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LINEAR RELATIONS OF COMPOSITION OPERATORS

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Abstract. We will characterize the compactness of linear combinations of composition operators on the Banach algebra of bounded analytic functions on the open unit disk.

1 Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk and $\mathcal{H}(\mathbb{D})$ the space of all analytic functions on $\mathbb{D}$. Denote by $\mathcal{S}(\mathbb{D})$ the set of analytic self-maps of $\mathbb{D}$. Then, for $\varphi \in \mathcal{S}(\mathbb{D})$, the composition operator C_φ is defined by

$$C_\varphi f(z) = (f \circ \varphi)(z)$$

for $z \in \mathbb{D}$ and $f \in \mathcal{H}(\mathbb{D})$. During the past few decades, many authors have investigated operator theoretic properties of composition operator C_φ on various analytic function spaces using function theoretic properties of symbol φ. For an overview of the study of compostion operators, we refer to the books [2], [14] and [17].

Presently some of the long standing open questions in this field are related to the topological structure of the set of compostion operators. For a Banach space $\mathcal{X}$ in $\mathcal{H}(\mathbb{D})$, we write $\mathcal{C}(\mathcal{X})$ for the set of composition operators on $\mathcal{X}$ with the operator norm topology. Berkson [1] focused

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attention on the topological structure with his isolation results on compostion opaerators on the Hardy spaces. In the case of the Hilbert Hardy space, Shapiro and Sundberg [15] gave further progress, obtained results on compact differences and isolation and suggested questions.in the case of the Hilbert Hardy space.

The problems are the following in the general case:

1. Characterize the components of $\mathcal{C}(\mathcal{X})$.
2. Which composition operators are isolated in $\mathcal{C}(\mathcal{X})$?
3. Which composition differences are compact on $\mathcal{X}$?

One conjecture was proposed : for φ and $\psi \in \mathcal{S}(\mathbb{D})$, $C_\varphi - C_\psi$ is compact on $\mathcal{X}$ if and only if C_φ and C_ψ are in the same component in $\mathcal{C}(\mathcal{X})$. The topological structure of $\mathbb{C}(\mathcal{X})$ has been studied on various analytic function spaces $\mathcal{X}$. These problems seem quite hard.

In view of the other, for φ and $\psi \in \mathcal{S}(\mathbb{D})$, it holds that $C_\varphi C_\psi = C_{\psi \circ \varphi}$, that is, the product of two composition operators becomes a composition operator. But the sum $C_\varphi + C_\psi$ is not necessarily a composition operator. The set of composition operators has no obvious additive or linear structure. Note that Toeplitz-Hankel operators have additive and linear structure but their products are not clear.

Let $\mathcal{B}(\mathcal{X})$ be the set of bounded linear operators on $\mathcal{X}$ and $\mathcal{K}$ the set of all compact operators on $\mathcal{X}$. Then $\mathcal{B}(\mathcal{X})/\mathcal{K}$ is called the Calkin algebra. The compactness of $C_\varphi - C_\psi$ is that $C_\varphi \equiv C_\psi \quad (mod.\mathcal{K})$. Topological structure problem (compact difference problem) implies linear relations problem. That is, $\sum_{i=1}^{N} \lambda_i C_{\varphi_i} - C_\psi$ is compact if and only if $\sum_{i=1}^{N} \lambda_i C_{\varphi_i} \equiv C_\psi$ $(mod.\mathcal{K})$.

In a recent paper, MacCluer, Zhao and the author [12] studied the topological structure of the set $\mathcal{C}(H^\infty)$ of composition operators on the Banach space H^∞ of bounded analytic functions on $\mathbb{D}$. In [7], Hosokawa, Izuchi and Zheng showed that C_φ is not isolated in $\mathcal{C}(H^\infty)$ if and only if φ is not an extreme point of the closed unit ball of H^∞, and that C_φ and C_ψ are in the same connected component in $\mathcal{C}(H^\infty)$ if and only if C_φ and C_ψ are in the same connected component in $\mathcal{C}(H^\infty)/\mathcal{K}$. In [6], Hosokawa and Izuchi studied the estimate of the essential norm which is the norm in $\mathcal{B}(H^\infty)/\mathcal{K}$.

