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Equivariant cohomology determines hypertoric manifold

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ABSTRACT. In this article, we prove that if two equivariant cohomologies of hypertoric manifolds are isomorphic then these hypertoric manifolds are equivariantly diffeomorphic.

1. Introduction

In [BD00], Bielowsky and Dancer introduce the *hypertoric variety*[1] as the hyperKähler analogue of symplectic toric variety. The hypertoric variety is defined by the hyperKähler quotient of the standard torus action on $\mathbb{H}^m$ where $\mathbb{H}$ is the quaternionic space, and belongs to the class (M^{4n}, T^n), i.e., $4n$-dimensional space with n-dimensional torus action. This notion is different from the toric variety which belongs to the class (M^{2n}, T^n); however, there are some similar properties in toric and hypertoric varieties. For example, the hypertoric variety is determined by the combinatorial data of the hyperplane arrangement as well as the symplectic toric varieties are determined by the combinatorial data of polytopes. In the paper [M08], Masuda proved that the equivariant cohomology of the non-singular toric variety (toric manifold) determines the polytope; therefore, the equivariant cohomology also determines the equivariant types of toric manifolds. The aim of this article is to prove the hypertoric analogue of the Masuda's theorem. The following theorem for two hypertoric manifolds (M, T) and (M', T) are the main theorem of this article.

THEOREM 1.1. *If $H_T^*(M;\mathbb{Z}) \simeq H_T^*(M';\mathbb{Z})$, i.e., they are isomorphic up to $H^*(BT)$-algebra, then $(M,T) \equiv (M',T)$, i.e., they are T-equivariantly isometric.*

By Theorem 1.1, we can easily show the following corollary.

COROLLARY 1.2. If $H_T^*(M;\mathbb{Z}) \simeq H_T^*(M';\mathbb{Z})$, then $(M,T) \cong (M',T)$, i.e., they are T-equivariantly diffeomorphic.

The organization of this article is as follows. In Section 2, we recall the definition of the hypertoric manifolds and their basic properties. In Section 3, we give an outline of the proof of the main theorem. In the final section (Section 4), we point out the Nishimura's suggestion and give the problem for the case of the (hyper)toric orbifold.

2. The hypertoric variety and hyperplane arrangement

In this section, we recall the definition of the hypertoric variety and how to define hyperplane arrangement from the hypertoric variety (see [BD00], [Ko08] or [P08] for detail). We assume throughout this article that $\mathbb{R}$ is the real space, $\mathbb{C}$ is the complex space and $\mathbb{H}$ is the quaternionic space, i.e., $\mathbb{H} \simeq \mathbb{R}^4$ as the $\mathbb{R}$-vector space and basis $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ except 1 satisfy the following multiplicative relations:

$$\mathbf{ijk} = \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1.$$

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[1]The former terminology was *toric hyperKäler*.

2.1. The definition of the hypertoric variety. Assume $\mathbb{H}^m$ is the m-dimensional quaternionic vector space with the left $\mathbb{H}$-scaler product. The m-dimensional torus T^m acts on $\mathbb{H}^m$ via the left scaler product, i.e., we can denote it explicitly as follows:

$$\begin{array}{ccc} \mathbb{H}^m & \longrightarrow & \mathbb{H}^m \\ \cup\!\!\!\!\shortmid & & \cup\!\!\!\!\shortmid \\ z + \mathbf{j}w & \overset{t\cdot}{\longmapsto} & tz + \mathbf{j}t^{-1}w \end{array}$$

for $z,\ w \in \mathbb{C}^m$ and $t \in T^m$. By using this torus action on $\mathbb{H}^m \simeq \mathbb{C}^m \oplus \mathbb{C}^m$, we can regard $\mathbb{H}^m$ as $T^*\mathbb{C}^m$, i.e., the cotangent bundle of $\mathbb{C}^m$; or $\mathbb{C}^m \oplus \overline{\mathbb{C}^m}$, where $\overline{\mathbb{C}^m}$ is the orientation reversing space of $\mathbb{C}^m$.

Now we can define three complex structures on $\mathbb{H}^m$, we denote them by I, J and K. These complex structures determine not only the hyperKähler structure but also three symplectic structures on $\mathbb{H}^m$. We denote three symplectic structures by ω_I, ω_J and ω_K, respectively. Moreover, the holomorphic two form $\omega_{\mathbf{C}} = \omega_J + \sqrt{-1}\omega_K$ gives the holomorphic symplectic structure on $\mathbb{H}^m$ if we regard the complex structure on $\mathbb{H}^m$ as I. Then the above T^m-action on $\mathbb{H}^m$ preserves symplectic structures $\omega_I = \omega_{\mathbf{R}}$ and $\omega_{\mathbf{C}}$, and determines the hyperKähler moment map

$$\mu_{\mathbf{R}} \oplus \mu_{\mathbf{C}} : \mathbb{H}^m \longrightarrow (\mathfrak{t}^m)^* \oplus (\mathfrak{t}^m_{\mathbf{C}})^*$$

such that

$$\mu_{\mathbf{R}}(z, w) = \frac{1}{2}\sum_{i=1}^{m}(|z_i|^2 - |w_i|^2)\partial_i$$

and

$$\mu_{\mathbf{C}}(z, w) = 2\sqrt{-1}\sum_{i=1}^{m} z_i w_i \partial_i,$$

where $z = (z_1, \ldots, z_m) \in \mathbb{C}^m$ and $w = (w_1, \ldots, w_m) \in \overline{\mathbb{C}^m}$ and ∂_i $(i = 1,\ \ldots,\ m)$ is the basis in $(\mathfrak{t}^m)^*$ and $(\mathfrak{t}^m_{\mathbf{C}})^*$.

