$$

where $q = R_u^\Phi$ is the réduite of u on $\Phi \setminus E$ with respect to Ω and the notation h_ζ has the same meaning as before.

The function p is a potential in Ω, i.e. its greatest harmonic minorant in Ω is 0 (note that Φ is thin at every minimal point with pole in $E \setminus K$ so that q is a potential in Ω). The capacitary density assumption for E implies that $p \geq c$ on Φ: in fact for $x \in [C_\zeta^s \setminus \{0\}] \cap \overline{B}(0,R)$ the harmonic measure of $(x - C^+) \cap E$ is $\geq c$ at x, (first for x sufficiently close to ζ and then by Harnack for $|x - \zeta| \leq R$).

Thus, since K is of zero harmonic measure for $\mathbb{R}^d \setminus \Phi$ (Dahlberg's theorem), K is also of zero harmonic measure in $\Omega \setminus \Phi$ and

$$p(x) \geq c\,\omega(x; \Phi; \Omega \setminus \Phi) \geq c\,\omega(x; K; \Omega) \tag{4.2}$$

by the maximum principle and the definition of harmonic measure. But since the greatest harmonic minorant of p in Ω must be zero it follows that

$\omega^{\Omega}(K) = 0$ which is a contradiction. $\square$

B. Second case : The compact set K has a zero lower capacitary density at ω-a.e. $\zeta \in K$ (otherwise we are easily reduced to the first case).

In this case, we may reduce ourselves to the case where $H_{d-1}(E) = 0$ the lower capacitary density of E vanishes everywhere, $R_{h_\zeta}^{C_\zeta^+} \neq h_\zeta$ for all $\zeta \in K$, $K \subset E$ and $\omega(K) > 0$.

Then as already seen in section 2 (case **B**) -and using similar notations- a point $\zeta \in E$ is necessarily simple and for each $\zeta \in E$ there are arbitrarily small r such that $R_{h_\zeta}^{B_{\zeta,r}^+} \geq Ch_\zeta$ out of $T_{\zeta,r}$ (using the relation $R_{h_\zeta}^{B_{\zeta,r}^+}(A_{\zeta,r}^-) \geq ch_\zeta(A_{\zeta,r}^-)$ and then the boundary Harnack principle).

So, we get $R_{h_\zeta}^{C_\zeta^+} \geq Ch_\zeta$ in Ω, which is a contradiction when $\zeta \in K$. $\square$

5 Two open problems

1) The above does not answer the following very natural question for the $d = 2$ case. Consider the Brownian motion $X = \{X_t\}_{0 \leq t < S}$ in a planar Lipschitz Denjoy domain $\Omega = \mathbb{C} \setminus E$ starting from a point z_0 in Ω and conditioned to exit from Ω at some simple boundary point $\zeta \in E$. Let $\{\theta_t\}_{t<S}$ be the continuous determination of the argument of $\{X_t - \zeta\}_{0 \leq t < S}$ with say $0 \leq \theta_0 < 2\pi$. Is it true that for ω^{Ω} almost every ζ, $\limsup_{t \uparrow S} \theta_t = +\infty$ a.s. ?

Note that this is obviously true when Ω is symmetric w.r. to $x_2 = 0$ (the flat case) -using a simple Borel-Cantelli argument-. We remark also that it suffices to solve the question when $H_1(E) = 0$.

2) Another obvious question is : does Bishop's theorem 3 in [7] extends to Lipschitz Denjoy domains in the plane ? namely does the Burdzy-Lyons conjecture (see [7]) for a compact $E \subset \mathbb{C}$ with $\dim(E) < 1$ holds when $\Omega = \mathbb{C} \setminus E$ is a Lipschitz Denjoy domain in $\mathbb{C}$?

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