After these works, H^∞ has attracted much attention in the study of this area. In particular, Toews [16] extended the results of [12] and [8] to the setting of several variables. Gorkin, Mortini and Suárez [5] gave upper and lower bounds for the essential norm of difference of two composition operators on H^∞, where the setting is on the unit ball of $\mathbb{C}^n (n \geq 1)$. Now, furthermore, linear relations of compostion operators have been studied in some cases. In [4], Gorkin and Mortini studied norms and essential norms of linear combinations of endomorphisms on uniform algebras. Kriete and Moorhouse [11] considered linear relations of composition operators on the Hilbert Hardy space. Hosokawa, Nieminen and the author [9] have done in the Bloch space case.

In this article, we investigate properties of linear combinations of composition operators on H^∞. In the next section we will review on the results of compact differences on H^∞ to study the linear relations of composition operators. In Section 3 we will characterize the compactness of linear combinations of composition operators on H^∞. These results are due to a part of the joint-work [10] with K.J. Izuchi.

2 Reviews on results of compact differences

Let $H^\infty = H^\infty(\mathbb{D})$ be the space of all bounded analytic functions on the open unit disk $\mathbb{D}$. Then H^∞ is a Banach algebra with the supremum norm

$$\|f\|_\infty = \sup\{|f(z)| : z \in \mathbb{D}\}.$$

Denote by *ball* H^∞ the closed unit ball of H^∞. For $\varphi \in \mathcal{S}(\mathbb{D})$, we define the composition operator C_φ on H^∞ by

$$C_\varphi f = f \circ \varphi \quad \text{for } f \in H^\infty.$$

It is clear that C_φ is linear and bounded on H^∞. and that C_φ is compact on H^∞ if and only if $\|\varphi\|_\infty < 1$ ([13]).

Our results involve the pseudo-hyperbolic metric. For z and $w \in \mathbb{D}$, the pseudo-hyperbolic distance between z and w is given by

$$\rho(z, w) = \left| \frac{z - w}{1 - \bar{z}w} \right|.$$

MacCluer, Zhao and the author [12] showed the following.

Theorem 2.1. *Let φ and $\psi \in \mathcal{S}(\mathbb{D})$ with $\varphi \neq \psi$. Suppose that $\|\varphi\|_\infty = \|\psi\|_\infty = 1$. Then $C_\varphi - C_\psi$ is compact on H^∞ if and only if*

$$\limsup_{|\varphi(z)|\to 1} \rho\big(\varphi(z), \psi(z)\big) = \limsup_{|\psi(z)|\to 1} \rho\big(\varphi(z), \psi(z)\big) = 0.$$

Here we can show that the conjecture posed in Section 1 is not true for the case of H^∞.

Example 2.2. *Let*

$$\varphi(z) = sz + 1 - s, \qquad 0 < s < 1$$

and

$$\psi(z) = \varphi(z) + t(z-1)^b,$$

where $|t|$ is so small that ψ maps $\mathbb{D}$ into $\mathbb{D}$.

Then

(i) If $0 < b \leq 2$, then $C_\varphi - C_\psi$ is not compact on H^∞.

(ii) If $2 < b$, then $C_\varphi - C_\psi$ is compact on H^∞. But C_φ and C_ψ are not in the same component of $\mathcal{C}(H^\infty)$.

3 Linear combinations of composition operators

We here characterize the compactness of linear combinations of composition operators on H^∞. This work is a part of the joint-work [10] with K.J. Izuchi.

We shall need the following proposition whose proof is an easy modification of that of Proposition 3.11 in [2].