Put the subtorus $K \subset T^m$. Then there is the following sequence:

$$K \overset{\iota}{\longrightarrow} T^m \overset{\rho}{\longrightarrow} T^m/K \simeq T^n,$$

where ι is the natural embedding homomorphism, ρ is the projection to the cokernel of ι and $n = m - \dim K$. This sequence induces the following exact sequence of Lie algebras:

$$\{0\} \longrightarrow \mathfrak{k} \overset{\iota_*}{\longrightarrow} \mathfrak{t}^m \overset{\rho_*}{\longrightarrow} \mathfrak{t}^n \longrightarrow \{0\}.$$

By taking the dual of this sequence, we have the following exact sequence of the dual Lie algebras:

$$\{0\} \longrightarrow (\mathfrak{t}^n)^* \overset{\rho^*}{\longrightarrow} (\mathfrak{t}^m)^* \overset{\iota^*}{\longrightarrow} \mathfrak{k}^* \longrightarrow \{0\}. \tag{2.1}$$

By using ι^* and its complexification $\iota^*_{\mathbf{C}}$, we can define the hyperKähler moment map of K-action on $\mathbb{H}^m$ as follows:

$$\mu_{HK} : \mathbb{H}^m \xrightarrow{\mu_{\mathbf{R}} \oplus \mu_{\mathbf{C}}} (\mathfrak{t}^m)^* \oplus (\mathfrak{t}^m_{\mathbf{C}})^* \xrightarrow{\iota^* \oplus \iota^*_{\mathbf{C}}} \mathfrak{k}^* \oplus \mathfrak{k}^*_{\mathbf{C}}.$$

By the definition of μ_{HK}, we may take $(\alpha, 0) \in \mathfrak{k}^* \oplus \mathfrak{k}^*_{\mathbf{C}}$ for $\alpha \neq 0$ as the regular value of μ_{HK}. Hence, its inverse image $\mu_{HK}^{-1}(\alpha, 0)$ has the almost free K-action because μ_{HK} is the K-equivariant map and K acts on $\mathfrak{k}^* \oplus \mathfrak{k}^*_{\mathbf{C}}$ trivially. Therefore, if we take its quotient space $\mu_{HK}^{-1}(\alpha, 0)/K$ then this becomes an orbifold with dimension $4n$. Moreover, $\mu_{HK}^{-1}(\alpha, 0)/K$ has the $T^m/K = T^n$ action. We call $\mu_{HK}^{-1}(\alpha, 0)/K$ a *hypertoric variety*. If hypertoric variety is non-singular, then we call it a *hypertoric manifold*. The following proposition gives the criterion of the hypertoric manifold (see [**Ko00**, Proposition 2.2]).

PROPOSITION 2.1. The following two statements are equivalent.

(1) The action of K on $\mu_{HK}^{-1}(\alpha, 0)$ is free, i.e., $\mu_{HK}^{-1}(\alpha, 0)/K$ is a manifold.

(2) For any $J \subset \{1, \ldots, m\}$ such that $\{\iota^* u_j \mid j \in J\}$ forms a basis of $\mathfrak{k}^*$,

$$\mathfrak{t}^m_{\mathbb{Z}} = \mathfrak{k}_{\mathbb{Z}} \oplus \sum_{j \in J^c} \mathbb{Z}\partial_j \quad \text{as a } \mathbb{Z}\text{-module,}$$

where $\mathfrak{k} \subset \mathfrak{t}$ via ι_*, and $\mathfrak{k}_{\mathbb{Z}}$ and $\mathfrak{t}_{\mathbb{Z}}$ are their lattice subgroups.

Moreover, we have the following proposition (see [**P04**, Lemma 3.4]).

PROPOSITION 2.2. Let $\mu_{HK}^{-1}(\alpha, 0)/K$ and $\mu_{HK}^{-1}(\alpha', 0)/K$ be hypertoric manifolds defined by $K \subset T^m$ and two non-zero elements α, $\alpha' \in \mathfrak{k}^*$. Then

$$\mu_{HK}^{-1}(\alpha, 0)/K \cong \mu_{HK}^{-1}(\alpha', 0)/K$$

as T^n-equivariant diffeomorphism.

The following example is one of the standard examples in hypertoric manifolds.

EXAMPLE 2.3. Let Δ be the diagonal subgroup in T^{n+1}. Then the hypertoric variety induced by Δ is equivariantly diffeomorphic to $T^*\mathbb{C}P^n$ with the induced T^n action from the T^n-action on $\mathbb{C}P^n$.

2.2. The hyperplane arrangement. In this subsection, we introduce the hyperplane arrangement associated with hypertoric varieties.

First, we give the flow chart to define the hypertoric variety.

(1) Take a subgroup $K \subset T^m$.

$\Downarrow$

(2) Take a non-zero element $\alpha \in \mathfrak{k}^*$.

$\Downarrow$

(3) Take the hyperKähler quotient $\mu_{HK}^{-1}(\alpha, 0)/K$.

