数理解析研究所講究録
第 1669 巻 2009 年 68-80

Variable exponent version of Hedberg-Wolff inequalities

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Introduction.

Hedberg-Wolff gave the following inequalities in [HW]:

$$C^{-1}\int_{\mathbf{R}^N}[(G_\alpha*\mu)(x)]^{p'}\,dx \;\le\; \int_{\mathbf{R}^N}\mathcal{W}^\mu_{\alpha,p}(x,1)\,d\mu(x) \;\le\; C\int_{\mathbf{R}^N}[(G_\alpha*\mu)(x)]^{p'}\,dx \qquad (1)$$

for every nonnegative measure μ on $\mathbf{R}^N$ with a positive constant C independent of μ, where G_α is the Bessel kernel of order α $(0<\alpha<N)$ on $\mathbf{R}^N$, $1<p<\infty$, $1/p+1/p'=1$ and

$$\mathcal{W}^\mu_{\alpha,p}(x,R)=\int_0^R\left(\frac{\mu(B(x,r))}{r^{N-\alpha p}}\right)^{p'-1}\frac{dr}{r} \qquad (R>0).$$

The function $\mathcal{W}^\mu_{\alpha,p}(\cdot,R)$ is called the *Wolff-potential* of μ fo order (α,p). Inequalities (1) imply

$$\mu\in(\mathcal{L}^{\alpha,p}(\mathbf{R}^N))^* \quad\Leftrightarrow\quad \int_{\mathbf{R}^N}\mathcal{W}^\mu_{\alpha,p}(x,1)\,d\mu(x)<\infty, \qquad (2)$$

where

$$\mathcal{L}^{\alpha,p}(\mathbf{R}^N)=\{u=G_\alpha*f\,;\,f\in L^p(\mathbf{R}^N)\}$$

with the norm $\|u\|_{\alpha,p}=\|f\|_p$. Since $\mathcal{L}^{m,p}(\mathbf{R}^N)=W^{m,p}(\mathbf{R}^N)$ for $m\in\mathbf{N}$ (A.P. Calderón), (2) shows that

$$\mu\in(W^{m,p}(\mathbf{R}^N))^* \quad\Leftrightarrow\quad \int_{\mathbf{R}^N}\mathcal{W}^\mu_{m,p}(x,1)\,d\mu(x)<\infty, \qquad (2')$$

for $m\in\mathbf{N}$.

In [AH], the proof of (1) is given via the following inequalities

$$C^{-1}\|M_{\alpha,R}\,\mu\|_q\le\|G_\alpha*\mu\|_q\le C\|M_{\alpha,R}\,\mu\|_q \qquad (3)$$

for $0<q<\infty$ and $R>0$ with a positive constant C independent of μ, where

$$(M_{\alpha,R}\,\mu)(x)=\sup_{0<r<R}r^{\alpha-N}\mu(B(x,r)).$$

These results have been generalized to the case where G_α is replaced by a general convolution kernel satisfying certain conditions (cf. [JPW], [AE, Part II]).

In the present paper, we consider variable exponents $p(x)$ on $\mathbf{R}^N$ and show that inequalities (1) and (3) hold in some restricted forms, and relations (2) and (2′) still hold true for μ with finite total mass when we replace p by $p(x)$ satisfying certain conditions. We discuss these for convolution kernels.

2000 Mathematics Subject Classification : 26D10, 31C45, 46E30
Key words and phrases : Hedberg-Wolff inequalities, variable exponent, convolution potential, Wolff potential

1. Definitions

As a potential kernel function on $\mathbf{R}^N$, we consider $k(x) = k(|x|)$ (with the abuse of notation) with a nonnegative nonincreasing lower semicontinuous function $k(r)$ on $(0, \infty)$ such that

(k.1) there is $R_0 > 0$ such that $k(r)$ is positive and satisfies the doubling condition on $(0, R_0)$, i.e., $k(r) \le C_d k(2r)$ for $0 < r < R_0/2$;

(k.2) $\int_0^1 k(r) r^{N-1}\, dr < \infty$.

By (k.2), $k(x) \in L^1_{loc}(\mathbf{R}^N)$. The k-potential of a nonnegative measure μ on $\mathbf{R}^N$ is defined by

$$(k * \mu)(x) = \int k(x-y)\, d\mu(y).$$

For $R > 0$, the (k, R)-maximal function of μ is defined by

$$(M_{k,R}\,\mu)(x) = \sup_{0<r<R} k(r)\mu(B(x,r)).$$

We consider a variable exponent $p(x)$ on $\mathbf{R}^N$ such that

(P1) $1 < p^- := \inf p(\cdot) \le p^+ := \sup p(\cdot) < \infty$;

(P2) $p(\cdot)$ is log-Hölder continuous, namely

$$|p(x) - p(y)| \le \frac{C_p}{\log(1/|x-y|)} \quad \text{for } |x-y| \le \frac{1}{2}$$

with a constant $C_p \ge 0$, which is referred to as the constant of log-Hölder continuity.

We refer to [KR] for the definition of the $p(\cdot)$-norm $\|f\|_{p(\cdot)}$, the variable exponent Lebesgue space $L^{p(\cdot)}(\mathbf{R}^N)$ and the variable exponent Sobolev space $W^{m,p(\cdot)}(\mathbf{R}^N)$ ($m \in \mathbf{N}$).

