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Asymptotics of the free boundary of a Hele-Shaw flow with multiple point sources

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1 Introduction

In this paper we study the asymptotic behavior of a Hele-Shaw flow produced by the injection of fluid from a finite number of points at different injection speeds. We prove that, as time tends to infinity, the boundary of the fluid domain approaches the circle centered at the barycenter of the injection points with weights proportional to the injection rates. The distances from the barycenter to the boundary points are estimated both from above and below.

Hele-Shaw flows are fluid flows in an experimental device which consists of two closely-placed parallel plates. Since the gap between two plates is sufficiently narrow, one can regard them as two-dimensional flows. We consider a Hele-Shaw flow produced by the injection of incompressible viscous fluid into the device from multiple points. Let the fluid initially occupy a bounded domain $\Omega(0)\subset\mathbb{C}$ and $c_1,\ldots,c_l\in\Omega(0)$ be the injection points. From each point c_j, more fluid is injected at the rate $\alpha_j>0$ per unit time. The fluid domain at time $t>0$ is denoted by $\Omega(t)$ and its boundary by $\partial\Omega(t)$. We write n for the unit outer normal vector to $\partial\Omega(t)$. To formulate the mathematical problem, we now introduce a function T which is defined by $T(z):=\inf\{t\geq 0\,|\,z\in\overline{\Omega(t)}\}$ for each $z\in\mathbb{C}$, i.e., $T(z)$ denotes the first time when the boundary $\partial\Omega(t)$ touches z. Let $p=p(z,t)$ be the pressure of the fluid at position $z=x+iy\in\Omega(t)$ and time $t>0$, where $i=\sqrt{-1}$. By the theory of Hele-Shaw flows, p and T are assumed to satisfy the following equation and boundary conditions:

$$-\Delta p=\sum_{j=1}^{l}\alpha_j\delta_{c_j}\qquad \text{for } z\in\Omega(t),\ t>0; \tag{1.1}$$

$$p=0\qquad \text{for } z\in\partial\Omega(t),\ t>0; \tag{1.2}$$

$$\frac{\partial p}{\partial n}\cdot\frac{\partial T}{\partial n}=-1\qquad \text{for } z\in\partial\Omega(t),\ t>0, \tag{1.3}$$

where $\Delta:=\partial^2/\partial x^2+\partial^2/\partial y^2$ is the Laplacian in $\mathbb{R}^2$ and δ_c is the Dirac measure at c.

From (1.1) and (1.2), for each time $t > 0$ the function p can be represented by

$$p(z,t) = \sum_{j=1}^{l} \alpha_j G_{c_j,\Omega(t)}(z) \quad \text{for } z \in \Omega(t), \tag{1.4}$$

where $G_{c_j,\Omega(t)}$ is the Green's function of $\Omega(t)$ for the Laplacian under the homogeneous Dirichlet boundary condition with pole at c_j. By substituting (1.4) into (1.3), we obtain

$$\left(\sum_{j=1}^{l} \alpha_j \frac{\partial G_{c_j,\Omega(t)}}{\partial n}\right) \cdot \frac{\partial T}{\partial n} = -1 \quad \text{for } z \in \partial\Omega(t),\ t > 0. \tag{1.5}$$

Thus the Hele-Shaw problem is to find a monotone increasing family of domains $\{\Omega(t)\}_{t>0}$ with smooth boundaries such that the corresponding function T is smooth and satisfies (1.5). We call such a family $\{\Omega(t)\}_{t>0}$ a classical solution of the Hele-Shaw problem (see Sakai [11, Section 13]).

