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Lattices of Non-Compact Lie Groups

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1 Introduction

Consider solvable Lie groups of type $H = \mathbb{R}^m \ltimes_\psi \mathbb{R}^{n+1}$ $(n \geq m)$. Here ψ is a homomorphis from $\mathbb{R}^m$ to $GL(n+1, \mathbb{R})$ and the group structure of H is given by

$$(\mathbf{s}, \mathbf{x})(\mathbf{t}, \mathbf{y}) = (\mathbf{s}+\mathbf{t}, \mathbf{x}+\psi(\mathbf{t})(\mathbf{y})), \quad (\mathbf{s}, \mathbf{t} \in \mathbb{R}^m,\ \mathbf{x}, \mathbf{y} \in \mathbb{R}^{n+1}).$$

We call Lie groups of this type 1-step solvable Lie groups. In this paper we study about the automorphisms groups of lattices (cocompact discrete subgroups) of 1-step solvable Lie groups H.

The unimodularization of n products $\mathsf{Aff}^+(\mathbb{R})^n$ of the affine group $\mathsf{Aff}^+(\mathbb{R})$ is a 1-step solvable Lie group which takes the form of $\mathbb{R}^n \ltimes_\psi \mathbb{R}^{n+1}$. In this case, the homomorphism ψ is injective and splits as a direct sum of non-equivariant real 1-dimensional representations. Conversely, if the homomorphism ψ of $H = \mathbb{R}^n \ltimes_\psi \mathbb{R}^{n+1}$ has all of these properties, then H is isomorphic to $\mathsf{Aff}^+(\mathbb{R})^n$. Let Γ be a lattice of $H = \mathbb{R}^n \ltimes_\psi \mathbb{R}^{n+1} \cong \mathsf{Aff}^+(\mathbb{R})^n$. In [2], we defined an algebraic number field $k(\Gamma)$ of degree $n+1$ which is associated with a lattice Γ, and showed that the automorphism group $\mathrm{Aut}(\Gamma')$ of a lattice Γ' commensurable with Γ is essentially identified with a subgroup of the automorphism group $\mathrm{Aut}(k(\Gamma)/\mathbb{Q})$. More precisely, there is a surjection from the set $\{\mathrm{Aut}(\Gamma') \mid \Gamma' < H,\ \Gamma' \in Com(\Gamma)\}$ to the set $\{F \mid F < \mathrm{Aut}(k(\Gamma)/\mathbb{Q})\ \}$ (Theorem 1.2 in [2]). Here $Com(\Gamma)$ denotes the set of lattices Γ' which are commensurable with Γ (see §4). But, when $n > m$, we have quite different results from those in the case of $n = m$.

In the first half of this paper, we review basic facts about lattices of 1-step solvable Lie groups $H = \mathbb{R}^m \ltimes_\psi \mathbb{R}^{n+1}$, and in §4 we state an interesting Theorem 1.2 in [2]. In the last two sections, we study the case of $m < n$, especially the case of $n = m+1$.

From now on, let H denote 1-step solvable Lie groups of type $\mathbb{R}^m \ltimes_\psi \mathbb{R}^{n+1}$. Moreover we assume that ψ is *injective and splits as a direct sum of non-equivariant real 1-dimensional representations.*

2 Structure matrix of H

From the assumption that ψ splits as a direct sum of non-equivariant real 1-dimensional representations, for a basis $\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_m\}$ of $\mathbb{R}^m$, $A_j := \psi(\mathbf{e}_j)$ $(1 \leq j \leq m)$ are simultaneously conjugate to diagonal matrices diag $\left(e^{\lambda_{1j}}, e^{\lambda_{2j}}, \cdots, e^{\lambda_{n+1,j}}\right)$. Put

$$\Lambda_\psi := \begin{pmatrix} \lambda_{11} & \lambda_{12} & \cdots & \lambda_{1m} \\ \lambda_{21} & \lambda_{22} & \cdots & \lambda_{2m} \\ \vdots & \vdots & & \vdots \\ \lambda_{n+1,1} & \lambda_{n+1,2} & \cdots & \lambda_{n+1,m} \end{pmatrix} = \begin{pmatrix} \Lambda_1 \\ \Lambda_2 \\ \vdots \\ \Lambda_{n+1} \end{pmatrix},$$

and call Λ_ψ the *structure matrix* of $H = \mathbb{R}^m \ltimes_\psi \mathbb{R}^{n+1}$. Clearly H is determined by the structure matrix.