In the first step of this flow chart, we have the exact sequence (2.1):

$$\{0\} \longrightarrow (\mathfrak{t}^n)^* \xrightarrow{\rho^*} (\mathfrak{t}^m)^* \xrightarrow{\iota^*} \mathfrak{k}^* \longrightarrow \{0\}.$$

By the exactness of the above sequence, we can take the lift of α (in the second step of the above flow chart) as follows:

$$\begin{array}{ccc} (\mathfrak{t}^m)^* & \xrightarrow{\iota^*} & \mathfrak{k}^* \\ \cup\!\!\!\shortmid & & \cup\!\!\!\shortmid \\ \widetilde{\alpha} & \longmapsto & \alpha, \end{array}$$

i.e., $\iota^*(\widetilde{\alpha}) = \alpha$. Then we may define m hyperplanes in $(\mathfrak{t}^n)^*$ as follows:

$$H_i = \{x \in (\mathfrak{t}^n)^* \mid \langle \rho^*(x) + \widetilde{\alpha}, \mathbf{e}_i \rangle = 0\}$$

where $\mathbf{e}_i$ ($i = 1, \ldots, m$) is the basis of $\mathfrak{t}^m \simeq \mathbb{R}^m$. We call

$$\mathcal{H} = \{H_1, \ldots, H_m\}$$

a *hyperplane arrangement of* $\mu_{HK}^{-1}(\alpha, 0)/K$. Note that the combinatorial structure of $\mathcal{H}$ does not depend on the choice of the lift $\widetilde{\alpha}$; in fact, only the parallel translations of H_i's occur by changing lifts of α.

Now we show a hyperplane arrangement of $T^*\mathbb{C}P^n$.

EXAMPLE 2.4. Let $T^*\mathbb{C}P^n$ be the cotangent bundle over $\mathbb{C}P^n$. Due to Example 2.3, the subgroup $\Delta \simeq S^1$ defines $T^*\mathbb{C}P^n$. Therefore, we have the following exact sequence:

$$(\mathfrak{t}^n)^* \xrightarrow{\rho^*} (\mathfrak{t}^{n+1})^* \xrightarrow{\iota^*} \mathbb{R}$$

where $\mathbb{R}$ is the dual of Lie algebra of Δ. Because Δ is the diagonal subgroup, the representation ι^* is written as

$$\iota^*(x_1, \ldots, x_{n+1}) = x_1 + \cdots + x_{n+1} \in \mathbb{R}.$$

Because of the exactness, we may define the representation ρ^* as follows:

$$\rho^*(t_1, \ldots, t_n) = (t_1, \ldots, t_n, -t_1 - \cdots - t_n) \in (\mathfrak{t}^{n+1})^*$$

Take $\alpha = n+1 \in \mathbb{R}$. Then we can take its lift $\widetilde{\alpha}$ as $\widetilde{\alpha} = (1, \ldots, 1)$. By the definition of hyperplane arrangement of $\mu_{HK}^{-1}(\alpha, 0)/\Delta$, we have the following hyperplanes:

$$\begin{aligned} H_1 &= \{(t_1, \ldots, t_n) \in (\mathfrak{t}^n)^* \mid t_1 = -1\}; \\ &\vdots \\ H_n &= \{(t_1, \ldots, t_n) \in (\mathfrak{t}^n)^* \mid t_n = -1\}; \\ H_{n+1} &= \{(t_1, \ldots, t_n) \in (\mathfrak{t}^n)^* \mid t_1 + \cdots + t_n = 1\}. \end{aligned}$$

The following Figure 1 shows the case $n = 2$.

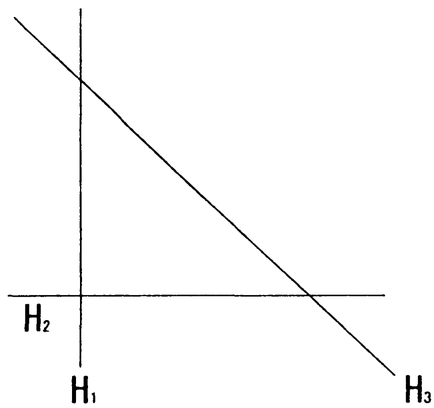

FIGURE 1. A hyperplane arrangement of $T^*\mathbb{C}P(2)$

By using the combinatorial data of $\mathcal{H}$, we can describe the ring structure of the equivariant cohomology (see Section 3.1) of hypertoric manifolds.

THEOREM 2.5 (Konno [**Ko99**]). *Let (M, T) be the hypertoric manifold and $\mathcal{H} = \{H_1, \ldots, H_m\}$ a hyperplane arrangement of M. Then its equivariant cohomology $H_T^*(M)$ is denoted as follows:*

$$H_T^*(M; \mathbb{Z}) \simeq \mathbb{Z}[\tau_1, \ldots, \tau_m]/\mathcal{I}$$

where $\deg \tau_i = 2$ *and $\mathcal{I}$ is the ideal in the polynomial ring $\mathbb{Z}[\tau_1, \ldots, \tau_m]$ generated by*

$$\prod_{i \in I} \tau_i \text{ for } \bigcap_{i \in I} H_i = \emptyset.$$

Here, I is the subset of $[m] = \{1, \ldots, m\}$.