For $R > 0$, we define the $(k, p(\cdot))$-Wolff potential of μ by

$$\mathcal{W}^{\mu}_{k,p(\cdot)}(x, R) = \int_0^R k(r)^{p(x)} \mu(B(x,r))^{p(x)-1} r^{N-1}\, dr.$$

Example. For $0 < \alpha < N$, the Riesz kernel $I_\alpha(x) = 1/|x|^{N-\alpha}$ and the Bessel kernel G_α of order α are typical examples of $k(x)$. For these kernels, we can take R_0 any positive value.

$$\mathcal{W}^{\mu}_{\alpha,p(\cdot)}(x, R) := \mathcal{W}^{\mu}_{I_\alpha,p(\cdot)'}(x, R) = \int_0^R \left(\frac{\mu(B(x,r))}{r^{N-\alpha p(x)}}\right)^{p(x)'-1} \frac{dr}{r}$$

and

$$\mathcal{W}^{\mu}_{G_\alpha,p(\cdot)}(x, R) \sim \mathcal{W}^{\mu}_{\alpha,p(\cdot)'}(x, R).$$

For a nonnegative measure μ and $R > 0$, let

$$M(\mu, R) := \sup_{x \in \mathbf{R}^N} \mu(B(x, R)).$$

It is easy to see that if $M(\mu, R) < \infty$ holds for some $R > 0$, then so holds for all $R > 0$.

Lemma 1. *If either $k * \mu \in L^{p(\cdot)}(\mathbf{R}^N)$ or $M_{k,R}\,\mu \in L^{p(\cdot)}(\mathbf{R}^N)$ or*

$$\int \mathcal{W}^{\mu}_{k,p(\cdot)}(x, R)\, d\mu(x) < \infty,$$

then $M(\mu, R) < \infty$ for all $R > 0$.

Proof. Suppose that $M(\mu, R) = \infty$ for some $R > 0$. As remarked above, we may assume $0 < R < R_0$ and $M(\mu, R/3) = \infty$. Then, for every $n \in \mathbf{N}$, there exists $\xi_n \in \mathbf{R}^N$ such that $\mu(B(\xi_n, R/3)) \geq n$. If $x \in B(\xi_n, R/3)$, then $\mu(B(x, 2R/3)) \geq n$, so that

$$(k * \mu)(x) \geq \int_{B(x,2R/3)} k(x-y)\, d\mu(y) \geq k(R)n$$

and

$$(M_{k,R}\,\mu)(x) \geq k(2R/3)\mu(B(x, 2R/3)) \geq k(R)n.$$

Thus

$$\int [(k * \mu)(x)]^{p(x)}\, dx \geq \int_{B(\xi,R/3)} [(k * \mu)(x)]^{p(x)}\, dx \geq C_1 n^{p^-}$$

with a constant $C_1 > 0$ independent of n. This shows that $k * \mu \notin L^{p(\cdot)}(\mathbf{R}^N)$. Similarly, we see that $M_{k,R}\,\mu \notin L^{p(\cdot)}(\mathbf{R}^N)$.

Also, if $x \in B(\xi_n, R/3)$, then

$$\begin{aligned}\mathcal{W}^{\mu}_{k,p(\cdot)}(x, R) &\geq \int_{2R/3}^{R} k(r)^{p(x)} \mu(B(x,r))^{p(x)-1} r^{N-1}\, dr \\ &\geq \int_{2R/3}^{R} k(R)^{p(x)} \mu(B(x, 2R/3))^{p(x)-1} r^{N-1}\, dr \geq C_2 n^{p^- - 1}\end{aligned}$$

with a constant $C_2 > 0$ independent of n, so that

$$\int \mathcal{W}^{\mu}_{k,p(\cdot)}(x, R)\, d\mu(x) \geq \int_{B(\xi_n,R/3)} \mathcal{W}^{\mu}_{k,p(\cdot)}(x, R)\, d\mu(x) \geq C_2 n^{p^-}$$

for all $n \in \mathbf{N}$.

We call $\int \mathcal{W}^{\mu}_{k,p(\cdot)}(x, R)\, d\mu(x)$ the $(k, p(\cdot))$-energy of μ.

2. Estimate of $(k, p(\cdot))$-energy by $p(\cdot)$-integral of k-potential

Theorem 1. *Let $M_0 \geq 1$, $0 < R < R_0/2$. Then*

$$\int \mathcal{W}^{\mu}_{k,p(\cdot)}(x, R)\, d\mu(x) \leq C \left(\mu(\mathbf{R}^N) + \int [(k * \mu)(x)]^{p(x)}\, dx \right)$$

for all nonnegative measure μ such that $M(\mu, R) \leq M_0$, with a constant $C > 0$ depending only on N, C_d, p^+, C_p, M_0 and $K_R := \int_0^R k(r) r^{N-1}\, dr$.

Proof. We consider a nonlinear potential

$$V^{\mu}_{k,p(\cdot)} = k * (k * \mu)^{p(\cdot)-1}.$$

Since

$$\int [(k * \mu)(x)]^{p(x)}\, dx = \int V^{\mu}_{k,p(\cdot)}(x)\, d\mu(x),$$

it suffices to show

$$\mathcal{W}^{\mu}_{k,p(\cdot)}(x, R) \le C\big(1 + V^{\mu}_{k,p(\cdot)}(x)\big). \tag{2.1}$$

for $0 < R < R_0/2$.