The problem has been investigated by many researchers with different methods. Elliott and Janovský [2] adopted a variational-inequality approach to the Hele-Shaw problem and proved the existence and uniqueness of global weak solutions. Sakai [11, 12] developed the theory of quadrature domains and applied it to the Hele-Shaw problem to obtain the existence and uniqueness of weak solutions and several properties. By this approach, Sakai [14] was able to obtain an estimate for the distances from a fixed point to the boundary points of $\Omega(t)$, which is stated as follows: Let $\Omega(0) \subset D(c,r)$ and $t\sum_{j=1}^{l} \alpha_j + m(\Omega(0)) \geq 4\pi r^2$, where $D(c,r)$ denotes the disk of radius r with center c and m two-dimensional Lebesgue measure. Then it holds that

$$\sqrt{\frac{t}{\pi}\sum_{j=1}^{l} \alpha_j + \frac{m(\Omega(0))}{\pi}} - r \leq |z-c| \leq \sqrt{\frac{t}{\pi}\sum_{j=1}^{l} \alpha_j + \frac{m(\Omega(0))}{\pi}} + r \tag{1.6}$$

for all $z \in \partial\Omega(t)$, $t > 0$. As a matter of fact, Sakai proved this result as a more general estimate on quadrature domains which we will define in the next section. By the estimate (1.6), we see that

$$\max_{z\in\partial\Omega(t)} |z-c| - \min_{z\in\partial\Omega(t)} |z-c| \leq 2r.$$

Another approach was taken by Escher and Simonett [3]. They converted the problem into a nonlinear evolution equation on a fixed domain and constructed a unique classical solution locally in time. Following this approach, in the case of a single injection point, Vondenhoff [16] recently proved the existence of a classical solution globally in time when the initial domain is sufficiently close to a disk centered at the injection point. Also he obtained detailed information on the asymptotic behavior of the Hele-Shaw flow by means of spectral analysis.

However, the method of spectral analysis [3, 16] seems to need nontrivial refinement in the case of multiple injection points to obtain the long time behavior of the solution, since it depends on the linearization of the evolution operator around an explicit solution. For this reason, here we consider the asymptotic behavior of the Hele-Shaw flow in the framework of weak solution in terms of quadrature domains. In the weak formulation we do not need to impose any restriction on the initial domain.

The aim of this paper is to present a more precise estimate for the asymptotic behavior of the interface of the Hele-Shaw flow in the case $l \geq 2$, in terms of the distances from a fixed point to the boundary points of $\Omega(t)$. To state our main theorem, we introduce the following important quantities:

$$w_l := \frac{\sum_{j=1}^{l} \alpha_j c_j}{\sum_{j=1}^{l} \alpha_j}, \tag{1.7}$$

$$r_0 := \inf \{ r \geq 0 \mid \Omega(0) \subset D(c, r) \text{ for some } c \in \mathbf{C} \}, \tag{1.8}$$

$$\Lambda := \sqrt{\frac{\pi}{\sum_{j=1}^{l} \alpha_j}} \cdot \min_{\sigma \in \mathfrak{S}_l} \left(\sum_{k=2}^{l} \frac{\alpha_{\sigma(k)} \sum_{j=1}^{k-1} \alpha_{\sigma(j)}}{\left(\sum_{j=1}^{k} \alpha_{\sigma(j)} \right)^2} \left| \frac{\sum_{j=1}^{k-1} \alpha_{\sigma(j)} c_{\sigma(j)}}{\sum_{j=1}^{k-1} \alpha_{\sigma(j)}} - c_{\sigma(k)} \right|^2 \right), \tag{1.9}$$

where the minimum is taken over the symmetric group $\mathfrak{S}_l$ on the finite set $\{1, \ldots, l\}$. Note that w_l is the barycenter of the injection points $c_1, \ldots, c_l$ with weights proportional to the respective injection rates $\alpha_1, \ldots, \alpha_l$, and r_0 is the smallest one among the radii of all disks containing $\Omega(0)$. The following is the main result in this paper.