The above generator τ_i $(i = 1, \ldots, m)$ corresponds with the line bundle of M which will be described as follows. Let $p_i : T^m \to T_i \simeq S^1$ be the natural projection to the i-th coordinate. Then we can define the $\mathbb{H}$-line bundle over $M = \mu_{HK}^{-1}(\alpha, 0)/K$ as follows:

$$\mu_{HK}^{-1}(\alpha, 0) \times_K \mathbb{H}_{p_i},$$

where $\mathbb{H}_{p_i}$ is the vector space which is isomorphic to $\mathbb{H}$ with the K-action via $K \subset T^m \xrightarrow{p_i} S^1$, and $\mu_{HK}^{-1}(\alpha, 0) \times_K \mathbb{H}_{p_i}$ is the orbit space $(\mu_{HK}^{-1}(\alpha, 0) \times \mathbb{H}_{p_i})/K$. Then this bundle splits into the following bundle:

$$\mu_{HK}^{-1}(\alpha, 0) \times_K \mathbb{H}_{p_i} \equiv \mu_{HK}^{-1}(\alpha, 0) \times_K (\mathbb{C}_{p_i} \oplus \overline{\mathbb{C}_{p_i}}).$$

Put $\mu_{HK}^{-1}(\alpha, 0) \times_K \mathbb{C}_{p_i} = \mathbb{L}_i$. The 1st chern class of $\mathbb{L}_i$ is the generator τ_i, i.e.,

$$c_1(\mathbb{L}_i) = \tau_i.$$

REMARK 2.6. For the toric manifold (M^{2n}, T^n) case, the S^1-invariant submanifold (characteristic submanifold) becomes a manifold with dimension $2n-2$; and the generators of $H^*_{T^n}(M^{2n})$ are expressed by taking the Poincaré dual of such invariant submanifold. On the other hand, for the hypertoric manifold (M^{4n}, T^n) case, such invariant submanifold becomes a manifold with dimension $4n-4$. Therefore, its Poincaré dual lives in $H^4_T(M)$. And this Poincaré dual becomes $c_2(\mathbb{L}_i \oplus \overline{\mathbb{L}_i}) = -\tau_i^2$ for $i = 1, \ldots, m$. This property of hypertoric manifolds is one of the different properties with toric manifolds.

EXAMPLE 2.7. For the cotangent bundle $T^*\mathbb{C}P^n$ over $\mathbb{C}P^n$ (see Example 2.3), by using Example 2.4 and Theorem 2.5, we have the following formula:

$$H^*_T(T^*\mathbb{C}P^n;\mathbb{Z}) \simeq \mathbb{Z}[\tau_1, \ldots, \tau_{n+1}]/\langle \tau_1 \cdots \tau_{n+1}\rangle$$

for the generators $\tau_i \in H^2_T(T^*\mathbb{C}P^n;\mathbb{Z})$.

3. Outline of the proof of the main theorem

Throughout of this section, we assume that (M, T) is a hypertoric manifold. The purpose of this section is to give the outline of the proof of Theorem 1.1 (see **[Ku2]** for detail).

3.1. Equivariant cohomology. In order to prove Theorem 1.1, first we recall the equivariant cohomology. Before we state its definition, we prepare some notations.

The symbol ET represents a universal space of T, i.e., ET satisfies the following two properties:

(1) ET is contractible;
(2) T acts on ET freely,

and BT represents its classifying space, i.e., $BT = ET/T$. Then the product space $ET \times M$ has the diagonal T-action, and we denote its orbit space $(ET \times M)/T$ by $ET \times_T M$. Because T acts freely on the ET factor in $ET \times M$, there is the following fibration:

$$M \xrightarrow{j} ET \times_T M \xrightarrow{\pi} BT. \tag{3.1}$$

We call the ordinary cohomology $H^*(ET \times_T M)$ the *equivariant cohomology of* (M, T) and denote it by $H^*_T(M)$. By using the fibration (3.1), we have the following homomorphism:

$$\pi^* : H^*(BT) \longrightarrow H^*_T(M).$$

Thus, we can regard $H^*_T(M)$ as not only the ring but also the $H^*(BT)$-algebra via π^*. Note that $H^*(BT;\mathcal{R})$ is isomorphic to the polynomial ring (see **[MT91]**), i.e.,

$$H^*(BT;\mathcal{R}) \simeq \mathcal{R}[x_1, \ldots, x_n]$$

for all coefficient ring $\mathcal{R}$, where $\dim T = n$ and $\deg x_i = 2$ $(i = 1, \ldots, n)$.

Due to the Konno's theorem (Theorem 2.5), we have the following exact sequence:

$$\{0\} \longrightarrow H^2(BT;\mathbb{Z}) \xrightarrow{\pi^*} H^2_T(M;\mathbb{Z}) \xrightarrow{j^*} H^2(M;\mathbb{Z}) \longrightarrow \{0\}. \tag{3.2}$$

Moreover, by using the similar argument in **[M08**, Proposition 2.2], the representation π^* in (3.2) can be expressed as the following proposition.

PROPOSITION 3.1. To each $i \in [m]$, there is a unique element $v_i \in H_2(BT;\mathbb{Z})$ such that

$$\pi^*(x) = \sum_{i=1}^{m} \langle x, v_i\rangle \tau_i$$

for any $x \in H^2(BT;\mathbb{Z})$, where $\langle , \rangle$ is the pairing of the cohomology and homology.