Proposition 3.1. *Let $\varphi_1, \varphi_2, \cdots, \varphi_N$ be distinct functions in $\mathcal{S}(\mathbb{D})$, and $\lambda_i \in \mathbb{C}$ with $\lambda_i \neq 0$ for every i. Then $\sum_{i=1}^N \lambda_i C_{\varphi_i}$ is compact on H^∞ if and only if whenever $\{f_n\}_n$ is a bounded sequence in H^∞ such that $\{f_n\}_n$ converges to 0 uniformly on any compact subset of $\mathbb{D}$, then $\|\sum_{i=1}^N \lambda_i C_{\varphi_i} f_n\|_\infty$ tends to 0 as $n \to \infty$.*

Let $\varphi_1, \varphi_2, \cdots, \varphi_N$ be distinct functions in $\mathcal{S}(\mathbb{D})$ and $N \geq 2$. Let $\mathcal{Z} = \mathcal{Z}(\varphi_1, \varphi_2, \cdots, \varphi_N)$ be the family of sequences $\{z_n\}_n$ in $\mathbb{D}$ satisfying the following three conditions;

(a) $|\varphi_i(z_n)| \to 1$ as $n \to \infty$ for some i,

(b) $\{\varphi_i(z_n)\}_n$ is a convergent sequence for every i,

(c)

$$\left\{ \frac{\varphi_j(z_n) - \varphi_i(z_n)}{1 - \overline{\varphi_j(z_n)}\varphi_i(z_n)} \right\}_n$$

is a convergent sequence for every i, j.

Condition (c) implies that

(c') $\{\rho(\varphi_i(z_n), \varphi_j(z_n))\}_n$ is a convergent sequence for every i, j.

Note that if $|\varphi_i(z_n)| \to 1$ as $n \to \infty$ for some i, then it is easy to see that there exists a subsequence $\{z_{n_j}\}_j$ of $\{z_n\}_n$ satisfying $\{z_{n_j}\}_j \in \mathcal{Z}$.

For $\{z_n\}_n \in \mathcal{Z}$, we write

$$I(\{z_n\}) = \{i : 1 \leq i \leq N, |\varphi_i(z_n)| \to 1 \text{ as } n \to \infty\}.$$

By condition (a), $I(\{z_n\}) \neq \emptyset$. By (b), there exists δ with $0 < \delta < 1$ such that $|\varphi_j(z_k)| < \delta < 1$ for every $j \notin I(\{z_n\})$ and k. For each $t \in I(\{z_n\})$, we write

$$(3.1) \quad I_0(\{z_n\}, t) = \{j \in I(\{z_n\}) : \rho(\varphi_j(z_n), \varphi_t(z_n)) \to 0 \text{ as } n \to \infty\}.$$

For $s, t \in I(\{z_n\})$, we have either $I_0(\{z_n\}, s) = I_0(\{z_n\}, t)$ or $I_0(\{z_n\}, s) \cap I_0(\{z_n\}, t) = \emptyset$. Hence there is a subset $\{t_1, t_2, \cdots, t_\ell\} \subset I(\{z_n\})$ such that

$$I(\{z_n\}) = \bigcup_{p=1}^{\ell} I_0(\{z_n\}, t_p)$$

and $I_0(\{z_n\}, t_p) \cap I_0(\{z_n\}, t_q) = \emptyset$ for $p \neq q$.

When we consider the compactness of linear combinations $\sum_{i=1}^N \lambda_i C_{\varphi_i}$, some C_{φ_i} could be compact, that is, $\|\varphi_i\|_\infty < 1$. We may exclude such trivial ones from our linear combinations.

Gorkin and Mortini [4, Theorem 11] characterized necessary conditions for linear combinations of composition operators to be compact on some uniform algebras. We here obtain necessary and sufficient conditions on the compactness.

Theorem 3.2. *Let $\varphi_1, \varphi_2, \cdots, \varphi_N$ $(N \geq 2)$ be distinct functions in $\mathcal{S}(\mathbb{D})$ with $\|\varphi_i\|_\infty = 1$, and $\lambda_i \in \mathbb{C}$ with $\lambda_i \neq 0$ for every i. Then the following conditions are equivalent.*

(i) $\sum_{i=1}^N \lambda_i C_{\varphi_i}$ *is compact on* H^∞.