Theorem 1.1. *Let $\Omega(0)$, c_j, α_j be as in the above setting and define w_l, r_0, Λ by (1.7), (1.8), (1.9), respectively. Suppose that $\{\Omega(t)\}_{t>0}$ is a classical solution of the Hele-Shaw problem. Then, there exist non-negative functions $\varepsilon_-(t)$, $\varepsilon_+(t)$ such that the inequality*

$$\sqrt{\frac{t}{\pi} \sum_{j=1}^{l} \alpha_j} - \varepsilon_-(t) \leq |z - w_l| \leq \sqrt{\frac{t}{\pi} \sum_{j=1}^{l} \alpha_j} + \varepsilon_+(t) \tag{1.10}$$

holds for all $z \in \partial\Omega(t)$, $t > 0$, and they have the following asymptotic behavior:

$$\varepsilon_-(t) = \Lambda t^{-1/2} + O\left(t^{-1}\right), \quad \varepsilon_+(t) = \left(\Lambda + \frac{r_0{}^2}{2} \sqrt{\frac{\pi}{\sum_{j=1}^{l} \alpha_j}} \right) t^{-1/2} + O\left(t^{-1}\right), \tag{1.11}$$

as $t \to \infty$.

By the estimates (1.10) and (1.11), we have

$$\max_{z \in \partial\Omega(t)} |z - w_l| - \min_{z \in \partial\Omega(t)} |z - w_l| \leq \varepsilon_+(t) + \varepsilon_-(t) = O\left(t^{-1/2}\right) \quad \text{as } t \to \infty.$$

Therefore, for the Hele-Shaw flow with multiple injection points, we see that the interface $\partial\Omega(t)$ of the fluid domain approaches the circle centered at the barycenter w_l as $t \to \infty$.

2 Weak formulation and Quadrature domains

In this section we observe that a classical solution of the Hele-Shaw problem satisfies an integral inequality for subharmonic functions. By the inequality, $\Omega(t)$ can be regarded as a quadrature domain of a positive measure, so that we will be concerned with the shape of quadrature domains in subsequent sections.

In the equation (1.5), the smoothness of the boundary $\partial\Omega(t)$ and of the function T are required. This is a difficulty in dealing with the equation (1.5). Following Sakai [11], we generalize the notion of classical solution so that it does not require any regularity of the boundary. Let $\{\Omega(t)\}_{t>0}$ be a classical solution of the Hele-Shaw problem. Then, for any subharmonic function s defined in $\Omega(t)$ which is integrable with respect to Lebesgue measure m, we see that

$$\begin{aligned}\int_{\Omega(t)\setminus\Omega(0)} s\,dm &= \int_0^t \int_{\partial\Omega(\tau)} s\cdot\frac{1}{\partial T/\partial n}\,d\sigma\,d\tau \\ &= \sum_{j=1}^{l}\alpha_j\int_0^t\int_{\partial\Omega(\tau)} s\cdot\left(-\frac{\partial G_{c_j,\Omega(\tau)}}{\partial n}\right)d\sigma\,d\tau \\ &\geq \sum_{j=1}^{l}\alpha_j\int_0^t s(c_j)\,d\tau = t\sum_{j=1}^{l}\alpha_j s(c_j).\end{aligned}$$

Therefore, any classical solution $\{\Omega(t)\}_{t>0}$ satisfies, for each $t>0$,

$$\int_{\Omega(0)} s\,dm + t\sum_{j=1}^{l}\alpha_j s(c_j) \leq \int_{\Omega(t)} s\,dm \tag{2.1}$$

for all integrable subharmonic functions s defined in $\Omega(t)$. In particular, since the constant functions $s=\pm1$ are integrable and subharmonic in $\Omega(t)$, we have

$$m\left(\Omega(t)\right) = t\sum_{j=1}^{l}\alpha_j + m\left(\Omega(0)\right).$$

In general, for a given finite (positive Borel) measure ν with compact support, a bounded open set Ω is called a quadrature domain of ν for subharmonic functions if $\nu(\mathbb{C}\setminus\Omega)=0$ and

$$\int s\,d\nu \leq \int_\Omega s\,dm$$

holds for all integrable subharmonic functions s defined in Ω. Quadrature domains for harmonic functions and for analytic functions are defined in the same way, but then we take equality instead of inequality in these definitions. From (2.1), for a classical solution $\{\Omega(t)\}_{t>0}$ of the Hele-Shaw problem, each $\Omega(t)$ can be interpreted as a quadrature domain of the measure $\chi_{\Omega(0)} + t\sum_{j=1}^{l}\alpha_j\delta_{c_j}$ for subharmonic functions, where $\chi_{\Omega(0)}$ denotes the characteristic function of $\Omega(0)$ and we regard it as the measure $\chi_{\Omega(0)}m$.