We note here some fundamental facts on Λ_ψ.

1. Changing bases of $\mathbb{R}^m$ and $\mathbb{R}^{n+1}$, the new structure matrix Λ'_ψ is written as $\Lambda'_\psi = T\Lambda_\psi P$, where T is a row exchanging matrix and P is an m-square non-singular matrix, that is $P \in GL(m, \mathbb{R})$. If $\Lambda'_\psi = T\Lambda_\psi P$ holds, then we say Λ_ψ and Λ'_ψ to be *equivalent* and identify Λ_ψ with Λ'_ψ.

2. Let $\Delta : G \to \mathbb{R}_+$ be the modular function of a Lie group G defined by $\Delta(g) = |\det Ad_g|$. For $H = \mathbb{R}^m \ltimes_\psi \mathbb{R}^{n+1}$, the modular function $\Delta : H \to \mathbb{R}_+$ is given by $\Delta(\mathbf{t}, \mathbf{x}) = \exp(\sum_{i=1}^{n+1} \Lambda_i \cdot \mathbf{t})$.

3. If there exists a cocompact discrete subgroup (i.e. a lattice) Γ of H, then $\Delta(\mathbf{t}, \mathbf{x}) = 1$ for ${}^\forall(\mathbf{t}, \mathbf{x}) \in H = \mathbb{R}^m \ltimes_\psi \mathbb{R}^{n+1}$. This shows $\sum_{i=1}^{n+1} \Lambda_i \cdot \mathbf{t} = 0 \quad ({}^\forall \mathbf{t} \in \mathbb{R}^m)$, and thus $\sum_{i=1}^{n+1} \Lambda_i = \mathbf{0}$.

In this paper, we study about lattices of H. Thus, from now on, we assume that *the structure matrix Λ_ψ satifies* $\sum_{i=1}^{n+1} \Lambda_i = 0$.

3 Lattices and algebraic number fields

In this section, we define the algebraic number field $k(\Gamma)$ associated with a lattice Γ of H. Let $H_0 := [H, H]$ and $H_1 := H/H_0$. Then $H_0 \cong \mathbb{R}^{n+1}$, $H_1 \cong \mathbb{R}^m$ and $H = \mathbb{R}^m \ltimes_\psi \mathbb{R}^{n+1} = H_1 \ltimes_\psi H_0$ holds. The following is a known result.

Lemma 3.1 ([3, Lemma 2.3]) *Let $\Gamma < H$ be a lattice. Put $\Gamma_0 := \Gamma \cap H_0 = \Gamma \cap \mathbb{R}^{n+1}$ and $\Gamma_1 := \Gamma/\Gamma_0$. Then Γ_0 and Γ_1 are lattices of $\mathbb{R}^{n+1}$ and $\mathbb{R}^m$, respectively.*

From Lemma 3.1, we can see $\Gamma_0 \cong \mathbb{Z}^{n+1}$ and $\Gamma_1 \cong \mathbb{Z}^m$. Moreover we have the exact sequences

$$\begin{array}{ccccccccc} 1 & \to & \mathbb{R}^{n+1} & \to & H & \to & \mathbb{R}^m & \to & 1 \\ & & \cup & & \cup & & \cup & & \\ 1 & \to & \Gamma_0 & \to & \Gamma & \to & \Gamma_1 & \to & 1 \end{array}$$

