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INVOLUTIONS ON COHOMOLOGY LENS SPACES AND PROJECTIVE SPACES

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ABSTRACT. We determine the possible mod 2 cohomology algebra of orbit spaces of free involutions on finitistic mod 2 cohomology lens spaces and projective spaces. We also give applications to $\mathbb{Z}_2$-equivariant maps from spheres to such spaces. Outlines of proofs are given and the details will appear elsewhere.

1. INTRODUCTION

Let G be a group acting continuously on a space X. Then there are two associated spaces, namely, the fixed point set and the orbit space. It has always been one of the basic problems in topological transformation groups to determine these two associated spaces. Determining these two associated spaces up to topological or homotopy type is often difficult and hence we try to determine the (co)homological type. The pioneering result of Smith [31], determining fixed point sets up to homology of prime periodic maps on homology spheres, was the first in this direction. More explicit relations between the space, the fixed point set and the orbit space were obtained by Floyd [10].

From now onwards, our focus will be on orbit spaces. One of the most famous conjectures in this direction was posed by Montgomery in 1950. Montgomery conjectured that:

For any action of a compact Lie group G on $\mathbb{R}^n$, the orbit space $\mathbb{R}^n/G$ is contractible.

The conjecture was reformulated by Conner [5] and is often called as the Conner conjecture. Even for a well understood space such as $\mathbb{R}^n$, it took more than 25 years to prove the above conjecture. It was due to work of Conner [5], Hsiangs [13, 14], Mostow [21] and finally that of Oliver [30], that the conjecture was proved in 1976. For spheres the orbit spaces of free actions of finite groups

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have been studied extensively by Livesay [19], Rice [23], Ritter [24], Rubinstein [26] and many others. However, very little is known if the space is a compact manifold other than a sphere. Myers [22] determined the orbit spaces of free involutions on three dimensional lens spaces. Tao [29] determined the orbit spaces of free involutions on $\mathbb{S}^1 \times \mathbb{S}^2$. Ritter [25] extended the results of Tao to free actions of cyclic groups of order 2^n. Recently Dotzel and others [9] determined completely the cohomology algebra of orbit spaces of free $\mathbb{Z}_p$ (p prime) and $\mathbb{S}^1$ actions on cohomology product of two spheres. Similar results for free involutions on cohomology projective spaces were obtained in [27].

In his study of fixed point theory, Swan [32] introduced a class of spaces known as finitistic spaces. A paracompact Hausdorff space X is said to be finitistic if its every open covering has a finite dimensional open refinement. Here the dimension of a covering is one less than the maximum number of members of the covering which intersect non-trivially. It is a large class of spaces including all compact Hausdorff spaces and all paracompact spaces of finite covering dimension. Bredon realized that the class of finitistic spaces was the most suitable for studying topological transformation groups and his book [3] contains an excellent account of results in this direction. Finitistic spaces behave nicely under group actions. More precisely, if G is a compact Lie group acting continuously on a space X, then the space X is finitistic if and only if the orbit space X/G is finitistic [7, 8]. Our results are also for finitistic spaces.

Recall that an involution on a topological space X is a continuous action of the cyclic group $G = \mathbb{Z}_2$ on X. In this note, we present some results on the mod 2 cohomology of orbit spaces of free involutions on finitistic mod 2 cohomology lens spaces and projective spaces. In Section 2, we briefly summarize the tools used in our work. In Section 3, we give examples of free involutions on lens spaces and projective spaces. In Section 4, we state our main results. In Section 5, we give applications to $\mathbb{Z}_2$-equivariant maps from spheres to such spaces. We end this note with some concluding remarks in Section 6. Detailed proofs of the results presented here will appear elsewhere [28].

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2. Preliminaries

Throughout we will use Čech cohomology. It is a well known fact that this is the most suitable cohomology for studying the cohomology theory of topological transformation groups on general spaces. Let G be a group and X be a G-space. Let

$$G \hookrightarrow E_G \longrightarrow B_G$$

be the universal principal G-bundle. Consider the diagonal action of G on $X \times E_G$. Let

$$X_G = (X \times E_G)/G$$

be the orbit space of the diagonal action on $X \times E_G$. Then the projection $X \times E_G \to E_G$ is G-equivariant and gives a fibration

$$X \hookrightarrow X_G \longrightarrow B_G$$

called the Borel fibration [4, Chapter IV]. Borel defined the equivariant cohomology of the G-space X to be any fixed cohomology (Čech cohomology in our case) of X_G. Our main tool is the Leray spectral sequence associated to the Borel fibration $X \hookrightarrow X_G \longrightarrow B_G$ given by the following proposition.