Here we summarize some elementary properties of quadrature domains (see Sakai [11, Section 1–3]):

(a) A quadrature domain for subharmonic functions is also one for harmonic functions. A quadrature domain for harmonic functions is also one for analytic functions.

(b) For any finite measure ν which is singular with respect to m, there exists a quadrature domain of ν for subharmonic functions. Let ν be a finite measure of the form $\nu = \chi_\Omega + \mu$, where Ω is a bounded domain and μ is a finite measure satisfying $\mu(\Omega) > 0$ and $\mu(\mathbb{C} \setminus \Omega) = 0$. Then there exists a quadrature domain of ν for subharmonic functions.

(c) If a measure ν satisfies one of the conditions in (b), then a quadrature domain of ν for subharmonic functions is uniquely determined up to a null set with respect to m. Moreover, the minimum quadrature domain $\Omega(\nu)$ exists, i.e., $\Omega(\nu) \subset \Omega$ holds for all quadrature domains Ω of ν for subharmonic functions.

(d) If measures ν_1 and ν_2 satisfy one of the conditions in (b) and $\nu_1 \leq \nu_2$, then $\Omega(\nu_1) \subset \Omega(\nu_2)$.

(e) For $\alpha > 0$ and $c \in \mathbb{C}$, a quadrature domain of the measure $\alpha\delta_c$ for subharmonic (also for harmonic and for analytic) functions is uniquely determined and is equal to $D(c, \sqrt{\alpha/\pi})$.

By the above properties of quadrature domains, we see that, for each $t > 0$, there exists the minimum quadrature domain of the measure $\chi_{\Omega(0)} + t\sum_{j=1}^{l} \alpha_j\delta_{c_j}$ for subharmonic functions. Sakai [11] defined a weak solution of the Hele-Shaw problem as the family of the minimum quadrature domains $\{\Omega(\chi_{\Omega(0)} + t\sum_{j=1}^{l} \alpha_j\delta_{c_j})\}_{t>0}$. There is another weak solution which is defined by using variational inequalities (see Gustafsson [4], and Elliott and Janovský [2]), but it was proved by Sakai [12] that these two weak solutions are equivalent. In the rest of the paper we work within the framework of quadrature domains and estimate them to prove Theorem 1.1. One of the advantages of dealing with quadrature domains is that we do not have to care about the smoothness of the free boundary $\partial\Omega(t)$ or topological changes of the domains $\{\Omega(t)\}_{t>0}$.

3 The Schwarz function

To prove Theorem 1.1, as a first step, we construct an explicit representation of the minimum quadrature domain of the measure $\pi(\alpha\delta_i + \beta\delta_{-i})$ for subharmonic functions, where α, $\beta > 0$. It will be given as a univalent rational map from the unit disk onto the quadrature domain, and we estimate the distances from the barycenter $(\alpha-\beta)i/(\alpha+\beta)$ to the boundary points of the quadrature domain. The construction of this rational map and its estimates will be discussed in the next section.

Let us introduce the notion of the Schwarz function and show relations between the Schwarz function and quadrature domains. We will see that the problem of finding a certain quadrature domain can be reduced to the construction of a domain with the corresponding Schwarz function.

The Schwarz function $S = S(z)$ of a curve Γ is defined as a holomorphic function on a neighborhood of Γ which satisfies

$$S(z) = \overline{z} \quad \text{for } z \in \Gamma,$$

where $\overline{z}$ is the complex conjugate of z. Note that the Schwarz function of Γ is uniquely determined for a given curve Γ by its analyticity.

