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On solutions of quasilinear elliptic equations with general structure

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§1. Introduction and preliminaries

Let G be an open set in $\mathbf{R}^N$ $(N \geq 2)$ and $1 < p < N$. We consider quasi-linear second order elliptic differential equations of the form

$$(\mathrm{E}_T) \qquad -\operatorname{div} \mathcal{A}(x, \nabla u) + \mathcal{B}(x, u) = T$$

in G. Here, T is a distribution, $\mathcal{A} : \mathbf{R}^N \times \mathbf{R}^N \to \mathbf{R}^N$ and $\mathcal{B} : \mathbf{R}^N \times \mathbf{R} \to \mathbf{R}$ satisfy the following conditions :

(A.1) $x \mapsto \mathcal{A}(x,\xi)$ is measurable on $\mathbf{R}^N$ for every $\xi \in \mathbf{R}^N$ and $\xi \mapsto \mathcal{A}(x,\xi)$ is continuous for a.e. $x \in \mathbf{R}^N$;

(A.2) $\mathcal{A}(x,\xi) \cdot \xi \geq \alpha_1 |\xi|^p$ for all $\xi \in \mathbf{R}^N$ and a.e. $x \in \mathbf{R}^N$ with a constant $\alpha_1 > 0$;

(A.3) $|\mathcal{A}(x,\xi)| \leq \alpha_2 |\xi|^{p-1}$ for all $\xi \in \mathbf{R}^N$ and a.e. $x \in \mathbf{R}^N$ with a constant $\alpha_2 > 0$;

(B.1) $x \mapsto \mathcal{B}(x,t)$ is measurable on $\mathbf{R}^N$ for every $t \in \mathbf{R}$ and $t \mapsto \mathcal{B}(x,t)$ is continuous for a.e. $x \in \mathbf{R}^N$;

(B.2) For any bounded open set D in $\mathbf{R}^N$, there is a constant $\alpha_3(D) \geq 0$ such that $|\mathcal{B}(x,t)| \leq \alpha_3(D)(|t|^{p-1} + 1)$ for all $t \in \mathbf{R}$ and a.e. $x \in D$;

A prototype of the equation (E_T) is

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) + b|u|^{p-2}u = T$$

with a locally bounded function b in G.

As a matter of fact, we treat the following two topics: (i) Hölder continuity of a solution of the equation of (E_T) (section 2); (ii) Integrability of the gradients of a solution of the equation of (E_T) (section 3).

Throughout this paper, we use some standard notation without explanation.

§2. Hölder continuity of a solution

In this section, we suppose the following monotoneity conditions on $\mathcal{A} : \mathbf{R}^N \times \mathbf{R}^N \to \mathbf{R}^N$ and $\mathcal{B} : \mathbf{R}^N \times \mathbf{R} \to \mathbf{R}$:

(A.4) $\big(\mathcal{A}(x,\xi_1) - \mathcal{A}(x,\xi_2)\big) \cdot \big(\xi_1 - \xi_2\big) > 0$ whenever $\xi_1,\ \xi_2 \in \mathbf{R}^N$, $\xi_1 \neq \xi_2$, for a.e. $x \in \mathbf{R}^N$;

(B.3) $t \mapsto \mathcal{B}(x,t)$ is nondecreasing on $\mathbf{R}$ for a.e. $x \in \mathbf{R}^N$.

We consider elliptic quasi-linear equations of the form

$$(\mathrm{E}_0) \qquad -\operatorname{div} \mathcal{A}(x,\ \nabla u(x)) + \mathcal{B}(x,\ u(x)) = 0.$$

For an open subset G of $\mathbf{R}^N$, we consider the Sobolev spaces $W^{1,p}(G)$, $W_0^{1,p}(G)$ and $W_{\mathrm{loc}}^{1,p}(G)$.

Let G be an open subset of $\mathbf{R}^N$. A function $u \in W_{\mathrm{loc}}^{1,p}(G)$ is said to be a (weak) *solution* of (E_0) in G if

$$\int_G \mathcal{A}(x, \nabla u) \cdot \nabla\varphi \, dx + \int_G \mathcal{B}(x,u)\varphi \, dx = 0$$

for all $\varphi \in C_0^\infty(G)$.