Proposition 2.1. [20, Theorem 5.2] *Let $G = \mathbb{Z}_2$ act on a space X and $X \hookrightarrow X_G \longrightarrow B_G$ be the associated Borel fibration. Then there is a first quadrant spectral sequence of algebras $\{E_r{}^{*,*}, d_r\}$, converging to $H^*(X_G; \mathbb{Z}_2)$ as an algebra, with*

$$E_2{}^{k,l} = H^k(B_G; \mathcal{H}^l(X; \mathbb{Z}_2)),$$

the cohomology of the base B_G with locally constant coefficients $\mathcal{H}^l(X; \mathbb{Z}_2)$ twisted by a canonical action of $\pi_1(B_G)$.

Let $h : X_G \to X/G$ be the map induced by the G-equivariant projection $X \times E_G \to X$. Then the following is true.

Proposition 2.2. [3, Chapter VII, Proposition 1.1] *Let $G = \mathbb{Z}_2$ act freely on a finitistic space X. Then*

$$h^* : H^*(X/G; \mathbb{Z}_2) \xrightarrow{\cong} H^*(X_G; \mathbb{Z}_2).$$

An action of a group on a space induces an action on the cohomology of the space. This induced action plays an important role in the cohomology theory of transformation groups. For $\mathbb{Z}_2$-actions on finitistic spaces the following result is true.

Proposition 2.3. [3, Chapter VII, Theorem 1.6] *Let $G = \mathbb{Z}_2$ act freely on a finitistic space X. Suppose that $\sum_{i \geq 0} rank\big(H^i(X; \mathbb{Z}_2)\big) < \infty$ and the induced action on $H^*(X; \mathbb{Z}_2)$ is trivial, then the Leray spectral sequence associated to $X \hookrightarrow X_G \longrightarrow B_G$ do not degenerate.*

3. Free involutions on lens spaces and projective spaces

Lens Spaces. These are odd dimensional spherical space forms described as follows. Let $p \geq 2$ be a positive integer and $q_1, q_2, \ldots, q_m$ be integers coprime to p, where $m \geq 1$. Let $\mathbb{S}^{2m-1} \subset \mathbb{C}^m$ be the unit sphere and let $\iota^2 = -1$. Then the map

$$(z_1, \ldots, z_m) \mapsto (e^{\frac{2\pi\iota q_1}{p}} z_1, \ldots, e^{\frac{2\pi\iota q_m}{p}} z_m)$$

defines a free action of the cyclic group $\mathbb{Z}_p$ on $\mathbb{S}^{2m-1}$. The orbit space is called as lens space (also called as generalized lens space) and is denoted by $L_p^{2m-1}(q_1, \ldots, q_m)$. It is a compact Hausdorff orientable manifold of dimension $2m-1$. Lens spaces are well understood and their homology groups can be easily computed using cell decomposition. They are important examples of spaces with finite fundamental group and $\pi_1\big(L_p^{2m-1}(q_1, \ldots, q_m)\big) = \mathbb{Z}_p$. The three dimensional lens spaces are of great importance and they appear frequently in works concerning three manifolds, surgery and knot theory. They are the first known examples of three manifolds which are not determined by their homology and fundamental group alone. From now onwards, for convenience, we write $L_p^{2m-1}(q)$ for $L_p^{2m-1}(q_1, \ldots, q_m)$.

We now construct a free involution on the lens space $L_p^{2m-1}(q)$. Let $q_1, \ldots, q_m$ be odd integers coprime to p. Consider the map $\mathbb{C}^m \to \mathbb{C}^m$ given by

$$(z_1, \ldots, z_m) \mapsto (e^{\frac{2\pi\iota q_1}{2p}} z_1, \ldots, e^{\frac{2\pi\iota q_m}{2p}} z_m).$$