Let us explain how the Schwarz function relates to quadrature domains (see Davis [1, Chapter 14] and Shapiro [15, Chapter 3]). Let $\Omega \subset \mathbf{C}$ be a bounded domain with smooth boundary and f a function holomorphic in a neighborhood of $\overline{\Omega}$, where $\overline{\Omega}$ denotes the closure of Ω. By the analyticity of f and Stokes' theorem, we see that

$$\int_{\Omega} f\,dm = \frac{1}{2i}\int_{\partial\Omega} f(z)\overline{z}\,dz,$$

where $\partial\Omega$ is positively oriented. Now assume that there exists the Schwarz function S of $\partial\Omega$ and it can be extended to a holomorphic function in $\Omega\setminus\{c_1,\ldots,c_l\}$ such that $c_j \in \Omega$ is a simple pole with residue $t\alpha_j/\pi$ for $j = 1, \ldots, l$. Then we have

$$\int_{\partial\Omega} f(z)\overline{z}\,dz = \int_{\partial\Omega} f(z)S(z)\,dz = 2it\sum_{j=1}^{l}\alpha_j f(c_j).$$

Thus,

$$\int_{\Omega} f\,dm = t\sum_{j=1}^{l}\alpha_j f(c_j) \tag{3.1}$$

holds for all holomorphic functions f defined in a neighborhood of $\overline{\Omega}$. From (3.1), Ω is expected to be a quadrature domain of the measure $t\sum_{j=1}^{l}\alpha_j\delta_{c_j}$ for subharmonic functions.

To obtain such a candidate for the quadrature domain, we therefore find a domain Ω such that the Schwarz function of $\partial\Omega$ has simple poles at $c_1, \ldots, c_l \in \Omega$ with respective residues $t\alpha_1/\pi, \ldots, t\alpha_l/\pi$. As we will see later, the domain Ω we found is in fact a quadrature domain of the measure $t\sum_{j=1}^{l}\alpha_j\delta_{c_j}$ for subharmonic functions. In order to find such a domain Ω, we assume that Ω can be represented as the image of the unit disk $D(0,1)$ by a rational function φ, i.e., $\Omega = \varphi(D(0,1))$, where φ is holomorphic and injective in a neighborhood of $\overline{D(0,1)}$. Then, the Schwarz function of $\partial\Omega$ is given by

$$S(z) := \overline{\varphi\left(1\Big/\overline{\varphi^{-1}(z)}\right)} \quad \text{for } z \text{ in a neighborhood of } \partial\Omega. \tag{3.2}$$

Moreover, if φ has only the simple poles at $w_1, \dots, w_l \in (\mathbb{C} \cup \{\infty\}) \setminus \overline{D(0,1)}$, then S can be meromorphically extended into Ω with simple poles at $\varphi(1/\overline{w_1}), \dots, \varphi(1/\overline{w_l})$. Hence, our task is to choose a rational function φ appropriately so that $\varphi(1/\overline{w_j}) = c_j$ and that the residue of the corresponding function S at c_j is $t\alpha_j/\pi$.

However, in general it is quite difficult to construct such a rational function φ. In particular, for $l \geq 3$, there are infinitely many possibilities of the disposition of $c_1, \dots, c_l$. In the case $l = 2$, as we will see later, by using translation, rotation and dilation we have only to consider the case where $c_1 = i$ and $c_2 = -i$.

4 Quadrature domains of two point masses

In this section, we deal with quadrature domains of the measure $\pi(\alpha\delta_i + \beta\delta_{-i})$. Note that the measure $\pi(\alpha\delta_i + \beta\delta_{-i})$ corresponds to a Hele-Shaw flow with two injection points. When the injection rates are the same, i.e., $\alpha = \beta$, Richardson [10] showed that the interface of the Hele-Shaw flow is a curve formed by inverting an ellipse with respect to the unit circle. Such a curve is called an elliptic lemniscate of Booth, which is named after the Reverend James Booth. Here we are also concerned with the case $\alpha \neq \beta$.

In Shapiro [15, Chapter 3], the rational function $\varphi_0(w) := 2Rw/(w^2 + R^2)$, where $R > 1$, is used to construct such a quadrature domain. To treat the case $\alpha \neq \beta$, we introduce a new rational function φ defined by

$$\varphi(w) = \varphi_{a,R,\eta}(w) := \frac{aR(w - i\eta)}{w^2 + R^2} + i\eta R. \tag{4.1}$$

Here, the function $\varphi = \varphi_{a,R,\eta}$ is parameterized by $a > 0$, $R > 1$ and $\eta \in \mathbb{R}$. For given α, $\beta > 0$, we choose a, R and η appropriately so that the domain $\Omega(a, R, \eta) := \varphi_{a,R,\eta}(D(0,1))$ is a quadrature domain of the measure $\pi(\alpha\delta_i + \beta\delta_{-i})$.