Since $k(r)$ is nonincreasing and $K_R < \infty$, $k(r) \le NK_R r^{-N}$ for $0 < r < R$. Hence, (P2) implies that

$$\big(k(y)\mu(B(x, |y|))\big)^{p(x)} \le C\big(k(y)\mu(B(x, |y|))\big)^{p(x-y)} \tag{2.2}$$

for $|y| \le R$ whenever $M(\mu, R) \le M_0$ and $k(y)\mu(B(x, |y|)) \ge 1$, with a constant $C = C(N, K_R, C_p, M_0) > 0$.

If $|y| \le R$, then $|x - y - \xi| \le 2|y|$ for $\xi \in B(x, |y|)$, so that

$$k(y)\mu(B(x, |y|)) \le C_d k(2y)\mu(B(x, |y|)) \le C_d(k * \mu)(x - y).$$

Hence, using (2.2) we have

$$\begin{aligned}
\mathcal{W}^{\mu}_{k,p(\cdot)}(x, R) &= \frac{1}{\sigma_N} \int_{\{|y|<R\}} k(y)\big(k(y)\mu(B(x, |y|))\big)^{p(x)-1}\, dy \\
&\le \frac{1}{\sigma_N} \int_{\{|y|<R\}} k(y)\, dy + C \int_{\{|y|<R\}} k(y)\big(k(y)\mu(B(x, |y|))\big)^{p(x-y)-1}\, dy \\
&\le K_R + C \int_{\{|y|<R\}} k(y)[(k * \mu)(x - y)]^{p(x-y)-1}\, dy \\
&\le K_R + C V^{\mu}_{k,p(\cdot)}(x),
\end{aligned}$$

with constants $C = C(N, C_d, K_R, p^+, C_p, M_0) > 0$, which shows (2.1). (Here, σ_N denotes the surface area of the unit sphere in $\mathbf{R}^N$.)

Remark. In Theorem 3, it is not known whether the term $\mu(\mathbf{R}^N)$ is really necessary.

On the other hand, for non-constant exponent $p(\cdot)$, the following inequality *does not hold* even if $M(\mu, R) \le M_0$:

$$\int [(k * \mu)(x)]^{p(x)}\, dx \le C \int \mathcal{W}^{\mu}_{k,p(\cdot)}(x, R)\, d\mu(x).$$

In fact, if $p(\cdot)$ is continuous and non-constant in $\mathbf{R}^N$, then we can find nonnegative measures $\{\mu_j\}$ such that $M(\mu_j, R) \to 0$ $(j \to \infty)$ and

$$\frac{\displaystyle\int \mathcal{W}^{\mu_j}_{G_\alpha,p(\cdot)}(x, R)\, d\mu_j(x)}{\displaystyle\int [(G_\alpha * \mu_j)(x)]^{p(x)}\, dx} \to 0 \tag{2.3}$$

as $j \to \infty$ for every $0 < \alpha < N$ and every $R > 0$.

Proof. We can choose two compact sets K_1 and K_2 and $1 < p_1 < p_2 < \infty$ such that $|K_1| > 0$, $|K_2| > 0$,

$$p(x) \le p_1 \quad \text{for } x \in K_1 \qquad \text{and} \qquad p(x) \ge p_2 \quad \text{for } x \in K_2.$$

Let $\mu_j = (1/j)\chi_{K_2}dx$, $j = 1, 2, \ldots$. Obviously, $M(\mu_j, R) \to 0$.

Since $\mu_j(B(x,r)) \le (1/j)c_N r^N$ for any $x \in \mathbf{R}^N$ and $r > 0$,

$$\begin{aligned}\mathcal{W}^{\mu_j}_{G_\alpha,p(\cdot)}(x,R) &\le C(\alpha,N,p^+)\int_0^R (r^{\alpha-N})^{p(x)}[(1/j)r^N]^{p(x)-1}r^{N-1}\,dr \\ &= C(\alpha,N,p^+)(1/j)^{p(x)-1}\int_0^R r^{\alpha p(x)-1}\,dr \le C(\alpha,N,p^+,p^-,R)(1/j)^{p(x)-1}\end{aligned}$$

with constants $C(\ldots) > 0$. If $x \in K_2$, then $(1/j)^{p(x)-1} \le (1/j)^{p_2-1}$, so that

$$\int \mathcal{W}^{\mu_j}_{G_\alpha,p(\cdot)}(x,R)\,d\mu_j(x) \le C(\alpha,N,p^+,p^-,R)(1/j)^{p_2}|K_2|. \tag{2.4}$$

On the other hand, since G_α is positive continuous on $\mathbf{R}^N$,

$$A = A(\alpha, K_1, K_2) := \inf\{G_\alpha(x-y); x \in K_1,\ y \in K_2\} > 0.$$

If $x \in K_1$, then

$$(G_\alpha * \mu_j)(x) = (1/j)\int_{K_2} G_\alpha(x-y)\,dy \ge (1/j)A|K_2|.$$

Thus,

$$\begin{aligned}\int (G_\alpha * \mu_j)^{p(x)}\,dx &\ge \int_{K_1} [(1/j)A|K_2|]^{p(x)}\,dx \\ &\ge (1/j)^{p_1}\min(A|K_2|,1)^{p_1}|K_1|.\end{aligned} \tag{2.5}$$

In view of (2.4) and (2.5), we obtain (2.3), since $p_2 > p_1$.