By taking each tensor product with $\mathbb{R}$ in the sequence (3.2), the sequence (3.2) induces the following exact sequence:

$$\{0\} \longrightarrow H^2(BT^n;\mathbb{R}) \xrightarrow{\pi^*_{\mathbb{R}}} H^2_T(M;\mathbb{R}) \xrightarrow{j^*_{\mathbb{R}}} H^2(M;\mathbb{R}) \longrightarrow \{0\}. \tag{3.3}$$

Because the above sequence (3.3) is the extension of the sequence (3.2), the representation $\pi_{\mathbb{R}}^*$ is also expressed as

$$\pi_{\mathbb{R}}^*(x) = \sum_{i=1}^{m} \langle x, v_i \rangle \tau_i \tag{3.4}$$

for a unique element $v_i \in H_2(BT; \mathbb{Z})$.

The key point of the proof is to construct the hyperplane arrangement in the equivariant cohomology $H_T^*(M; \mathbb{R})$. We will describe it in the next subsection.

3.2. Hyperplane arrangement in the equivariant cohomology. The goal of this section is to construct the hyperplane arrangement in $H^2(BT; \mathbb{R})$ by using the sequence (3.3). In order to construct it, we will prove the following key lemma.

LEMMA 3.2. *The following diagram is commute:*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (\mathfrak{t}^n)^* & \xrightarrow{\rho^*} & (\mathfrak{t}^m)^* & \xrightarrow{\iota^*} & \mathfrak{k}^* & \longrightarrow & 0 \\
 & & \downarrow J_n^* & & \downarrow J_m^* & & \downarrow J_K^* & & \\
0 & \longrightarrow & H^2(BT^n; \mathbb{R}) & \xrightarrow{\pi_{\mathbb{R}}^*} & H_T^2(M; \mathbb{R}) & \xrightarrow{j_{\mathbb{R}}^*} & H^2(M; \mathbb{R}) & \longrightarrow & 0
\end{array}$$

Here, the isomorphism J_n^* is defined by $H_2(BT; \mathbb{Z}) \simeq \mathrm{Hom}(S^1, T^n) \simeq \mathfrak{t}_{\mathbb{Z}}^n$, the isomorphism J_m^* is defined by $\mathbf{e}_i^* \mapsto \tau_i$ for $i = 1, \ldots, m$, and J_K^* is induced homomorphism from J_m^* and J_n^*. Now we may start to prove this lemma.

OUTLINE OF THE PROOF OF LEMMA 3.2. With the method similarly to show the equation (3.4), we have the following equations:

$$\rho^*(u) = \sum_{i=1}^{m} \langle u, \widetilde{v}_i \rangle \mathbf{e}_i,$$

for some unique element $\widetilde{v}_i \in \mathfrak{t}_{\mathbb{Z}}^n$ for $i = 1, \ldots, m$. Because of the equation (3.4), we have

$$\pi_{\mathbb{R}}^*(x) = \sum_{i=1}^{m} \langle x, v_i \rangle \tau_i.$$

Therefore, in order to have the commutativity of the first diagram, we need to prove that

$$\begin{array}{ccc}
\mathfrak{t}_{\mathbb{Z}}^n & \xrightarrow{(J_n)_*} & H_2(BT;\ \mathbb{Z}) \\
\Psi & & \Psi \\
\widetilde{v}_i & \longmapsto & v_i
\end{array}$$

for $i = 1, \ldots, m$. This fact is known by using the following fact: the image of two corresponding elements $f_{\widetilde{v}_i}, f_{v_i} \in \mathrm{Hom}(S^1, T^n)$ determine the same isotropy subgroup of characteristic submanifold M_i for $i = 1, \ldots, m$. Therefore, $(J_n)_* : \widetilde{v}_i \mapsto \pm v_i$. If $(J_n)_*(\widetilde{v}_i) = -v_i$, then we change τ_i to $-\tau_i$. Then we have $(J_n)_*(\widetilde{v}_i) = v_i$ and the commutativity of the first diagram.

For the second diagram, the isomorphism J_K^* is induced by the first diagram. Hence, the second diagram is commute. □

Now we may construct the hyperplane arrangement in $H^2(BT; \mathbb{R})$. First we recall the construction of the hyperplane arrangement in $(\mathfrak{t}^n)^*$. Because of the definition of the hypertoric manifolds, there is some non-zero element $\alpha \in \mathfrak{k}^*$ such that $M = \mu_{HK}^{-1}(\alpha, 0)/K$. According to the construction of the hyperplane in $(\mathfrak{t}^n)^*$, we can take its lift $\widetilde{\alpha} \in (\mathfrak{t}^m)^*$ such that this gives the hyperplane arrangement $\mathcal{H}_{\widetilde{\alpha}}$ in $(\mathfrak{t}^n)^*$.

Because J_K^* is isomorphism, we can take the non-zero element $\beta = J_K^*(\alpha) \in H^2(M)$. By taking $J_m^*(\widetilde{\alpha}) = \widetilde{\beta}$, we have $j_{\mathbb{R}}^*(\widetilde{\beta}) = \beta$. With the method similar to construct $\mathcal{H}_{\widetilde{\alpha}}$, we have the hyperplane $\mathcal{H}_{\widetilde{\beta}}$ in $H^2(BT; \mathbb{R})$. Then we have the following lemma.