4.1 Construction of a rational map

Lemma 4.1. *Let α, β be positive numbers such that $\alpha + \beta$ is sufficiently large. Then, by taking some $a > 0$, $R > 1$ and $\eta \in \mathbb{R}$ and defining a rational function φ by* (4.1), *the Schwarz function S of $\partial\Omega(a, R, \eta)$, where $\Omega(a, R, \eta) := \varphi(D(0,1))$, is meromorphic in a neighborhood of $\overline{\Omega(a, R, \eta)}$ having only simple poles at i, $-i$ with residues α, β, respectively.*

We give the outline of the proof of Lemma 4.1. For the time being let us assume that φ is holomorphic and injective in the disk $D(0, 2)$. Then the Schwarz function S of the closed curve $\partial\Omega(a, R, \eta)$ is given by (3.2), as mentioned in the previous section. Hence our task will be to choose a, R and η appropriately so that the Schwarz function S has simple poles at $\pm i$ with residues $\rho_1 = \alpha$, $\rho_2 = \beta$, respectively.

Since φ has two simple poles at $\pm iR$, the function S is meromorphic in $\Omega(a, R, \eta)$ with only two simple poles at

$$\varphi\left(\frac{1}{\mp iR}\right) = \frac{iaR^2(\pm 1 - \eta R)}{R^4 - 1} + i\eta R.$$

Hence, we take $a > 0$ to be $(R^4 - 1)/R^2$ so that the poles of S are at $\pm i$. Moreover, some elementary computations show that

$$\rho_1 = \frac{1}{2R^3} \cdot \left(R^5 + R + 2\eta^2 R - \eta R^4 - \eta - 2\eta R^2\right),$$
$$\rho_2 = \frac{1}{2R^3} \cdot \left(R^5 + R + 2\eta^2 R + \eta R^4 + \eta + 2\eta R^2\right).$$

Therefore we need to solve the following system of algebraic equations for R and η:

$$\alpha + \beta = \rho_1 + \rho_2 = \frac{1}{R^2} \cdot (R^4 + 1 + 2\eta^2), \tag{4.2}$$
$$\beta - \alpha = \rho_2 - \rho_1 = \frac{\eta}{R^3} \cdot (R^2 + 1)^2. \tag{4.3}$$

In fact, we obtain a solution R and η with the following estimates:

$$R = \sqrt{\alpha + \beta} + O\left(\frac{1}{\sqrt{\alpha + \beta}}\right) \quad \text{as } \alpha + \beta \to \infty, \tag{4.4}$$
$$\eta = (\beta - \alpha)\left\{\frac{1}{\sqrt{\alpha + \beta}} + O\left((\alpha + \beta)^{-3/2}\right)\right\} \quad \text{as } \alpha + \beta \to \infty. \tag{4.5}$$

By taking a, R and η as above, we can show that φ is holomorphic and injective in the disk $D(0, 2)$ when $\alpha + \beta$ is sufficiently large. This completes the proof.

By virtue of Lemma 4.1 and (3.1), we see that the domain $\Omega(a, R, \eta)$ satisfies

$$\int_{\Omega(a,R,\eta)} f\, dm = \pi\alpha f(i) + \pi\beta f(-i)$$

for all holomorphic functions f defined in a neighborhood of $\overline{\Omega(a, R, \eta)}$. Now we confirm that the domain $\Omega(a, R, \eta)$ is indeed a quadrature domain for subharmonic functions.