A continuous solution of (E_0) in an open subset G is called $(\mathcal{A},\mathcal{B})$-*harmonic* in G. For any $(\mathcal{A},\mathcal{B})$-harmonic functions, the following locally Hölder continuity estimate holds ([7; Theorem 4.7] or [8; Proposition 2.1]) :

Proposition 1. *Let G be an open set. Then there are constants c and $0 < \lambda \le 1$ such that for $B(x_0,R) \Subset G$ and for every $(\mathcal{A},\ \mathcal{B})$-harmonic function h in G with $|h| \le L$ in $B(x_0,R)$,*

$$osc(h, B(x_0,r)) \le c\left(\frac{r}{R}\right)^{\lambda}\big(osc(h, B(x_0,R)) + R\big),$$

whenever $0 < r < R \le 1$. Here c depends only on $N, p, \alpha_1, \alpha_2, \alpha_3(G)$ and L and λ depends only on N, p, α_1, α_2 and $\alpha_3(G)$.

In the case of $\mathcal{A}(x,\xi) = |\xi|^{p-2}\xi$ and $\mathcal{B} = 0$, namely for the p-Laplace equation, we can choose $\lambda = 1$ ([3; Lemma 2.1]).

Suppose that ν is a signed Radon measure on G. Hölder continuity of a solution to the equation of the form

$$(\mathrm{E}_\nu) \qquad -\operatorname{div} \mathcal{A}(x,\ \nabla u(x)) + \mathcal{B}(x,\ u(x)) = \nu$$

was investigated in [9], [2] and [3]. In [6], Kilpeläinen and Zhong showed that, for the equation

$$(1) \qquad -\operatorname{div} \mathcal{A}(x,\ \nabla u(x)) = \nu$$

and for the case ν is a nonnegative Radon measure, if there exist constants $M > 0$ and $0 < \beta < \lambda$ with

$$\nu(B(x_0, r)) \le M\ r^{N-p+\beta(p-1)}$$

whenever $B(x, 3r) \subset G$, where λ is the number in Proposition 1 above, then a solution to the equation (1) is Hölder continuous with the same exponent β. We can extend this result to the case of the equation (E_ν) and of the signed Radon measure ([8]).

Theorem 1. *Let G be an open set and $u \in W^{1,p}_{\mathrm{loc}}(G)$ be a solution of (E_ν) in G. If ν is a signed Radon measure on G such that there exist constants $M > 0$ and $0 < \beta < \lambda$, where $\lambda = \lambda(N, p, \alpha_1, \alpha_2, \alpha_3(G)) > 0$ is the number in Proposition 1 above, with*

$$|\nu|(B(x, r)) \le M\ r^{N-p+\beta(p-1)}$$

whenever $B(x, 3r) \subset G$, then u is locally Hölder continuous in G with the exponent β.

§3. Global integrability of the gradient of a solution

In this section, we treat the higher integrability of the gradient of a solution of (E_T) in a bounded open set G. In [9], Rakotoson and Ziemer showed the local integrability of the gradient of a solution of (E_T) with $T \in W^{-1,p'+\delta}_{\mathrm{loc}}(G)$ for some $\delta > 0$. In [4], Kilpeläinen and Koskela treated the global integrability of the gradient of a solution of the equation (1) in the previous section under the condition the complement of G satisfies the uniformly thickness.

A set E is said to be *uniformly p-thick* with constants c_0 and $r_0 > 0$, if

$$\mathrm{cap}_p\big(\overline{B}(x_0, r) \cap E, B(x_0, 2r)\big) \ge c_0\, \mathrm{cap}_p\big(\overline{B}(x_0, r), B(x_0, 2r)\big)$$

for all $x_0 \in E$ and for all $0 < r < r_0$. For the notion of p-capacity cap_p, we refer to [1; Chapter 2].

We can show the following global integrability of the gradient of a solution of (E_T).

Theorem 2. *Suppose that G is a bounded open set, $\complement G$ is uniformly p-thick with constants $c_0, r_0 > 0$ and u is a solution of (E_T) in G such that $u - \theta \in W^{1,p}_0(G)$. Then there exists $\delta_0 = \delta(N, p, \alpha_1, \alpha_2, \alpha_3(G), c_0)$ such that $|\nabla u| \in L^{p+\delta}(G)$ whenever $T \in W^{-1,p'+\delta}(G)$ and $|\nabla\theta| \in L^{p+\delta}(G)$ for $0 < \delta < \delta_0$.*

Remark 1. The uniformly thickness condition cannot be suppressed in Theorem 2. (see [4; Remark 3.3])

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