LEMMA 3.3. *The isomorphism $J_n^* : (\mathfrak{t}^n)^* \to H^2(BT; \mathbb{R})$ preserves $\mathcal{H}_{\widetilde{\alpha}}$ to $\mathcal{H}_{\widetilde{\beta}}$.*

OUTLINE OF PROOF. A hyperplane $H_i \in \mathcal{H}_{\widetilde{\alpha}}$ is written as follows:

$$H_i = \{u \in (\mathfrak{t}^n)^* \mid \langle \rho^*(u) + \widetilde{\alpha}, \mathbf{e}_i \rangle = 0\}.$$

This hyperplane goes to the following set by using J_n^*.

$$H_i' = \{x = J_n^*(u) \in H^2(BT;\mathbb{R}) \mid \langle \rho^*(u) + \widetilde{\alpha}, \mathbf{e}_i \rangle = 0\}.$$

Then we have

$$\begin{aligned}
&\langle \rho^*(u) + \widetilde{\alpha}, \mathbf{e}_i \rangle = 0 \\
&\langle \rho^*(u) + \widetilde{\alpha}, (J_m)_*(\mu_i) \rangle = 0 \\
&\langle J_m^* \circ \rho^*(u) + J_m^*(\widetilde{\alpha}), \mu_i \rangle = 0 \\
&\langle \pi_{\mathbb{R}}^* \circ J_n^*(u) + \widetilde{\beta}, \mu_i \rangle = 0
\end{aligned}$$

where $\mu_i \in H_2^T(M;\mathbb{R})$ is the element which corresponds to $\tau_i \in H_T^2(M;\mathbb{R})$. Therefore, H_i' is the element in $\mathcal{H}_{\widetilde{\beta}}$. □

This means that we show the following theorem.

THEOREM 3.4. *Let (M,T) be the hypertoric manifold and $\mathcal{H}$ be a hyperplane arrangement of M. Then we can define the hyperplane arrangement $\mathcal{H}'$ in $H^2(BT;\mathbb{R})$ such that $J_n^* : (\mathfrak{t}^n)^* \to H^2(BT;\mathbb{R})$ preserves $\mathcal{H}$ to $\mathcal{H}'$.*

3.3. Outline of the proof of the main theorem. In this section, we prove Theorem 1.1. Let (M,T) and (M',T) be hypertoric manifolds. Assume $H_T^*(M;\mathbb{Z}) \simeq H_T^*(M;\mathbb{Z})$ as the $H^*(BT;\mathbb{Z})$-algebra, that is, there is the ring isomorphism $f_T : H_T^*(M;\mathbb{Z}) \to H_T^*(M;\mathbb{Z})$ such that

$$f_T(rx) = rf_T(x)$$

for all $x \in H_T^*(M;\mathbb{Z})$ and $r \in H^*(BT;\mathbb{Z})$. Note that we denote the induced ring isomorphism $H^*(M;\mathbb{Z}) \to H^*(M';\mathbb{Z})$ by f. Then we have the following commutative diagrams:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^2(BT^n;\mathbb{R}) & \xrightarrow{\pi_{\mathbb{R}}^*} & H_T^2(M;\mathbb{R}) & \xrightarrow{j_{\mathbb{R}}^*} & H^2(M;\mathbb{R}) & \longrightarrow & 0 \\
 & & \downarrow_{id} & & \downarrow_{f_T} & & \downarrow_{f} & & \\
0 & \longrightarrow & H^2(BT^n;\mathbb{R}) & \xrightarrow{\pi_{\mathbb{R}}^*} & H_T^2(M';\mathbb{R}) & \xrightarrow{j_{\mathbb{R}}^*} & H^2(M';\mathbb{R}) & \longrightarrow & 0
\end{array} \tag{3.5}$$

Let $\beta \in H^2(M;\mathbb{R})$ be a non-zero element and $\widetilde{\beta} \in H_T^2(M;\mathbb{R})$ be its lift. The goal of this section is to show that $\mathcal{H}_{\widetilde{\beta}}$ and $\mathcal{H}_{f_T(\widetilde{\beta})}$ are precisely same hyperplane arrangement. In order to show this fact, it is sufficient to prove the following proposition.

PROPOSITION 3.5. If f_T is an $H^*(BT;\mathbb{Z})$-algebra isomorphism between $H_T^*(M;\mathbb{Z})$ to $H_T^*(M';\mathbb{Z})$, then f_T preserves $\{\tau_1, \ldots, \tau_m\}$ to $\{\tau_1', \ldots, \tau_m'\}$ up to signs. In other wards, $\mathcal{H}_{\widetilde{\beta}}$ and $\mathcal{H}_{f_T(\widetilde{\beta})}$ are precisely same hyperplane arrangement up to coorientations of hyperplanes.