This map commutes with the $\mathbb{Z}_p$-action on $\mathbb{S}^{2m-1}$ defining the lens space and hence descends to a map

$$\alpha : L_p^{2m-1}(q) \to L_p^{2m-1}(q)$$

such that $\alpha^2 =$ identity. Thus α is an involution. Denote an element of $L_p^{2m-1}(q)$ by $[z]$ for $z = (z_1, \ldots, z_m) \in \mathbb{S}^{2m-1}$. If $\alpha([z]) = [z]$, then

$$(e^{\frac{2\pi\iota q_1}{2p}} z_1, \ldots, e^{\frac{2\pi\iota q_m}{2p}} z_m) = (e^{\frac{2\pi\iota k q_1}{p}} z_1, \ldots, e^{\frac{2\pi\iota k q_m}{p}} z_m)$$

for some integer k. Let $1 \leq i \leq m$ be an integer such that $z_i \neq 0$, then $e^{\frac{2\pi\iota q_i}{2p}} z_i = e^{\frac{2\pi\iota k q_i}{p}} z_i$ and hence $e^{\frac{2\pi\iota q_i}{2p}} = e^{\frac{2\pi\iota k q_i}{p}}$. This implies

$$\frac{q_i}{2p} - \frac{kq_i}{p} = \frac{q_i(1-2k)}{2p}$$

is an integer, a contradiction. Hence the involution α is free. Note that the orbit space of the above involution is $L_{2p}^{2m-1}(q)$.

Real Projective Spaces. Observe that $L_2^{2m-1}(q) = \mathbb{R}P^{2m-1}$, the odd dimensional real projective space. Recall that $\mathbb{R}P^{2m-1}$ is the orbit space of the antipodal involution on $\mathbb{S}^{2m-1}$ given by

$$(x_1, x_2, ..., x_{2m-1}, x_{2m}) \mapsto (-x_1, -x_2, ..., -x_{2m-1}, -x_{2m}).$$

If we denote an element of $\mathbb{R}P^{2m-1}$ by $[x_1, x_2, ..., x_{2m-1}, x_{2m}]$, then the map

$$[x_1, x_2, ..., x_{2m-1}, x_{2m}] \mapsto [-x_2, x_1, ..., -x_{2m}, x_{2m-1}]$$

defines a free involution on $\mathbb{R}P^{2m-1}$. Infact this is the involution α for $p = 2$.

Complex Projective Spaces. Similarly, the complex projective space $\mathbb{C}P^{2m-1}$ admit free involutions. Recall that $\mathbb{C}P^{2m-1}$ is the orbit space of the free $\mathbb{S}^1$-action on $\mathbb{S}^{4m-1}$ given by

$$(z_1, z_2, ..., z_{2m-1}, z_{2m}) \mapsto (\zeta z_1, \zeta z_2, ..., \zeta z_{2m-1}, \zeta z_{2m}) \text{ for } \zeta \in \mathbb{S}^1.$$

If we denote an element of $\mathbb{C}P^{2m-1}$ by $[z_1, z_2, ..., z_{2m-1}, z_{2m}]$, then the map

$$[z_1, z_2, ..., z_{2m-1}, z_{2m}] \mapsto [-\overline{z}_2, \overline{z}_1, ..., -\overline{z}_{2m}, \overline{z}_{2m-1}]$$

defines an involution on $\mathbb{C}P^{2m-1}$. If

$$[z_1, z_2, ..., z_{2m-1}, z_{2m}] = [-\overline{z}_2, \overline{z}_1, ..., -\overline{z}_{2m}, \overline{z}_{2m-1}],$$

then

$$(\lambda z_1, \lambda z_2, ..., \lambda z_{2m-1}, \lambda z_{2m}) = (-\overline{z}_2, \overline{z}_1, ..., -\overline{z}_{2m}, \overline{z}_{2m-1})$$

for some $\lambda \in \mathbb{S}^1$, which gives $z_1 = z_2 = ... = z_{2m-1} = z_{2m} = 0$, a contradiction. Hence the involution is free.

4. Orbit spaces of free involutions on lens spaces and projective spaces

Before proceeding further, we set some notations. If X and Y are spaces, then $X \simeq_2 Y$ mean that X and Y have isomorphic mod 2 cohomology algebras. If X is a space such that $X \simeq_2 L_p^{2m-1}(q)$, we say that X is a mod 2 cohomology lens space and refer to dimension of $L_p^{2m-1}(q)$ as its dimension.

Involutions on lens spaces have been studied in detail, particularly on three dimensional lens spaces [11, 15, 16, 17, 22]. Hodgson and Rubinstein [11] obtained a classification of smooth involutions on three dimensional lens spaces having one dimensional fixed point sets. Kim [17] obtained a classification of free involutions on three dimensional lens spaces whose orbit spaces contains Klein bottles. Kim [16] showed that, if $p = 4k$ for some k, then the orbit space of any sense preserving free involution on $L_p^3(1, q)$ is the lens space $L_{2p}^3(1, q')$, where $q'q \equiv \pm 1$ or $q' \equiv \pm q$ mod p. Here an involution is sense preserving if the induced map on $H_1(L_p^3(1, q); \mathbb{Z})$ is the identity map. Myers [22] showed that every free involution on a three dimensional lens space is conjugate to an orthogonal free involution, in which case, the orbit space is again a lens space.