Lemma 4.2. *Let α, β be positive numbers such that $\alpha + \beta$ is sufficiently large. Then, the domain $\Omega(a, R, \eta)$ constructed in Lemma 4.1 is a unique quadrature domain of the measure $\pi(\alpha\delta_i + \beta\delta_{-i})$ for subharmonic functions.*

To prove Lemma 4.2, we make use of the approximation theorem by Sakai [11, Lemma 7.3], which states that any integrable harmonic function h defined in $\Omega(a, R, \eta)$ can be approximated in $L^1(\Omega(a, R, \eta))$ by linear combinations of $\mathrm{Re}(1/(\cdot - \zeta))$, $\mathrm{Im}(1/(\cdot -$

ζ)) and $\log|\cdot-\zeta|$ with $\zeta \in \mathbb{C}\setminus\Omega(a,R,\eta)$. Combining the approximation theorem with the fact that $\Omega(a,R,\eta)$ is a smooth simply-connected domain, we see that

$$\int_{\Omega(a,R,\eta)} h\,dm = \pi\alpha h(i) + \pi\beta h(-i)$$

holds for all integrable harmonic functions h defined in $\Omega(a,R,\eta)$, i.e., $\Omega(a,R,\eta)$ is a quadrature domain of $\pi(\alpha\delta_i+\beta\delta_{-i})$ for harmonic functions.

To finish the proof, we have to show that $\Omega(a,R,\eta)$ is, in fact, a unique quadrature domain for subharmonic functions. We have already seen that there exists the minimum quadrature domain of $\pi(\alpha\delta_i+\beta\delta_{-i})$ for subharmonic functions. Let us denote it by Ω_0 and show that $\Omega(a,R,\eta)=\Omega_0$. Since Ω_0 is also a quadrature domain for harmonic functions, it suffices to show the uniqueness of quadrature domains of $\pi(\alpha\delta_i+\beta\delta_{-i})$ for harmonic functions. This uniqueness property is provided by an adaptation of maximum principles, due to Sakai [11] (see also Shapiro [15, Proposition 4.8 and Theorem 4.9] for the proof). Therefore, $\Omega(a,R,\eta)=\Omega_0$ and hence $\Omega(a,R,\eta)$ is a unique quadrature domain of $\pi(\alpha\delta_i+\beta\delta_{-i})$ for subharmonic functions.

4.2 Estimates of Quadrature domains

By Lemma 4.2, we see that a unique quadrature domain $\Omega(\alpha,\beta)$ of the measure $\pi(\alpha\delta_i+\beta\delta_{-i})$ for subharmonic functions is represented as $\Omega(\alpha,\beta)=\varphi_{a,R,\eta}(D(0,1))$. On the other hand, $a>0$, $R>1$ and $\eta\in\mathbb{R}$ are estimated in the proof of Lemma 4.1. In the following theorem, we proceed to the calculation of the distance from the point $(\alpha-\beta)i/(\alpha+\beta)$ to a boundary point $z\in\partial\Omega(\alpha,\beta)$, and obtain the asymptotics of the quadrature domain $\Omega(\alpha,\beta)$ when $(\alpha+\beta)\cdot\min\{\alpha,\beta\}\to\infty$. Note that

$$\sqrt{(\alpha+\beta)\cdot\min\{\alpha,\beta\}}\le\alpha+\beta.$$

Hence, $(\alpha+\beta)\cdot\min\{\alpha,\beta\}\to\infty$ implies $\alpha+\beta\to\infty$.

Theorem 4.3. *For α, $\beta>0$ such that $\alpha+\beta$ is sufficiently large, let $\Omega(\alpha,\beta)$ be a unique quadrature domain of the measure $\pi(\alpha\delta_i+\beta\delta_{-i})$ for subharmonic functions. Then, as $(\alpha+\beta)\cdot\min\{\alpha,\beta\}\to\infty$,*

$$\min_{z\in\partial\Omega(\alpha,\beta)}\left|z-\frac{\alpha-\beta}{\alpha+\beta}i\right| = \sqrt{\alpha+\beta-2}+\frac{(\alpha-\beta)^2}{(\alpha+\beta)^{5/2}}+(\alpha-\beta)^2\cdot O\left((\alpha+\beta)^{-7/2}\right), \tag{4.6}$$

$$\begin{aligned}\max_{z\in\partial\Omega(\alpha,\beta)}\left|z-\frac{\alpha-\beta}{\alpha+\beta}i\right| &= \sqrt{\alpha+\beta+2}-\frac{(\alpha-\beta)^2}{(\alpha+\beta)^{5/2}}+\frac{8\alpha\beta|\alpha-\beta|}{(\alpha+\beta)^4}\\ &\quad+(\alpha-\beta)^2\cdot O\left((\alpha+\beta)^{-7/2}\right)+(\alpha-\beta)\cdot O\left((\alpha+\beta)^{-3}\right).\end{aligned} \tag{4.7}$$