(ii) $\sum \{\lambda_i : i \in I_0(\{z_n\}, t)\} = 0$ *for every* $\{z_n\}_n \in \mathcal{Z} = \mathcal{Z}(\varphi_1, \varphi_2, \cdots, \varphi_N)$ *and* $t \in I(\{z_n\})$.

Proof. (i) $\Rightarrow$ (ii). Suppose that $\sum_{i=1}^N \lambda_i C_{\varphi_i}$ is compact on H^∞. Let $\{z_n\}_n \in \mathcal{Z}$ and $t \in I(\{z_n\})$. For each positive integer k, we write

$$f_k(z) = \frac{1 - |\varphi_t(z_k)|^2}{1 - \overline{\varphi_t(z_k)} z} \prod_{j \notin I_0(\{z_n\}, t)} \frac{\varphi_j(z_k) - z}{1 - \overline{\varphi_j(z_k)} z}.$$

Then $f_k \in H^\infty$, $\|f_k\|_\infty \leq 2$, and $\{f_k\}_k$ converges to 0 uniformly on every compact subset of $\mathbb{D}$. We have

$$\begin{aligned}
&\Big\| \sum_{i=1}^N \lambda_i C_{\varphi_i} f_k \Big\|_\infty \\
&\geq \Big| \sum_{i=1}^N \lambda_i f_k(\varphi_i(z_k)) \Big| \\
&= \Big| \sum_{i \in I_0(\{z_n\}, t)} \lambda_i \frac{1 - |\varphi_t(z_k)|^2}{1 - \overline{\varphi_t(z_k)} \varphi_i(z_k)} \prod_{j \notin I_0(\{z_n\}, t)} \frac{\varphi_j(z_k) - \varphi_i(z_k)}{1 - \overline{\varphi_j(z_k)} \varphi_i(z_k)} \Big|.
\end{aligned}$$

Here

$$\frac{1 - |\varphi_t(z_k)|^2}{1 - \overline{\varphi_t(z_k)} \varphi_i(z_k)} = 1 + \overline{\varphi_t(z_k)} \frac{\varphi_i(z_k) - \varphi_t(z_k)}{1 - \overline{\varphi_t(z_k)} \varphi_i(z_k)}.$$

For $i \in I_0(\{z_n\}, t)$, by (3.1) $\rho\big(\varphi_i(z_k), \varphi_t(z_k)\big) \to 0$ as $k \to \infty$. Hence

$$\frac{1 - |\varphi_t(z_k)|^2}{1 - \overline{\varphi_t(z_k)} \varphi_i(z_k)} \to 1$$

as $k \to \infty$.

On the other hand,

$$\frac{\varphi_j(z_k)-\varphi_i(z_k)}{1-\overline{\varphi_j(z_k)}\varphi_i(z_k)}-\frac{\varphi_j(z_k)-\varphi_t(z_k)}{1-\overline{\varphi_j(z_k)}\varphi_t(z_k)}$$

$$=\frac{\varphi_t(z_k)-\varphi_i(z_k)}{1-\overline{\varphi_t(z_k)}\varphi_i(z_k)}\frac{\left(1+\overline{\varphi_t(z_k)}\frac{\varphi_j(z_k)-\varphi_t(z_k)}{1-\overline{\varphi_t(z_k)}\varphi_j(z_k)}\right)}{1+\overline{\varphi_t(z_k)}\frac{\varphi_i(z_k)-\varphi_t(z_k)}{1-\overline{\varphi_t(z_k)}\varphi_i(z_k)}}$$

$$\times\left(1+\overline{\varphi_j(z_k)}\frac{\varphi_i(z_k)-\varphi_j(z_k)}{1-\overline{\varphi_j(z_k)}\varphi_i(z_k)}\right).$$

Since $\rho\big(\varphi_i(z_k),\varphi_t(z_k)\big)\to 0$, by (c) we have

$$\lim_{k\to\infty}\frac{\varphi_j(z_k)-\varphi_i(z_k)}{1-\overline{\varphi_j(z_k)}\varphi_i(z_k)}=\lim_{k\to\infty}\frac{\varphi_j(z_k)-\varphi_t(z_k)}{1-\overline{\varphi_j(z_k)}\varphi_t(z_k)}.$$

Since $j\notin I_0(\{z_n\},t)$, by (3.1) and (c)

$$\lim_{k\to\infty}\frac{\varphi_j(z_k)-\varphi_t(z_k)}{1-\overline{\varphi_j(z_k)}\varphi_t(z_k)}=\beta_{j,t}\neq 0$$

for some $\beta_{j,t}\in\mathbb{C}$.