The work in this note is motivated by the work of Kim [16] and Myers [22]. We consider free involutions on finitistic mod 2 cohomology lens spaces and determine the possible mod 2 cohomology algebra of orbit spaces. Note that the lens space $L_p^{2m-1}(q)$ is a compact Hausdorff space and hence is finitistic. If X/G denotes the orbit space, then we prove the following theorem.

Theorem 4.1. [28, Main Theorem] *Let $G = \mathbb{Z}_2$ act freely on a finitistic space $X \simeq_2 L_p^{2m-1}(q)$. If $4 \nmid m$, then $H^*(X/G;\mathbb{Z}_2)$ is isomorphic to one of the following graded algebras:*

(1) $\mathbb{Z}_2[x]/\langle x^{2m}\rangle$,
where $deg(x) = 1$.
(2) $\mathbb{Z}_2[x, y]/\langle x^2, y^m\rangle$,
where $deg(x) = 1$ *and* $deg(y) = 2$.
(3) $\mathbb{Z}_2[x, y, z]/\langle x^4, y^2, z^{\frac{m}{2}}, x^2y - \lambda x^3\rangle$,
where $deg(x) = 1$, $deg(y) = 1$, $deg(z) = 4$, $\lambda \in \mathbb{Z}_2$, $m > 2$ *is even.*

Proof. The proof involves computations in the Leray spectral sequence of Propostion 2.1 and usage of equivariant cohomology along with Propositions 2.2 and 2.3. □

Since $L_2^{2m-1}(q) = \mathbb{R}P^{2m-1}$, as a side product of above computations, we determine the mod 2 cohomology algebra of orbit spaces of free involutions on mod 2 cohomology real projective spaces without any condition on m. More precisely, we prove the following.

Theorem 4.2. [28, Proposition 5.2] *Let $G = \mathbb{Z}_2$ act freely on a finitistic space $X \simeq_2 \mathbb{R}P^{2m-1}$. Then*

$$H^*(X/G;\mathbb{Z}_2) \cong \mathbb{Z}_2[x, y]/\langle x^2, y^m\rangle,$$

where $deg(x) = 1$ *and* $deg(y) = 2$.

Similar computations yield the following result for the complex case, which was first obtained in [27].

Theorem 4.3. [27, Theorem 1] *Let $G = \mathbb{Z}_2$ act freely on a finitistic space $X \simeq_2 \mathbb{C}P^{2m-1}$. Then*

$$H^*(X/G;\mathbb{Z}_2) \cong \mathbb{Z}_2[x, y]/\langle x^3, y^m\rangle,$$

where $deg(x) = 1$ *and* $deg(y) = 4$.

Remark 4.4. It is easy to see that, when n is even, then $\mathbb{Z}_2$ cannot act freely on a finitistic space $X \simeq_2 \mathbb{R}P^n$ or $\mathbb{C}P^n$. For, if n is even, then the Floyd's Euler characteristic formula [3, p.145]

$$\chi(X) + \chi(X^{\mathbb{Z}_2}) = 2\chi(X/\mathbb{Z}_2)$$

gives a contradiction.

Remark 4.5. Let $X \simeq_2 \mathbb{H}P^n$ be a finitistic space, where $\mathbb{H}P^n$ is the quaternionic projective space. For $n = 1$, $X \simeq_2 \mathbb{S}^4$, which is well studied. And for $n \geq 2$ there is no free involution on X, which follows from the stronger fact that such spaces have the fixed point property.

Remark 4.6. Let $X \simeq_2 \mathbb{O}P^2$ be a finitistic space, where $\mathbb{O}P^2$ is the Cayley projective plane. Just as in Remark 4.4, it follows from the Floyd's Euler characteristic formula that there is no free involution on X.