3. Estimate of $p(\cdot)$-integral of (k,R)-maximal function by $(k,p(\cdot))$-energy

Theorem 2. *Let $M_0 \ge 1$ and $0 < R < R_0/3$. Then*

$$\int [(M_{k,R}\,\mu)(x)]^{p(x)}\,dx \le C\left(\mu(\mathbf{R}^N) + \int \mathcal{W}^{\mu}_{k,p(\cdot)}(x,3R)\,d\mu(x)\right)$$

for all nonnegative measure μ such that $M(\mu, 3R) \le M_0$, with a constant $C > 0$ depending only on N, C_d, K_R, p^+, C_p and M_0.

Proof. Let $0 < R < R_0/3$. For $0 < r < R$,

$$\begin{aligned}\int_0^{3R/2} k(t)^{p(x)}\mu(B(x,t))^{p(x)}\frac{dt}{t} &\ge \int_r^{3r/2} k(2r)^{p(x)}\mu(B(x,r))^{p(x)}\frac{dt}{t} \\ &\ge \log(3/2)C_d^{-p^+}[k(r)\mu(B(x,r))]^{p(x)}.\end{aligned}$$

Hence

$$\left[(M_{k,R}\,\mu)(x)\right]^{p(x)} \leq 3C_d^{p^+} \int_0^{3R/2} k(t)^{p(x)} \mu(B(x,t))^{p(x)} \frac{dt}{t},$$

and so

$$\int \left[(M_{k,R}\,\mu)(x)\right]^{p(x)} dx \leq 3C_d^{p^+} \int_0^{3R/2} \left(\int k(t)^{p(x)} \mu(B(x,t))^{p(x)} \, dx \right) \frac{dt}{t}.$$

Now,

$$\begin{aligned} &\int k(t)^{p(x)} \mu(B(x,t))^{p(x)} \, dx \\ &\quad = \int k(t)^{p(x)} \mu(B(x,t))^{p(x)-1} \left(\int \chi_{B(x,t)}(y) \, d\mu(y) \right) dx \\ &\quad = \int \left(\int \chi_{B(y,t)}(x) k(t)^{p(x)} \mu(B(x,t))^{p(x)-1} \, dx \right) d\mu(y) \\ &\quad = \int \left(\int_{B(y,t)} k(t)^{p(x)} \mu(B(x,t))^{p(x)-1} \, dx \right) d\mu(y). \end{aligned}$$

As in the proof of Theorem 1, we have

$$\left[k(t)\mu(B(x,t))\right]^{p(x)-1} \leq C\left[k(t)\mu(B(x,t))\right]^{p(y)-1} \leq C\left[k(t)\mu(B(y,2t))\right]^{p(y)-1}$$

whenever $|x-y| < t < 3R/2$, $M(\mu, 3R) \leq M_0$ and $k(t)\mu(B(x,t)) \geq 1$, where constants C depend only on N, C_d, K_{3R}, p^+, C_p and M_0. Thus,

$$\int k(t)^{p(x)} \mu(B(x,t))^{p(x)} \, dx \leq |B(0,t)| \left(k(t)\mu(\mathbf{R}^N) + C \int k(t)^{p(y)} \mu(B(y,2t))^{p(y)-1} \, d\mu(y) \right).$$

Therefore

$$\begin{aligned} &\int \left[(M_{k,R}\,\mu)(x)\right]^{p(x)} dx \\ &\quad \leq C \left(\mu(\mathbf{R}^N) \int_0^{3R/2} k(t) t^{N-1} \, dt + \int_0^{3R/2} t^{N-1} \left(\int k(t)^{p(y)} \mu(B(y,2t))^{p(y)-1} \, d\mu(y) \right) dt \right) \\ &\quad \leq C \left(\mu(\mathbf{R}^N) + \int \left(\int_0^{3R/2} k(t)^{p(y)} \mu(B(y,2t))^{p(y)-1} t^{N-1} \, dt \right) d\mu(y) \right) \\ &\quad \leq C \left(\mu(\mathbf{R}^N) + \int \mathcal{W}^{\mu}_{k,p(\cdot)}(y, 3R) \, d\mu(y) \right) \end{aligned}$$

with constants C depending only on N, C_d, K_{3R}, p^+, C_p and M_0.