In general, Γ is not a semi-direct product group. But the restriction $\psi_{|\Gamma_1}$ becomes a homomorphism from Γ_1 to $Aut(\Gamma_0)$, and hence, $\psi(\mathbf{t}) \in SL(n+1, \mathbb{Z}) \quad (\mathbf{t} \in \Gamma_1)$. Thus we may assume that, in the structure matrix

$$\Lambda_\psi = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \cdots & \lambda_{1m} \\ \lambda_{21} & \lambda_{22} & \cdots & \lambda_{2m} \\ \vdots & \vdots & & \vdots \\ \lambda_{n+1,1} & \lambda_{n+1,2} & \cdots & \lambda_{n+1,m} \end{pmatrix},$$

the numbers $e^{\lambda_{1j}}, e^{\lambda_{2j}}, \cdots, e^{\lambda_{n+1,j}}$ are eigenvalues of an integer matrix $A_j = \psi(\mathbf{t}_j) \in SL(n+1, \mathbb{Z})$, that is, those numbers are algebraic integers. Here $\{\mathbf{t}_1, \mathbf{t}_2, \cdots, \mathbf{t}_m\}$ is a $\mathbb{Z}$-basis of $\Gamma_1 \cong \mathbb{Z}^m$.

We suppose the following condtions on Λ_ψ.

Assumtion A on ψ (i.e. on Λ_ψ)

1. ψ is injective.
2. There exists $\mathbf{t}_0 \in \Gamma_1$ such that each eigenvalue $\alpha_1, \alpha_2, \cdots, \alpha_{n+1}$ of $\psi(\mathbf{t}_0) = A$ is an algebraic integer of degree $n+1$. Here $\alpha_1, \alpha_2, \cdots, \alpha_{n+1}$ are each other conjugate elements.

Remark 3.1.

(1) When $n = m$, the assumption 2 is automatically derived from the assumption 1.

(2) For each $\mathbf{t} \in \Gamma_1$, the matrix $\psi(\mathbf{t})$ can be described as $g(A)$ $(g[X] \in \mathbb{Q}[X])$ because $\psi(\mathbf{t})$ and $\psi(\mathbf{t}_0) = A$ are commutative ([2, Corollary 3.2]).

Under Assumption A, we can assign a totally real algebraic number field $k(\Gamma) = \mathbb{Q}(\alpha)$ of degree $n+1$ to a lattice $\Gamma < H$, where $\alpha = \alpha_1$ in the above assumption 2. Call $k(\Gamma)$ the *algebraic number field associated with* Γ. We note that, from Remark 3.1-(2), $k(\Gamma)$ does not depends on the choice of $\mathbf{t}_0$.

Lattices Γ and Γ' are called to be *commensurable* and denoted by $\Gamma \overset{com}{\sim} \Gamma'$ if $|\Gamma : \Gamma \cap \Gamma'| < \infty$ and $|\Gamma' : \Gamma \cap \Gamma'| < \infty$. From Remark 3.1-(2), it follows that $k(\Gamma) = k(\Gamma')$ if $\Gamma \overset{com}{\sim} \Gamma'$. Furthermore we say that Γ and Γ' are *weakly commensurable* if there exists $\varphi \in \mathrm{Aut}(H)$ such that $\varphi(\Gamma) \overset{com}{\sim} \Gamma'$. When $n = m$, $k(\Gamma') = \varphi_*(k(\Gamma))$ holds ([2, Lemma 3.3]). From those facts, we obtain the following theorem.

Theorem 3.2 *Suppose $n = m$. Then the map*

$$\left\{\begin{array}{c}\text{the set of all weakly}\\ \text{commensurable classes}\\ \text{of lattices of } H\end{array}\right\} \to \left\{\begin{array}{c}\text{the set of all isomorphism}\\ \text{classes of totally real algebraic}\\ \text{number fields of degree } n+1\end{array}\right\}$$

induced from the map $\Gamma \to k(\Gamma)$ is bijective.

4 Aut(Γ)

Let Γ be a lattice of H, and take $\varphi \in \mathrm{Aut}(\Gamma)$. Then the following hold.