By condition (i) and Proposition 3.1,

$$\Big\|\sum_{i=1}^{N}\lambda_i C_{\varphi_i}f_k\Big\|_\infty\to 0$$

as $k\to\infty$. Therefore we get

$$\Big(\sum_{i\in I_0(\{z_n\},t)}\lambda_i\Big)\prod_{j\notin I_0(\{z_n\},t)}\beta_{j,t}=0.$$

Consequently, we have

$$\sum_{i\in I_0(\{z_n\},t)}\lambda_i=0.$$

(ii) $\Rightarrow$ (i). Suppose that $\sum_{i=1}^{N}\lambda_i C_{\varphi_i}$ is not compact on H^∞. Then there exists a sequence $\{f_n\}_n$ in *ball* H^∞ such that $f_n\to 0$ uniformly on every compact subset of $\mathbb{D}$ and

$$\Big\|\sum_{i=1}^{N}\lambda_i f_n\circ\varphi_i\Big\|_\infty\not\to 0$$

as $n \to \infty$. For some $\varepsilon > 0$, considering a subsequence of $\{f_n\}_n$, we may assume that

$$\left\| \sum_{i=1}^{N} \lambda_i f_n \circ \varphi_i \right\|_\infty > \varepsilon > 0$$

for every n. Take $z_n \in \mathbb{D}$ with $|z_n| \to 1$ and

$$\left| \sum_{i=1}^{N} \lambda_i f_n(\varphi_i(z_n)) \right| > \varepsilon.$$

Considering subsequence of $\{z_n\}_n$, we may assume that $\varphi_i(z_n) \to \alpha_i$ as $n \to \infty$ for every i. Since $f_n \to 0$ uniformly on every compact subset of $\mathbb{D}$, $|\alpha_i| = 1$ for some i. Moreover we may assume that $\{z_n\}_n \in \mathcal{Z}$. Also we have

$$\liminf_{k\to\infty} \left| \sum_{i\in I(\{z_n\})} \lambda_i f_k(\varphi_i(z_k)) \right| \geq \varepsilon. \tag{3.2}$$

Recall that there exists a subset $\{t_1, t_2, \cdots, t_\ell\} \subset I(\{z_n\})$ such that

$$I(\{z_n\}) = \bigcup_{p=1}^{\ell} I_0(\{z_n\}, t_p)$$

and $I_0(\{z_n\}, t_p) \cap I_0(\{z_n\}, t_q) = \emptyset$ for $p \neq q$. Let $i \in I_0(\{z_n\}, t_p)$. Then $\rho(\varphi_i(z_k), \varphi_{t_p}(z_k)) \to 0$ as $k \to \infty$. By Schwarz's lemma, see [3, p. 2],

$$\rho\big(f_k(\varphi_i(z_k)), f_k(\varphi_{t_p}(z_k))\big) \leq \rho\big(\varphi_i(z_k), \varphi_{t_p}(z_k)\big) \to 0 \tag{3.3}$$

as $k \to \infty$. Since $\{f_k(\varphi_i(z_k))\}_k$ is bounded, considering a subsequence of $\{z_k\}_k$, we may assume that $f_k(\varphi_i(z_k)) \to \beta_i$ as $k \to \infty$ for every i. By (3.3), $\beta_i = \beta_{t_p}$ for every $i \in I_0(\{z_n\}, t_p)$. Therefore

$$\begin{aligned}
\lim_{k\to\infty} \sum_{i\in I(\{z_n\})} \lambda_i f_k(\varphi_i(z_k)) &= \lim_{k\to\infty} \sum_{p=1}^{\ell} \sum_{i\in I_0(\{z_n\},t_p)} \lambda_i f_k(\varphi_i(z_k)) \\
&= \sum_{p=1}^{\ell} \sum_{i\in I_0(\{z_n\},t_p)} \lambda_i \beta_{t_p} \\
&= \sum_{p=1}^{\ell} \beta_{t_p} \sum_{i\in I_0(\{z_n\},t_p)} \lambda_i \\
&= 0 \qquad \text{by condition (ii).}
\end{aligned}$$