4. Estimate of $p(\cdot)$-norm of convolution potential by $p(\cdot)$-norm of k-maximal function

The example given in the Remark in section 2 also shows that the following (modular) inequality *does not hold* whenever $p(\cdot)$ is continuous and non-constant:

$$\int_{\mathbf{R}^N} (G_\alpha * \mu)^{p(x)}\, dx \le C \int_{\mathbf{R}^N} (M_{\alpha,R}\,\mu)^{p(x)}\, dx.$$

However, we obtain norm inequality under an additional conditions on $p(x)$:

Theorem 3. *Suppose* $k(r)$ *in addition satisfies*

(k.3) $\int_1^\infty k(r) r^{N-1}\, dr < \infty$;

(k.4) There is a constant $C_k > 0$ such that

$$\int_0^r k(t) t^{N-1}\, dt \le C_k r^N k(r) \quad \text{for } 0 < r < R_0;$$

and suppose $p(x)$ *in addition satisfies* (P3) (log-Hölder continuity at ∞)

$$|p(x) - p(y)| \le \frac{C_\infty}{\log(e + |x|)} \quad \text{for } |y| > |x|.$$

Then, for $0 < R < R_0/2$,

$$\|k * \mu\|_{p(\cdot)} \le C \|M_{k,R}\,\mu\|_{p(\cdot)}$$

with a constant $C > 0$ *depending only on* N, C_d, C_k, $k(R)$, K, p^+, p^-, C_{lh}, C_∞ *and* R.

Note that the Bessel kernal G_α satisfies (k.3) and (k.4).

To prove Theorem 3, given $R > 0$, let

$$k_R(r) = k(r)\chi_{(0,R)}(r) \quad \text{and} \quad \tilde{k}_R(r) = k(r)\chi_{[R,\infty)}(r).$$

We treat $k_R * \mu$ and $\tilde{k}_R * \mu$ separately. First, we show

Proposition 1. *Suppose* $k(r)$ *satisfies* (k.1), (k.2) *and* (k.4), *and suppose* $p(x)$ *satisfies* (P1), (P2) *and* (P3). *Then, for* $0 < R < R_0/2$,

$$\|k_R * \mu\|_{p(\cdot)} \le C \|M_{k,R}\,\mu\|_{p(\cdot)}$$

with a constant $C > 0$ *depending only on* N, C_d, C_k, $k(R)$, p^+, p^-, C_{lh}, C_∞ *and* R.

We prove this proposition applying the following theorem due to D. Cruz-Uribe, A. Fiorenza, J.M. Martell and C. Pérez [CFMP]:

C-F-M-P Theorem. *Let* $\mathcal{F}$ *be a family of ordered pairs* (f, g) *of nonnegative measurable functions on* $\mathbf{R}^N$. *Suppose that for some* p_0, $0 < p_0 < \infty$,

$$\int_{\mathbf{R}^N} f(x)^{p_0} w(x)\, dx \le C_0 \int_{\mathbf{R}^N} g(x)^{p_0} w(x)\, dx$$

for all $(f,g) \in \mathcal{F}$ and for all A_1-weights w, where C_0 depends only on p_0 and the A_1-constant of w. Let $p(\cdot)$ satisfy (P1), (P2) *and* (P3), *and assume further that $p^- > p_0$. Then*

$$\|f\|_{p(\cdot)} \leq C\|g\|_{p(\cdot)}$$

for all $(f,g) \in \mathcal{F}$.

Remark. In [CFMP], the last phrase in the above theorem is "for all $(f,g) \in \mathcal{F}$ such that $f \in L^{p(\cdot)}(\mathbf{R}^N)$". By examining its proof, we see that $g \in L^{p(\cdot)}(\mathbf{R}^N)$ (i.e., $\|g\|_{p(\cdot)} < \infty$) implies $f \in L^{p(\cdot)}(\mathbf{R}^N)$, and hence we do not need "such that $f \in L^{p(\cdot)}(\mathbf{R}^N)$".

Thus the proof of Proposition 1 is reduced to the verification of

Proposition 1′. *Let $1 < q < \infty$. Under the assumptions on k in Proposition 1, for $0 < R < R_0/2$,*

$$\int_{\mathbf{R}^N} (k_R * \mu)^q w\,dx \leq C \int_{\mathbf{R}^N} (M_{k,R}\mu)^q w\,dx$$

for all A_1-weights w, where C depends only on N, q, C_d, C_k, R and the A_1-constant of w.

In the case $k(x) = G_\alpha$, this proposition is given in [T]. For general kernels k, we can prove this proposition by combining the arguments given in [T] and [AE, Part II]. Since our setting is different from either of them, we here give details of a proof.

First we recall some properties of A_1-weights w. w is, by definition, a nonnegative locally integrable function on $\mathbf{R}^N$ such that

$$\int_B w(x)\,dx \leq A_1|B| \operatorname*{ess\,inf}_B w$$

for every ball (or cube) B. The constant A_1 is called the A_1-constant of w. For a measurable set E in $\mathbf{R}^N$, we write $w(E) = \int_E w(x)\,dx$. An A_1-weight satisfies the A_∞-condition:

$$w(E) \leq C_w \left(\frac{|E|}{|Q|}\right)^\sigma w(Q) \tag{4.1}$$

for every cube Q and every measurable subset E of Q, where $C_w > 0$ and $\sigma > 0$ are constants depending only on N and the A_1-constant of w (see, e.g., [T; Theorem 1.2.9] or [HKM; Chap.15]).