1. φ naturally induces automorphisms $\varphi_1 : \Gamma_1 \to \Gamma_1$ and $\varphi_0 : \Gamma_0 \to \Gamma_0$.
2. $\psi(\varphi_1(\mathbf{t})) = \varphi_0 \psi(\mathbf{t}) \varphi_0^{-1} \quad ({}^\forall \mathbf{t} \in \Gamma_1 = \mathbb{Z}^m)$.

The equality 2 follows from that φ is a homomorphism. We call this equality in 2 the *compatibility condition* for (φ_1, φ_0).

Remark 4.1. It is known that $\varphi \in \mathrm{Aut}(\Gamma)$ is uniquely extended to $\tilde{\varphi} \in \mathrm{Aut}(H)$ (e.g. [1][2]). Clearly the compatibility condition holds for $(\tilde{\varphi}_1, \tilde{\varphi}_0)$.

Using 1 and 2 above, we can define a homomorphism

$$A_\Gamma : \mathrm{Aut}(\Gamma) \longrightarrow \mathrm{Aut}(k(\Gamma)/\mathbb{Q})$$

by

$$A_\Gamma(\varphi)(\psi(\mathbf{t}_0)) = \psi(\varphi_1(\mathbf{t}_0)) = \varphi_0 \psi(\mathbf{t}_0) \varphi_0^{-1}.$$

From the definition, the map $A_\Gamma(\varphi)$ induces a permutation of the set $\{\alpha = \alpha_1, \alpha_2, \cdots, \alpha_{n+1}\}$.

Theorem 4.1 *Suppose $m = n$. Let Γ be a lattice of H. Then, for each subgroup $F < Aut(k(\Gamma)/\mathbb{Q})$, there exists a lattice $\Gamma' < H$ such that*

(1) Γ' *is commensurable with* Γ,

(2) $A_{\Gamma'}(Aut(\Gamma')) = F$.

Outline of the proof. Let k be a totally real algebraic number field of degree $n+1$ and let $\{f^{(1)}, f^{(2)}, \cdots, f^{(n+1)}\}$ be the set of all imbeddings of k into $\mathbb{R}$. Let $\mathcal{O}(k)$ be the subring of algebraic integers in k. The ring $\mathcal{O}(k)$ is isomorphic to $\mathbb{Z}^{n+1}$ as additive groups. Denote by $\mathcal{E}(k)$ the unit group of $\mathcal{O}(k)$ and put

$$\mathcal{E}^+(k) := \{\varepsilon \in \mathcal{E}(k) \mid f^{(i)}(\varepsilon) > 0 \ \ (1 \le i \le n+1) \ \}.$$

Define an injective map $\ell_k : \mathcal{E}^+(k) \to \mathbb{R}^{n+1}$ by

$$\ell_k(\varepsilon) = \left(\log\left(f^{(1)}(\varepsilon)\right),\ \log\left(f^{(2)}(\varepsilon)\right), \cdots,\ \log\left(f^{(n+1)}(\varepsilon)\right)\right)$$

The Dirichlet's unit theorem asserts that $\ell_k(\mathcal{E}^+(k))$ is a lattice of $V = \{(x_1, x_2, \cdots, x_{n+1}) \in \mathbb{R}^{n+1} \mid \sum_{i=1}^{n+1} x_i = 0 \ \}$. Put

$$\Gamma_k = \ell_k(\mathcal{E}^+(k)) \ltimes_{\psi_k} \mathcal{O}(k), \quad H_k = \left(\ell_k(\mathcal{E}^+(k)) \otimes \mathbb{R}\right) \ltimes_{\tilde{\psi_k}} (\mathcal{O}(k) \otimes \mathbb{R})\,.$$

The homomorphism $\psi_k : \ell_k(\mathcal{E}^+(k)) \to \mathrm{Aut}(\mathcal{O}(k))$ is given by $\psi_k \circ \ell_k = \iota_k$, where ι_k is the tautological map defined by $\iota_k(\varepsilon)(\gamma) = \varepsilon\gamma$ $(\gamma \in k)$. The homomorphism $\tilde{\psi_k}$ is the naural extension of ψ_k.