This contradicts condition (3.2). □

The following corollaries follow from Theorem 3.2.

Corollary 3.3. *Let $\varphi_1, \varphi_2, \cdots, \varphi_N$ ($N \geq 2$) be distinct functions in $\mathcal{S}(\mathbb{D})$ with $\|\varphi_i\|_\infty = 1$, and $\lambda_i \in \mathbb{C}$ with $\lambda_i \neq 0$ for every i. If $\sum_{i \in J} \lambda_i \neq 0$ for every subset J of $\{1, 2, \cdots, N\}$, then $\sum_{i=1}^N \lambda_i C_{\varphi_i}$ is not compact on H^∞.*

This says that the sum $\sum_{i=1}^N C_{\varphi_i}$ is never compact on H^∞ for every $\varphi_i \in \mathcal{S}(\mathbb{D})$ with $\|\varphi_i\|_\infty = 1, i = 1, \ldots, N$.

Corollary 3.4. *Let $\varphi_1, \varphi_2, \cdots, \varphi_N$ ($N \geq 2$) be distinct functions in $\mathcal{S}(\mathbb{D})$ with $\|\varphi_i\|_\infty = 1$, and $\lambda_i \in \mathbb{C}$ with $\lambda_i \neq 0$ for every i. Suppose that $\sum_{i=1}^N \lambda_i = 0$ and $\sum_{i \in J} \lambda_i \neq 0$ for every non-empty proper subset J of $\{1, 2, \cdots, N\}$. Then $\sum_{i=1}^N \lambda_i C_{\varphi_i}$ is compact on H^∞ if and only if $C_{\varphi_i} - C_{\varphi_j}$ is compact on H^∞ for every i, j with $i \neq j$.*

Proof. Suppose that $\sum_{i=1}^N \lambda_i C_{\varphi_i}$ is compact on H^∞. Then by Theorem 3.2 (ii), for every $\{z_n\}_n \in \mathcal{Z}$, $I(\{z_n\}) = \{1, 2, \cdots, N\}$ and $I_0(\{z_n\}, t) = \{1, 2, \cdots, N\}$ for every $t \in I(\{z_n\})$. Hence

$$\lim_{|\varphi_i(z)| \to 1} \rho\big(\varphi_i(z), \varphi_j(z)\big) = 0.$$

By [12], $C_{\varphi_i} - C_{\varphi_j}$ is compact for every i, j.

Suppose that $C_{\varphi_i} - C_{\varphi_j}$ is compact for every i, j. Since

$$\sum_{i=1}^N \lambda_i C_{\varphi_i} = \Big(\sum_{i=1}^N \lambda_i\Big) C_{\varphi_1} + \sum_{i=2}^N \lambda_i (C_{\varphi_i} - C_{\varphi_j}) = \sum_{i=2}^N \lambda_i (C_{\varphi_i} - C_{\varphi_j}),$$

we have that $\sum_{i=1}^N \lambda_i C_{\varphi_i}$ is compact. □

We recall that the Bloch space $\mathcal{B}$ consists of all analytic functions f on $\mathbb{D}$ such that $\|f\|_{\mathcal{B}} = \sup_{z \in \mathbb{D}} (1 - |z|^2)|f'(z)| < \infty$. It is well known that $\mathcal{B}$ is a Banach space under the norm $\|f\| = |f(0)| + \|f\|_{\mathcal{B}}$. Then, under the assumption of Corollary 3.4, we obtain the following by Theorem 3 in [12].