The following is the key lemma (cf. [T; Lemma 3.1.3] and [AE; Lemma 4.3.2]):

Lemma 2. *Suppose $k(r)$ satisfies* (k.1), (k.2) *and* (k.4). *Let $0 < R < R_0/2$ and w be an A_1-weight. Set $a = 4C_d^2$. Then for every $\eta > 0$ there exists $\varepsilon \in (0,1]$, depending only on N, the A_1-constant of w, R, C_d, C_k and η, such that*

$$\begin{aligned} &w(\{x; (k_R * \mu)(x) > a\lambda\}) \\ &\qquad \leq \eta\, w(\{x; (k_R * \mu)(x) > \lambda\}) + w(\{x; (M_{k,R}\,\mu)(x) > \varepsilon\lambda\}) \end{aligned}$$

for all $\lambda > 0$.

Proof. For $\lambda > 0$, let

$$E_\lambda = \{x; (k_R * \mu)(x) > \lambda\}.$$

It is an open set, since $k_R * \mu$ is lower-semicontinuous. Let $\{Q_j\}$ be the Whitney decomposition of E_λ into closed dyadic cubes; namely, the interiors of Q_j and $Q_{j'}$ are disjoint for $j \neq j'$, $E_\lambda = \bigcup_j Q_j$ and

$$\operatorname{diam} Q_j \leq \operatorname{dist}(Q_j, E_\lambda^c) \leq 4 \operatorname{diam} Q_j$$

for each j. If $\operatorname{diam} Q_j > R/8$, then subdivide it into dyadic cubes with diameter $\leq R/8$ but $> R/16$. We denote this modified decomposition by $\{Q_j\}$ again.

Let $Q \in \{Q_j\}$, $d = \operatorname{diam} Q$ and let $B = B(x_Q, 6d)$, where x_Q be the center of Q. Note that $8d \leq R$. Let $\mu_1 = \mu|_B$ and $\mu_2 = \mu - \mu_1$. For every $x \in Q$, $B \subset B(x, 7d)$. Hence,

$$\begin{aligned}
\int_Q (k_R * \mu_1)(\xi)\, d\xi &= \int_Q \left(\int_B k_R(\xi - y)\, d\mu(y) \right) d\xi = \int_B \left(\int_Q k_R(\xi - y)\, d\xi \right) d\mu(y) \\
&\leq \int_{B(x,7d)} \left(\int_{B(0,7d)} k(\xi)\, d\xi \right) d\mu(y) \\
&= C(N) \left(\int_0^{7d} k(t) t^{N-1}\, dt \right) \mu(B(x, 7d)) \\
&\leq C_1(N, C_k) |Q| k(7d) \mu(B(x, 7d)) \qquad \text{(by (k.4))} \\
&\leq C_1(N, C_k) |Q| (M_{k,R}\,\mu)(x).
\end{aligned}$$

Let an A_1-weight w and $\eta > 0$ be given. Then, by (4.1), we can find $\varepsilon \in (0, 1]$ depending only on η, N, C_d, C_k and the A_1-constant of w such that if $E \subset Q$ and $|E| \leq C_1(N, C_k)(2\varepsilon/a)|Q|$ then $w(E) \leq \eta w(Q)$. If there exists $x \in Q$ such that $(M_{k,R}\,\mu)(x) \leq \varepsilon\lambda$, then the above inequalities imply

$$\begin{aligned}
&\left|\{\xi \in Q; (k_R * \mu_1)(\xi) > \frac{a}{2}\lambda\}\right| \\
&\qquad \leq \frac{2}{a\lambda} \int_Q (k_R * \mu_1)(\xi)\, d\xi \leq C_1(N, C_k)(2\varepsilon/a)|Q|,
\end{aligned}$$

so that

$$w(\{\xi \in Q;\ (k_R * \mu_1)(\xi) > \frac{a}{2}\lambda\}) \leq \eta\, w(Q).$$

Thus,

$$w(\{x \in Q;\ (k_R * \mu_1)(x) > \frac{a}{2}\lambda,\ (M_{k,R}\,\mu)(x) \leq \varepsilon\lambda\}) \leq \eta\, w(Q) \tag{4.2}$$

Next, we show

$$\begin{aligned}
&\{x \in Q; (k_R * \mu)(x) > a\lambda,\ (M_{k,R}\,\mu)(x) \leq \varepsilon\lambda\} \\
&\qquad \subset \{x \in Q;\ (k_R * \mu_1)(x) > \frac{a}{2}\lambda,\ (M_{k,R}\,\mu)(x) \leq \varepsilon\lambda\}
\end{aligned} \tag{4.3}$$

If Q is one of undivided Whitney cubes, then $\operatorname{dist}(Q, E_\lambda^c) \leq 4d$, so that $B \cap E_\lambda^c \neq \emptyset$. Let $x' \in B \cap E_\lambda^c$. Note that $d \leq R/8$, so that $12d < R_0$. If $x \in Q$ and $y \in B^c$, then

$|x-x'| \le 7d$ and $|x-y| \ge 5d$, so that $|x-y| \ge (5/12)|x'-y|$. Hence, if $x \in Q$ and $(M_{k,R}\,\mu)(x) \le \varepsilon\lambda$, then

$$\begin{aligned}(k_R * \mu_2)(x) &= \int_{B(x,R)} k(x-y)\,d\mu_2(y) \le \int_{B(x,R)} k\big((5/12)(x'-y)\big)\,d\mu_2(y)\\ &\le C_d^2 \int_{B(x,R)} k(x'-y)\,d\mu_2(y)\\ &\le C_d^2 \int_{B(x',R)} k(x'-y)\,d\mu(y) + C_d^2 \int_{B(x,R)\setminus B(x',R)} k(x'-y)\,d\mu(y)\\ &\le C_d^2 (k_R * \mu)(x') + C_d^2 k(R)\mu(B(x,R))\\ &\le C_d^2(1+\varepsilon)\lambda \le 2C_d^2\lambda \le \frac{a}{2}\lambda.\end{aligned}$$

Thus we have (4.3) in this case.