Now take groups H and Γ in the theorem. Then we can construct an isomorphism Ψ_Γ from H to H_k such that $\varphi_k(\Psi_\Gamma(\Gamma)) \overset{com}{\sim} \Gamma_k$ for $\varphi_k \in \mathrm{Aut}(H_k)$ ([2, Lemma 3.6]). From this, to prove the theorem, we may assume $\Gamma = \Gamma_k \subset H_k = H$.

Suppose $H = H_k, \Gamma = \Gamma_k$. From the definition, $A_{\Gamma_k}(\mathrm{Aut}(\Gamma_k)) = \mathrm{Aut}(k/\mathbb{Q})$ holds. Take a subgroup $\mathcal{E}_1 < \mathcal{E}^+(k)$ with $|\mathcal{E}^+(k) : \mathcal{E}_1| < \infty$, and put $\Gamma' := \ell_k(\mathcal{E}_1) \ltimes_{\psi_k} \mathcal{O}(k)$. Clearly Γ_k and Γ' are commensurable. Moreover it is seen that

$$Ad(\iota_k^{-1})A_{\Gamma'}(\mathrm{Aut}(\Gamma')) = \{\sigma \in \mathrm{Aut}(k/\mathbb{Q}) \mid \sigma(\mathcal{E}_1) = \mathcal{E}_1 \ \}.$$

Thus, for a given $F < \mathrm{Aut}(k/\mathbb{Q})$, we only have to construct $\mathcal{E}_1$ such that

$$F = \{\ \sigma \in \mathrm{Aut}(k/\mathbb{Q}) \mid \sigma(\mathcal{E}_1) = \mathcal{E}_1 \ \}.$$

Such an $\mathcal{E}_1$ can be constructed by using Artin's theorem on relative fundamental units (e.g., [2, Theorem 4.1]). $\square$

When $n = m$, we showed that if $\varphi \in Ker A_\Gamma$, then $\varphi^2 = Ad(h_0)$ for some $h_0 \in H$ ([2, Corollary 3.11]).

5 Aut(Γ) when $n > m$

In the rest of paper, we treat the case where $n > m$, that is, $H = \mathbb{R}^m \ltimes_\psi \mathbb{R}^{n+1} (n > m)$. We add one more assumption on the structure matrix Λ_ψ.

<u>Assumption B on Λ_ψ</u>

Every m row vectors $\Lambda_{i_1}, \Lambda_{i_2}, \cdots, \Lambda_{i_m}$ of Λ_ψ are linearly independent over $\mathbb{R}$.

Remark 5.1. When $n = m$, Assumption B is automatically derived from the injectivity of ψ.

Let Γ be a lattice of H, and take $\varphi \in \mathrm{Aut}(\Gamma)$. Then, from the compatibility condition $\psi(\varphi_1(\mathbf{t})) = \varphi_0\psi(\mathbf{t})\varphi_0^{-1}$, the map $A_\Gamma(\varphi) \in \mathrm{Aut}(k(\Gamma)/\mathbb{Q})$ induces a permutation $\sigma \in S_{n+1}$. Moreover $\varphi \in \mathrm{Aut}(\Gamma)$ acts on the strucure matrix Λ_ψ as follows.

$$T_\sigma \Lambda_\psi = T_\sigma \begin{pmatrix} \Lambda_1 \\ \Lambda_2 \\ \vdots \\ \Lambda_{n+1} \end{pmatrix} = \begin{pmatrix} \Lambda_1 \\ \Lambda_2 \\ \vdots \\ \Lambda_{n+1} \end{pmatrix} P_\sigma \qquad (5.1)$$

where T_σ is the row exchanging matrix corresponding to σ and $P_\sigma \in GL(m, \mathbb{Z})$. Let

$$\begin{pmatrix} \Lambda_1 \\ \Lambda_2 \\ \vdots \\ \Lambda_{n+1} \end{pmatrix} = \begin{pmatrix} I_m \\ c_{11}, c_{12}, \cdots, c_{1m} \\ \cdots\cdots\cdots\cdots \\ c_{p1}, c_{p2}, \cdots, c_{pm} \end{pmatrix} \begin{pmatrix} \Lambda_1 \\ \Lambda_2 \\ \vdots \\ \Lambda_m \end{pmatrix}.$$

where $p = n + 1 - m$.