Let M^T be the set of T-fixed points in M. As is well known, it consists of finitely many points. For $\xi \in H_T^2(M;\mathbb{Z})$, we denote its restriction to $p \in M^T$ by $\xi|_p$ and define

$$Z(\xi) := \{p \in M^T \mid \xi|_p = 0\}.$$

LEMMA 3.6. *Express $\xi = \sum_{i=1}^m a_i\tau_i$ with integers a_i. If $a_i \neq 0$ for some i, then $Z(\xi) \subset Z(\tau_i)$. Moreover, if $a_i \neq 0$ and $a_j \neq 0$ for some different i and j, then $Z(\xi) \subset Z(\tau_i)$ and $Z(\xi) \neq Z(\tau_i)$.*

PROOF. Let $p \in M^T$. Recall $\mathbb{L}_i = \mu_{HK}(\alpha, 0) \times_K \mathbb{C}_{p_i}$ (see Section 2). This line bundle $\mathbb{L}_i$ satisfies that $\mathbb{L}_i \oplus \overline{\mathbb{L}_i}|_{M_i}$ is the normal bundle of M_i and $\mathbb{L}_i|_{M\setminus M_i}$ is the trivial bundle by the definition, where M_i is the characteristic submanifold. Since $\tau_i = c_1(\mathbb{L}_i)$, we have that $\tau_i|_p = 0$ if $p \notin M_i$. Moreover, if $p \in M_i$, then

$$\tau_i|_p = c_1(\mathbb{L}_i|_p) \in H_T^2(p;\mathbb{Z}) = H^2(BT;\mathbb{Z}).$$

This implies that

$$\tau_i|_p = 0 \quad \text{if and only if} \quad p \notin M_i \tag{3.6}$$

and that there are exactly n number of M_i's containing p and $\{\tau_i|_p \mid p \in M_i\}$ forms a basis of $H^2(BT;\mathbb{Z})$.

Suppose $p \in Z(\xi)$. Then $0 = \xi|_p = \sum_{i=1}^m a_i\tau_i|_p$ and it follows from the observation above that $\tau_i|_p = 0$ if $a_i \neq 0$. Therefore, we have $Z(\xi) \subset Z(\tau_i)$ (former statement).

If both a_i and a_j are non-zero, then $Z(\xi) \subset Z(\tau_i) \cap Z(\tau_j)$ by the former statement in the lemma. Therefore, it suffices to prove that $Z(\tau_i) \cap Z(\tau_j)$ is properly contained in $Z(\tau_i)$. Suppose that $Z(\tau_i) \cap Z(\tau_j) = Z(\tau_i)$. Then $Z(\tau_j) \supset Z(\tau_i)$, so $M_j^T \subset M_i^T$ by (3.6). This implies that $M_j = M_i$, a contradiction. □

Let $S = H^*(BT;\mathbb{Z})\backslash\{0\}$ and let $S^{-1}H_T^*(M;\mathbb{Z})$ denote the localized ring of $H_T^*(M;\mathbb{Z})$ by S, i.e.,

$$S^{-1}H_T^*(M;\mathbb{Z}) = \{\frac{r}{s} \mid r \in H_T^*(M;\mathbb{Z}),\ s \in S\}/\sim$$

where

$$\frac{r_1}{s_1} \sim \frac{r_2}{s_2} \iff (r_1s_2 - r_2s_1)t = 0 \ \text{ for some } t \in S.$$

Since $H^{odd}(M;\mathbb{Z}) = 0$, $H_T^*(M;\mathbb{Z})$ is free as a module over $H^*(BT;\mathbb{Z})$. Hence, the natural map

$$H_T^*(M;\mathbb{Z}) \longrightarrow S^{-1}H_T^*(M;\mathbb{Z}) \cong S^{-1}H_T^*(M^T;\mathbb{Z}) = \bigoplus_{p\in M^T} S^{-1}H_T^*(p;\mathbb{Z})$$

is injective, where the above isomorphism is induced from the inclusion map from M^T to M and is a consequence of the Localization Theorem in equivariant cohomology ([**H75**, p.40]). The annihilator

$$\mathrm{Ann}(\xi) := \{\eta \in S^{-1}H_T^*(M;\mathbb{Z}) \mid \eta\xi = 0\} \subset \bigoplus_{p\in M^T} S^{-1}H_T^*(p;\mathbb{Z})$$

of ξ is nothing but the sum of $S^{-1}H_T^*(p;\mathbb{Z})$ over p with $\xi|_p = 0$, because if $\xi|_p \neq 0$ then we have $\eta|_p = 0$. Therefore, it is a free $S^{-1}H^*(BT;\mathbb{Z})$ module of rank $|Z(\xi)|$. Since $\mathrm{Ann}(\xi)$ is defined using the algebra structure of $H_T^*(M;\mathbb{Z})$, $|Z(\xi)|$ is an invariant of ξ depending only on the algebra structure of $H_T^*(M;\mathbb{Z})$. We note that $|Z(\xi)|$ is invariant under an algebra isomorphism. We call $|Z(\xi)|$ the *zero-length* of ξ.

Now we may start to prove Proposition 3.5.

PROOF OF PROPOSITION 3.5. Let $\mathcal{T}_1$ be the set of τ_i's in $H_T^2(M)$ with largest zero-length, and let $\mathcal{T}_2$ be the set of τ_i's in $H_T^2(M)$ with second largest zero-length, and so on. Similarly we define $\mathcal{T}_1'$, $\mathcal{T}_2'$ and so on for τ_i''s in $H_T^2(M')$.