Next let Q be one of divided cubes. Recall that $R/16 < d \le R/8$. If $x \in Q$ and $y \in B^c$, then $|y-x| \ge 5d > (5/16)R > R/4$. Hence,

$$\begin{aligned}(k_R * \mu_2)(x) &\le \int_{\{R/4<|y-x|<R\}} k(x-y)\,d\mu(y)\\ &\le k(R/4)\mu(B(x,R)) \le C_d^2 k(R)\mu(B(x,R)) \le \frac{a}{2}(M_{k,R}\,\mu)(x).\end{aligned}$$

Hence, if $(M_{k,R}\,\mu)(x) \le \varepsilon\lambda$, then

$$(k_R * \mu_2)(x) \le \frac{a}{2}\varepsilon\lambda \le \frac{a}{2}\lambda,$$

which implies (4.3).

Now, from (4.2) and (4.3) we see that

$$\begin{aligned}&w(\{x \in Q; (k_R * \mu)(x) > a\lambda\})\\ &\qquad \le \eta\, w(Q) + w(\{x \in Q; (M_{k,R}\,\mu)(x) > \varepsilon\lambda\}).\end{aligned}$$

for every $Q \in \{Q_j\}$. Summing up over all Q, we obtain Lemma 2.

Proof of Proposition 1′: Let a be as in the above lemma and E_λ be as in the above proof, i.e., $E_\lambda = \{x;\ (k_R * \mu)(x) > \lambda\}$ $(\lambda > 0)$. First assume that μ has compact support. Then, $k_R * \mu$ has compact support, too, and hence $\lambda \mapsto w(E_\lambda)$ is a bounded function on $(0,\infty)$. Applying Lemma 2 with $\eta = a^{-q}/2$, we have, for any $L > 0$,

$$\begin{aligned}\int_0^{aL} w(E_\lambda)\lambda^{q-1}\,d\lambda &= a^q \int_0^L w(E_{a\lambda})\lambda^{q-1}\,d\lambda\\ &\le \frac{1}{2}\int_0^L w(E_\lambda)\,\lambda^{q-1}\,d\lambda + a^q \int_0^L w(\{x;\ (M_{k,R}\,\mu)(x) > \varepsilon\lambda\})\lambda^{q-1}\,d\lambda\end{aligned}$$

with $\varepsilon > 0$ depending only on N, the A_1-constant of w, R, C_d and C_k. Hence,

$$\int_0^{aL} w(E_\lambda)\lambda^{q-1}\,d\lambda \le 2a^q\varepsilon^{-q} \int_0^{\varepsilon L} w(\{x;\ (M_{k,R}\,\mu)(x) > \lambda\})\lambda^{q-1}\,d\lambda.$$

Now, letting $L \to \infty$, we have

$$\int_{\mathbf{R}^N} (k_R * \mu)^q w\,dx \le 2a^q \varepsilon^{-q} \int_{\mathbf{R}^N} (M_{k,R}\,\mu)^q w\,dx.$$

If μ does not have compact support, let $\mu_m = \chi_{B(0,m)}\mu$ and apply the above result to μ_m, and then let $m \to \infty$. Since $k_R * \mu_m \uparrow k_R * \mu$, the required result follows by the monotone convergence theorem.

To treat $\tilde{k}_R * \mu$, we prepare another lemma. For nonnegative measure μ on $\mathbf{R}^N$ and $R > 0$, let

$$\widetilde{M}_R\,\mu(x) = \sup_{r \ge R} r^{-N} \mu(B(x,r)).$$

Lemma 3. *If $k(R) > 0$, then*

$$\widetilde{M}_R\,\mu \le C(N, R, k(R))\mathcal{M}(M_{k,R}\,\mu),$$

where, $\mathcal{M}(f)$ denotes the Hardy-Littlewood maximal function of f.

Proof. Fix $x \in \mathbf{R}^N$ and let $r \ge R > 0$. We can find a finite number of $y_j \in B(x,r)$ such that

$$B(x,r) \subset \bigcup_j B(y_j, R/2) \quad \text{and} \quad \sum_i \chi_{B(y_j,R/2)} \le A(N) < \infty.$$

If $y \in B(y_j, R/2)$, then $B(y_j, R/2) \subset B(y, R)$, so that

$$\mu(B(y_j, R/2)) \le \mu(B(y,R)) \le \frac{(M_{k,R}\,\mu)(y)}{k(R)}.$$

Since $B(y_j, R/2) \subset B(x, r + R/2) \subset B(x, 2r)$,

$$\begin{aligned}
\mu(B(x,r)) &\le \sum_j \mu(B(y_j, R/2)) \\
&\le \frac{1}{k(R)|B(0,R/2)|} \sum_j \int_{B(y_j,R/2)} (M_{k,R}\,\mu)(y)\,dy \\
&\le \frac{2^N A(N)}{k(R)|B(0,R)|} \int_{B(x,2r)} (M_{k,R}\,\mu)(y)\,dy \\
&\le \frac{4^N A(N)}{k(R)R^N} r^N \mathcal{M}(M_{k,R}\,\mu)(x),
\end{aligned}$$

so that

$$r^{-N}\mu(B(x,r)) \le C(N, R, k(R))\mathcal{M}(M_{k,R}\,\mu)(x)$$

for $r \ge R$. Thus, we obtain the required estimate.