Putting $P'_\sigma = \begin{pmatrix} \Lambda_1 \\ \Lambda_2 \\ \vdots \\ \Lambda_m \end{pmatrix} P_\sigma \begin{pmatrix} \Lambda_1 \\ \Lambda_2 \\ \vdots \\ \Lambda_m \end{pmatrix}^{-1}$, the above relation (5.1) is re-written as

$$T_\sigma \begin{pmatrix} I_m \\ c_{11}, c_{12}, \cdots, c_{1m} \\ \cdots\cdots\cdots\cdots \\ c_{p1}, c_{p2}, \cdots, c_{pm} \end{pmatrix} = \begin{pmatrix} I_m \\ c_{11}, c_{12}, \cdots, c_{1m} \\ \cdots\cdots\cdots\cdots \\ c_{p1}, c_{p2}, \cdots, c_{pm} \end{pmatrix} P'_\sigma. \qquad (5.2)$$

From the condition $\sum_{i=1}^{n+1} \Lambda_i = \mathbf{0}$, we have

$$1 + \sum_{i=1}^{p} c_{ij} = 0 \quad (1 \leq j \leq m). \qquad (5.3)$$

Remark 5.2. When $n = m$, for every permutation $\sigma \in S_{n+1}$, the conditions (5.1)(5.2) are satisfied.

Divide each permutation $\sigma \in S_{n+1}$ into a product of distinct cylic permutations, $\sigma = \sigma_1\sigma_2\cdots\sigma_k$. We say cyclic permutations $\sigma_i = (i_1, i_2, \cdots, i_\ell)$ and

$\sigma_j = (j_1, j_2, \cdots, j_k)$ to be distinct if $i_s \neq j_t$ $(i \neq j)$ $(1 \leq s \leq \ell,\ 1 \leq t \leq k)$, and define the length of $\sigma_i = (i_1, i_2, \cdots, i_\ell)$ to be ℓ. For example, (123) and (4567) are distinct cyclic permutations, and (123) and (3456) are not distinct.

The following propositions are given by simple calculations (see [5] for the details). In the propositions and corollaries below, we suppose Assumption B.

Proposition 5.1 ([5]) *Let $\sigma = \sigma_1\sigma_2\cdots\sigma_k \in S_{n+1}$ ($\sigma \neq$ trivial) be the product of distinct cyclic permutations. Suppose that, for the $\sigma = \sigma_1\sigma_2\cdots\sigma_k$, there exists P'_σ satisfying the condtions (5.2)(5.3). Then all σ_i have the same length, or the length of σ_1 is 1 and the other σ_i's have the same length.*

When each non-trivial cyclic permutation σ_i of $\sigma = \sigma_1\sigma_2\cdots\sigma_k$ has the same length, we also say the length to be the *size* of σ. For example, the size of $\sigma = (123)(456)(789)$ is 3.

Proposition 5.2 ([5]) *If $(n+1) - m = s$ is even, then the size of σ satisfying the conditions (5.2) (5.3) is not larger than s.*

Corollary 5.3 ([5]) *Suppose that $n+1$ is even and $(n+1) - m = 2$. Then a permutation σ satisfying the conditions (5.2)(5.3) is one of*

$$(1)\quad trivial,\qquad (2)\quad \sigma = (12)(34)\cdots(m+1, m+2),$$

by renumbering row vectors $\Lambda_1, \Lambda_2, \cdots, \Lambda_{n+1}$ of Λ_ψ if necessary. Moreover, in the case of (2), the structure matrix Λ_ψ is equivalent to the form

$$\begin{pmatrix} & & & I_m & & & \\ a_1 & b_1 & a_2 & b_2 & \cdots & a_k & b_k \\ b_1 & a_1 & b_2 & a_2 & \cdots & b_k & a_k \end{pmatrix} \begin{pmatrix} \Lambda_1 \\ \Lambda_2 \\ \vdots \\ \Lambda_m \end{pmatrix}. \qquad \text{(S)}$$

We say Λ_ψ to be type (S) if it is equivalent to the above form (S). Corolally 5.3 implies that $\mathrm{Aut}(\Gamma)$ in the case $n > m$ is quite different from that in the case $n = m$.