Let m_k (resp. m_k') be the zero-length of elements in $\mathcal{T}_k$ (resp. $\mathcal{T}_k'$). Since both f_T and f_T^{-1} preserve zero-length and are isomorphisms, $m_1 = m_1'$ and f_T maps $\mathcal{T}_1$ to $\mathcal{T}_1'$ bijectively up to sign by Lemma 3.6. Take an element τ_i from $\mathcal{T}_2$. Since $\mathcal{T}_1$ and $\mathcal{T}_1'$ are preserved under f_T and f_T^{-1}, $f_T(\tau_i)$ is not a linear combination of elements in $\mathcal{T}_1'$. This together with Lemma 3.6 means that $m_2 \leq m_2'$. The same argument for f_T^{-1} instead of f_T shows that $m_2' \leq m_2$, so that $m_2 = m_2'$. Again, this together with Lemma 3.6 implies that f maps $\mathcal{T}_2$ to $\mathcal{T}_2'$ bijectively up to sign. The lemma follows by repeating this argument. □

Now we have the following proposition (see [**P08**, Lemma 3.5]).

PROPOSITION 3.7. The hypertoric manifold (M,T) is independent, up to T^n-equivariant isometry, of the coorientation of the hyperplane arrangement $\mathcal{H}$ of M.

By using Proposition 3.5 and 3.7, we have Theorem 1.1.

4. Nishimura's suggestion and the future prospects

Several days later after the author's talk in RIMS, Nishimura suggested that the set of hypertoric manifolds up to T^n-equivariant diffeomorphism is the very special case in the set of hypertoric varieties. We will introduce about that in this final section, and give the problem for the case of all (hyper)toric varieties.

Let $+k_i$ (resp. $-k_i$) be the number of hyperplanes whose coorientation vector is e_i (resp. $-e_i$) in $\mathbb{R}^n$, where e_i (resp. $-e_i$) is the canonical basis such that the i-th coordinate 1 (resp. -1) and the other coordinates are 0 in $\mathbb{R}^n$. Let k_0 be the number of hyperplanes whose coorientation vector is $\sum_{i=1}^{n} v_i$, where $v_i = e_i$ or $-e_i$. Now we may define two types of hypertoric manifolds by using these hyperplanes as follows:

(1) $M_0(k_1, \ldots, k_n)$;
(2) $M_1(k_0, \pm k_1, \ldots, \pm k_n)$,

where the hyperplane of M_1 which corresponds to k_0 is determined by the sign of $\pm k_i$ for all $i = 1, \ldots, n$. Because of Proposition 2.1 and 2.2, we can denote all hypertoric manifolds up to T^n-equivariant diffeomorphism as one of the above manifolds (up to simultaneous sign changing). Therefore, the fact that the T^n-equivariant diffeomorphism types of hypertoric manifolds are determined by the $H^*(BT;\mathbb{Z})$-algebraic types of $H_T^*(M;\mathbb{Z})$ (see Corollary 1.2) is almost trivial. However, as we seen in Section 3, two hyperplane arrangements determined by $H_T^*(M;\mathbb{Z})$ and $H_T^*(M';\mathbb{Z})$ (they are algebraic isomorphic) are same not only their combinatorial types but also their position of hyperplanes. It follows that the $H^*(BT;\mathbb{Z})$-algebraic structure of $H_T^*(M;\mathbb{Z})$ can determine not only the T-equivariant diffeomorphism of (M,T) but also T-equivariant isometry of (M,T) (see Theorem 1.1).

According to the above comments by Nishimura, we know that the really important objects in hypertoric varieties are orbifolds. Fortunately, if we take the coefficient as the rational number $\mathbb{Q}$ then Theorem 2.5 is true for the hypertoric orbifolds. However, we can easy to construct two distinct hyperplanes (angles of intersections of hyperplanes are different) from two $H_T^*(M;\mathbb{Q})$ and $H_T^*(M';\mathbb{Q})$ (they are same up to $H^*(BT;\mathbb{Q})$-algebra). It follows that the orbifold analogue of Corollary 1.2 for $\mathbb{Q}$-coefficient does not hold. Moreover, to compute $H_T^*(M;\mathbb{Z})$ is very complicated for hypertoric orbifolds as well as toric orbifolds. Because of the singularity of orbifolds, there is the torsion element appears in $H_T^*(M;\mathbb{Z})$.

In order to consider the space with singularities, we can use the *intersection cohomology* $IH^*(M)$ or the *equivariant intersection cohomology* $IH_T^*(M)$. The intersection cohomology is considered as the "true" cohomology theory for the spaces with singularities. Actually, the *equivariantly formality* satisfies for IH_T^* but it does not satisfy for H_T^* if the space has singularities (see [**GKM98**], [**BP09**]). In this year (2009), Braden-Proudfoot determines the equivariant intersection cohomology of hypertoric varieties $IH_T^*(M)$ by using the functorial method in [**BP09**]. So, finally, we may ask the following problem as the orbifold analogue of Theorem 1.2 by using the equivariant intersection cohomology.

PROBLEM 4.1. *Does equivariant intersection cohomology determine (hyper)toric orbifold? In other words, if $IH_T^*(M) \simeq IH_T^*(M')$ satisfies for two (hyper)toric orbifolds then is there a T-equivariant map $f : M \longrightarrow M'$ such that f is a homeomorphism which preserves the singularities?*

If we have the affirmative answer in this problem, it corresponds to the generalizations of the main results in [**M08**] and [**Ku2**] to the orbifold case.

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