Proposition 2. *Suppose $k(r)$ satisfies* (k.1), (k.2) *and* (k.3), *and suppose $p(x)$ satisfies* (P1), (P2) *and* (P3). *Then, for $0 < R < R_0$,*

$$\|\tilde{k}_R * \mu\|_{p(\cdot)} \le C\|M_{k,R}\mu\|_{p(\cdot)}$$

with a constant $C > 0$ depeding only on N, R, $k(R)$, C_{lh}, C_∞, p^+, p^- and

$$K := \int_0^\infty k(r) r^{N-1}\,dt.$$

Proof.

$$\begin{aligned}(\tilde{k}_R * \mu)(x) &= \int_{\mathbf{R}^N \setminus B(x,R)} k(x-y)\,d\mu(y) \\ &= \int_{[R,\infty)} k(r)\,d[\mu(B(x,\cdot))](r) \\ &\le \limsup_{r\to\infty} k(r)\mu(B(x,r)) + \int_{(R,\infty)} \mu(B(x,r))\,d(-k)(r).\end{aligned}$$

Note that (k.3) implies that $k(r) \le r^{-N}$ for $r \ge r_0$. Thus, if $r > \max(r_0, R)$, we have

$$k(r)\mu(B(x,r)) \le r^N k(r)(\widetilde{M}_R\,\mu)(x) \le (\widetilde{M}_R\,\mu)(x).$$

Hence,

$$\limsup_{r\to\infty} k(r)\mu(B(x,r)) \le (\widetilde{M}_R\,\mu)(x).$$

On the other hand,

$$\begin{aligned}\int_{(R,\infty)} \mu(B(x,r))\,d(-k)(r) &\le \left(\int_{(R,\infty)} r^N\,d(-k)(r)\right)(\widetilde{M}_R\,\mu)(x) \\ &\le \left(R^N k(R) + N\int_R^\infty k(r) r^{N-1}\,dr\right)(\widetilde{M}_R\,\mu)(x) \\ &\le \big(R^N k(R) + NK\big)(\widetilde{M}_R\,\mu)(x).\end{aligned}$$

Hence

$$(\tilde{k}_R * \mu)(x) \le C(N, R, k(R), K)(\widetilde{M}_R\,\mu)(x).$$

Thus, by Lemma 2, we have

$$(\tilde{k}_R * \mu)(x) \le C\mathcal{M}(M_{k,R}\,\mu)(x)$$

with a constant $C = C(N, R, k(R), K) > 0$, which implies

$$\|\tilde{k}_R * \mu\|_{p(\cdot)} \le C\|\mathcal{M}(M_{k,R}\,\mu)\|_{p(\cdot)}$$

with $C = C(N, R, k(R), K, p^+) > 0$. Now, under our assumptions on $p(\cdot)$, we know (see [CFN]) that

$$\|\mathcal{M}(f)\|_{p(\cdot)} \le C\|f\|_{p(\cdot)},$$

and hence we obtain the required estimate.

Combining Propositions 1 and 2, we obtain Theorem 3.

From Theorems 1, 2 and 3, we derive

Corollary 1. *Suppose* $p(\cdot)$ *satisfies* (P1), (P2) *and* (P3), *and* $k(r)$ *satisfies* (k.1), (k.2), (k.3) *and* (k.4). *Then, for nonnegative measures* μ *in* $\mathbf{R}^N$ *with* $\mu(\mathbf{R}^N) < \infty$,

$$k * \mu \in L^{p(\cdot)}(\mathbf{R}^N) \quad \textit{if and only if} \quad \int \mathcal{W}^{\mu}_{k,p(\cdot)}(x, R)\, d\mu(x) < \infty .$$

It is known (see [GHN]) that if $p(\cdot)$ satisfies (P1), (P2) and (P3), then

$$W^{m,p(\cdot)}(\mathbf{R}^N) = \{u = G_m * f \,;\, f \in L^{p(\cdot)}(\mathbf{R}^N)\}$$

for $m \in \mathbf{N}$. Thus we can state

Corollary 2. *If* $p(\cdot)$ *satisfies* (P1), (P2) *and* (P3), *then for nonnegative measures* μ *on* $\mathbf{R}^N$ *with* $\mu(\mathbf{R}^N) < \infty$,

$$\mu \in (W^{m,p(\cdot)}(\mathbf{R}^N))^* \quad \textit{if and only if} \quad \int \mathcal{W}^{\mu}_{m,p(\cdot)}(x, R)\, d\mu(x) < \infty$$

for $m \in \mathbf{N}$ *with* $0 < m < N$.

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