Corollary 5.4 ([5]) *Let Γ be a lattice of $H = \mathbb{R}^m \ltimes_\psi \mathbb{R}^{n+1}$. Suppose that $n+1$ is even and $(n+1) - m = 2$. Then*

$$|A_\Gamma(Aut(\Gamma))| \leq 2.$$

6 Case of $H = \mathbb{R}^2 \ltimes_\psi \mathbb{R}^4$

In this section we treat $H = \mathbb{R}^2 \ltimes_\psi \mathbb{R}^4$, and give two examples of Γ such that $|A_\Gamma(\mathrm{Aut}(\Gamma))| = 2$ and $|A_\Gamma(\mathrm{Aut}(\Gamma))| = 1$. See [4] for another examples.

Let A_j $(1 \leq j \leq 3)$ be the 4×4 integer matrices given by

$$A_1 := \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -13 \\ 0 & 0 & 1 & 7 \end{pmatrix}, \quad A_2 := \begin{pmatrix} 5 & 1 & 0 & -1 \\ -12 & -2 & 1 & 7 \\ 7 & 1 & 2 & -12 \\ -1 & 0 & 1 & 5 \end{pmatrix},$$

$$A_3 := \begin{pmatrix} -9 & -104 & -575 & -2742 \\ 69 & 719 & 3921 & 18619 \\ -153 & -1283 & -6756 & -31725 \\ 104 & 575 & 2742 & 12438 \end{pmatrix}.$$

Then we can see the following.

1. $\det A_j = 1$ $(j = 1, 2, 3)$, that is, $A_j \in SL(4, \mathbb{Z})$.
2. Let $f_2(x) = -x^3 + 7x^2 - 12x + 5$, $f_3(x) = 104x^3 - 153x^2 + 69x - 9$. Then $A_2 = f_2(A_1)$, $A_3 = f_3(A_1)$.
3. Let $g_j(x)$ be the characteristic polynomial of A_j. Each $g_j(x)$ is given as

$$\begin{aligned} g_1(x) &= x^4 - 7x^3 + 13x^2 - 7x + 1, \\ g_2(x) &= (x^2 - 3x + 1)^2, \\ g_3(x) &= x^4 - 6392x^2 + 1515658x^2 - 11717x + 1, \end{aligned}$$

   and thus all of the eigenvalues of the matrices A_j $(1 \leq j \leq 3)$ are positive real numbers.

4. The eigenvalues of A_1 are

$$\begin{aligned} \alpha_1 &= \tfrac{7-\sqrt{5}}{4} - \tfrac{1}{2}\sqrt{\tfrac{19-7\sqrt{5}}{2}} \\ \alpha_2 &= \tfrac{7-\sqrt{5}}{4} + \tfrac{1}{2}\sqrt{\tfrac{19-7\sqrt{5}}{2}} \\ \alpha_3 &= \tfrac{7-\sqrt{5}}{4} - \tfrac{1}{2}\sqrt{\tfrac{19+7\sqrt{5}}{2}} \\ \alpha_4 &= \tfrac{7+\sqrt{5}}{4} + \tfrac{1}{2}\sqrt{\tfrac{19+7\sqrt{5}}{2}} \end{aligned}$$

   Clearly the eigenvalues of A_2 and A_3 are $f_2(\alpha_i)$ and $f_3(\alpha_i)$ $(1 \leq i \leq 4)$. The numerical values of α_i are

$$\begin{aligned} \alpha_1 &\doteqdot 0.544113 \\ \alpha_2 &\doteqdot 1.8378528 \\ \alpha_3 &\doteqdot 0.227777 \\ \alpha_4 &\doteqdot 4.390257 \end{aligned}$$