5. Application to $\mathbb{Z}_2$-equivariant maps

Let $\mathbb{S}^k$ be the unit k-sphere equipped with the antipodal involution and X be a paracompact Hausdorff space with a fixed free involution. Conner and Floyd [6] proposed an invariant of the involution, which they called the index of the involution and defined as

$$\mathrm{ind}(X) = max\ \{\ k \mid \text{there exist a } \mathbb{Z}_2\text{-equivariant map } \mathbb{S}^k \to X\}.$$

It is natural to look for a purely cohomological criteria to study the above invariant. The best known and most easily managed cohomology class are the characteristic classes with coefficients in $\mathbb{Z}_2$. Generalizing the Yang's index [33], Conner and Floyd defined

$$\text{co-ind}_{\mathbb{Z}_2}(X) = max\ \{\ k \mid w^k \neq 0\},$$

where $w \in H^1(X/G; \mathbb{Z}_2)$ is the Whitney class of the principal G-bundle

$$X \to X/G.$$

Since X is paracompact Hausdorff, we can take a classifying map

$$c : X/G \to B_G$$

for the principal G-bundle $X \to X/G$. The image of the Whitney class of the universal principal G-bundle $G \hookrightarrow E_G \longrightarrow B_G$ under the classifying map c^* is the Whitney class of the principal G-bundle $X \to X/G$.

If $X \simeq_2 L_p^{2m-1}(q)$ is a finitistic space with a free involution, then our computations determine the Whitney class explicitly and also $\text{co-ind}_{\mathbb{Z}_2}(X)$. Since $\text{co-ind}_{\mathbb{Z}_2}(\mathbb{S}^k) = k$, by [6, (4.5)], we have

$$\mathrm{ind}(X) \leq \text{co-ind}_{\mathbb{Z}_2}(X).$$

As a consequence we have the following result.

Proposition 5.1. [28, Theorem 6.1] *Let $X \simeq_2 L_p^{2m-1}(q)$ be a finitistic space with a free involution. If $4 \nmid m$, then there does not exist any $\mathbb{Z}_2$-equivariant map from $\mathbb{S}^k \to X$ for $k \geq 2$.*

The above result extends a result of Jaworowski [15] (proved for lens spaces) to cohomology lens spaces. For cohomology real projective spaces we have the following result without any condition on m.

Proposition 5.2. [28, Theorem 6.1] *Let $X \simeq_2 \mathbb{R}P^{2m-1}$ be a finitistic space with a free involution. Then there does not exist any $\mathbb{Z}_2$-equivariant map from $\mathbb{S}^k \to X$ for $k \geq 2$.*

For cohomology complex projective spaces we deduce the following result, which was first obtained in [27].

Proposition 5.3. [27, Corollary] *Let $X \simeq_2 \mathbb{C}P^{2m-1}$ be a finitistic space with a free involution. Then there does not exist any $\mathbb{Z}_2$-equivariant map from $\mathbb{S}^k \to X$ for $k \geq 3$.*

6. Some concluding remarks

Let F be either the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers. Let $FG_{n,k}$ be the Grassmann manifold of all k-dimensional vector subspaces in F^n. In particular, $\mathbb{R}G_{n,1} = \mathbb{R}P^{n-1}$, the real projective space of dimension $n-1$. Similarly, $\mathbb{C}G_{n,1} = \mathbb{C}P^{n-1}$, the complex projective space of complex dimension $n-1$. The Grassmann manifolds are important objects of study in topology and geometry. It is interesting to note that they admit free involutions. For example, if $n = 2k$, then the map $FG_{n,k} \to FG_{n,k}$ sending any k-dimensional vector subspace in F^n to its orthogonal complement, defines a free involution on $FG_{n,k}$. See [1] for some more examples of free involutions on $FG_{n,k}$ for n even. Since, the Grassmann manifolds are generalizations of projective spaces, it is tempting to ask the following question:

Question. *Can we describe the cohomology algebra of orbit spaces of free involutions on cohomology Grassmann manifolds?*

Though the cohomology algebra of Grassmann manifolds is well understood, the computations in spectral sequences seems to be complicated in this case, since the induced action in the cohomology may not be trivial.

There are situations in topology, where one needs to obtain cohomological information of the total space of a fiber bundle if the cohomological information of its fiber is given. In general, it is difficult to obtain information about the total space of a fiber bundle. However, for certain types of fibers, some results are known [2, 12, 18]. Horanská and Korbaš [12, 18] have studied this problem for fiber bundles with Grassmann manifolds as fibers. Consider fiber bundles with cohomology Grassmann manifolds as fibers. Let the bundles be equipped with fiber preserving free involutions. With an answer to the above question, one would be able to describe the equivariant cohomology of such fiber bundles. This would extend the results of Horanská and Korbaš to the equivariant setting.

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