Corollary 3.5. *Let $\varphi_1, \varphi_2, \cdots, \varphi_N$ ($N \geq 2$) be distinct functions in $\mathcal{S}(\mathbb{D})$ with $\|\varphi_i\|_\infty = 1$, and $\lambda_i \in \mathbb{C}$ with $\lambda_i \neq 0$ for every i. Suppose that $\sum_{i=1}^N \lambda_i = 0$ and $\sum_{i \in J} \lambda_i \neq 0$ for every non-empty proper subset J of $\{1, 2, \cdots, N\}$. Then the following conditions are equivalent.*

(i) $\sum_{i=1}^N \lambda_i C_{\varphi_i} : H^\infty \to H^\infty$ *is compact.*

(ii) $\sum_{i=1}^{N} \lambda_i C_{\varphi_i} : \mathcal{B} \to H^\infty$ *is compact.*

It would be another problem to characterize the boundedness and compactness of $\sum_{i=1}^{N} \lambda_i C_{\varphi_i}$ acting from $\mathcal{B}$ to H^∞ in general. The boundedness and compactness of the differences of two composition operators acting from $\mathcal{B}$ to H^∞ is concerning to the component problem of the set $\mathcal{C}(H^\infty)$ of composition operators on H^∞ ([12]).

Example 3.6. *We show the existence of* $\varphi_1, \varphi_2, \varphi_3 \in \mathcal{S}(\mathbb{D})$ *with* $\|\varphi_i\|_\infty = 1$ *such that* $C_{\varphi_1} - C_{\varphi_2} - C_{\varphi_3}$ *is compact.*

Let $\sigma(z) = (1+z)/(1-z)$ and

$$\varphi_1(z) = \frac{\sqrt{\sigma(z)} - 1}{\sqrt{\sigma(z)} + 1}$$

be a lens map ([14]). Also let

$$\varphi_2(z) = 1 - \sqrt{1-z}.$$

Denote by $\partial\mathbb{D}$ the boundary of $\mathbb{D}$. Then $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{D})$, $\varphi_1(\pm 1) = \pm 1$, $|\varphi_1(e^{i\theta})| < 1$ for $e^{i\theta} \in \partial\mathbb{D}$ with $e^{i\theta} \neq \pm 1$, $\varphi_2(1) = 1$, and $|\varphi_2(e^{i\theta})| < 1$ for $e^{i\theta} \in \partial\mathbb{D}$ with $e^{i\theta} \neq 1$. As Example (i) in [7, p. 513],

$$\begin{aligned} \rho(\varphi_1(z), \varphi_2(z)) &= \left| \frac{\sqrt{\sigma(z)}(1-\varphi_2(z)) - (1+\varphi_2(z))}{\overline{\sqrt{\sigma(z)}}(1-\varphi_2(z)) + (1+\varphi_2(z))} \right| \\ &= \left| \frac{\sqrt{1+z} - (1+\varphi_2(z))}{\sqrt{1+\bar{z}}\frac{\sqrt{1-z}}{\sqrt{1-\bar{z}}} + (1+\varphi_2(z))} \right|. \end{aligned}$$

Since

$$Re\, \frac{\sqrt{1-z}}{\sqrt{1-\bar{z}}} > 0 \quad for \quad z \in \mathbb{D},$$

we have

$$\lim_{z\to 1} \rho(\varphi_1(z), \varphi_2(z)) = 0.$$

Let

$$\varphi_3(z) = -1 + \sqrt{1+z}.$$

Then $\varphi_3 \in \mathcal{S}(\mathbb{D})$, $\varphi_3(-1) = -1$, and $|\varphi_3(e^{i\theta})| < 1$ for $e^{i\theta} \in \partial\mathbb{D}$ with $e^{i\theta} \neq -1$. Similarly we have

$$\lim_{z\to -1} \rho(\varphi_1(z), \varphi_3(z)) = 0.$$

Hence by Theorem 3.2. $C_{\varphi_1} - C_{\varphi_2} - C_{\varphi_3}$ is compact.

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