5. Let Λ be the 4×2 matrix whose (ij) entry is $\log(f_j(\alpha_i))$ $(1 \le i \le 4, 1 \le j \le 2)$. (We put $f_1(x) := x$). Then the numerical value of Λ is the following:

$$\Lambda \doteqdot \begin{pmatrix} -0.608598 & -0.962424 \\ 0.608598 & -0.962424 \\ -1.47939 & 0.962424 \\ 1.47939 & 0.9692424 \end{pmatrix}$$

6. Let Λ_u be the "upper half" of Λ. That is, let Λ_u be the 2×2 matrix whose (ij) entry is $\log(f_j(\alpha_i))$ $(1 \le i \le 2, 1 \le j \le 2)$. Then we have

$$\Lambda(\Lambda_u)^{-1} \doteqdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0.71541 & -1.71541 \\ -1.71541 & 0.71541 \end{pmatrix}.$$

7. Let Λ' be the 4×2 matrix whose $(i1)$ entry is $\log \alpha_i$ and $(i2)$ entry is $\log f_3(\alpha_i)$ $(1 \le i \le 4)$. Then the numerical value of Λ' is the following:

$$\Lambda' \doteqdot \begin{pmatrix} -0.608598 & -9.35757 \\ 0.608598 & 5.50787 \\ -1.47939 & -4.87376 \\ 1.47939 & 8.72346 \end{pmatrix}$$

Lemma 6.1 (1) Λ *in 5 is of type* (S), (2) Λ' *in 7 is not of type* (S).

Proof It is seen that

$$\Lambda = \begin{pmatrix} \lambda_1 & \mu_1 \\ -\lambda_1 & \mu_1 \\ \lambda_2 & \mu_2 \\ -\lambda_2 & \mu_2 \end{pmatrix}, \quad (\mu_1 + \mu_2 = 0).$$

Thus

$$\Lambda(\Lambda_u)^{-1} = \begin{pmatrix} \lambda_1 & \mu_1 \\ -\lambda_1 & \mu_1 \\ \lambda_2 & \mu_2 \\ -\lambda_2 & \mu_2 \end{pmatrix} \frac{1}{2\lambda_1\mu_1} \begin{pmatrix} \mu_1 & -\mu_1 \\ \lambda_1 & \lambda_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \dfrac{\lambda_1\mu_2 + \lambda_2\mu_1}{2\lambda_1\mu_1} & \dfrac{\lambda_1\mu_2 - \lambda_2\mu_1}{2\lambda_1\mu_1} \\ \dfrac{\lambda_1\mu_2 - \lambda_2\mu_1}{2\lambda_1\mu_1} & \dfrac{\lambda_1\mu_2 + \lambda_2\mu_1}{2\lambda_1\mu_1} \end{pmatrix}.$$

We omit the proof of (2). □

Proposition 6.2 *Let* $H = \mathbb{R}^2 \ltimes_\psi \mathbb{R}^4$ *be a 1-step solvable Lie group such that the structure matrix* Λ_ψ *is* Λ *(resp.* Λ'*) in Lemma 6.1. Let* Γ *be the lattice of* H *given by* $\mathbb{Z}^2 \ltimes_\psi \mathbb{Z}^4$*. Then* $|A_\Gamma(Aut(\Gamma))| = 2$ *(resp.* $|A_\Gamma(Aut(\Gamma))| = 1$*) .*

Proof Let $\Lambda_\psi = \Lambda$, and let $\sigma = (12)(34)$. Then the homomorphism $\varphi \in \mathrm{Aut}(\Gamma)$ given by $\varphi_0 = T_\sigma$, $\varphi_1 = P_\sigma = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ clearly satisfies the relation $T_\sigma \Lambda = \Lambda P_\sigma$, and thus $A_\Gamma(\varphi) = \sigma$. Let $\Lambda_\psi = \Lambda'$. Then Corollary 5.3 and Lemma 6.1 show $|A_\Gamma(\mathrm{Aut}(\Gamma))| = 1$. □

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