In view of the representation $\partial\Omega(\alpha,\beta) = \varphi(\partial D(0,1))$, where $\varphi = \varphi_{a,R,\eta}$ with $a > 0$, $R > 1$ and $\eta \in \mathbb{R}$ defined in the proof of Lemma 4.1, it is sufficient to calculate the minimum and the maximum of the function

$$d(w) := \left|\varphi(w) - \frac{\alpha-\beta}{\alpha+\beta}i\right| \quad \text{for } w \in \partial D(0,1),$$

which is the distance from the point $i(\alpha-\beta)/(\alpha+\beta)$ to a boundary point $\varphi(w) \in \partial\Omega(\alpha,\beta)$. By elementary calculations with the aid of the equations (4.2), (4.3) and the estimates (4.4), (4.5), we can prove the estimates (4.6) and (4.7).

By an argument similar to the proof of Theorem 4.3, we estimate the distance from the point $-i$ to a boundary point of the quadrature domain $\Omega(\alpha,\beta)$, and show that the quadrature domain $\Omega(\alpha,\beta)$ approaches the disk centered at $-i$ when $\alpha > 0$ is fixed and $\beta \to \infty$.

Theorem 4.4. *Suppose that α is a fixed positive number. For sufficiently large $\beta > 0$, let $\Omega(\alpha,\beta)$ be a unique quadrature domain of the measure $\pi(\alpha\delta_i + \beta\delta_{-i})$ for subharmonic functions. Then, as $\beta \to \infty$,*

$$\min_{z\in\partial\Omega(\alpha,\beta)} |z+i| = \sqrt{\beta} + \frac{\alpha}{2\sqrt{\beta}} - \frac{2\alpha}{\beta} + \left(4\alpha - \frac{\alpha^2}{8}\right)\beta^{-3/2} + O\left(\beta^{-2}\right),$$
$$\max_{z\in\partial\Omega(\alpha,\beta)} |z+i| = \sqrt{\beta} + \frac{\alpha}{2\sqrt{\beta}} + \frac{2\alpha}{\beta} + \left(4\alpha - \frac{\alpha^2}{8}\right)\beta^{-3/2} + O\left(\beta^{-2}\right).$$

5 Quadratue domains of multiple point masses

In this section, we apply Theorem 4.3 and give an estimate for quadrature domains of a linear combination of the Dirac measures. Then, Theorem 1.1 is obtained as a consequence of the estimate combined with Theorem 4.4, as we will see in the next section. In what follows, we write $\Omega(\nu)$ for the minimum quadrature domain of the measure ν for subharmonic functions. First we state the following two lemmas without proof.

Lemma 5.1. *Let β_1, β_2 and κ be positive numbers and $c_1, c_2 \in \mathbb{C}$. Then,*

$$\Omega\left(\kappa^2\beta_1\delta_{\kappa c_1} + \kappa^2\beta_2\delta_{\kappa c_2}\right) = \{\kappa z \in \mathbb{C} \mid z \in \Omega\left(\beta_1\delta_{c_1} + \beta_2\delta_{c_2}\right)\}$$

holds.

By Lemma 5.1 and simple arguments concerning translation or rotation, we see that the estimates for any quadrature domains of two point masses are reduced to the estimates given by Theorem 4.3 and Theorem 4.4.

The next lemma shows that minimum quadrature domains possesses the semi-group property. Gustafsson and Sakai [5] have already proved this property for more general measures, but it is established for saturated (or maximum) quadrature domains (see [5, Theorem 2.2] for the detail). On the other hand, Sakai [11] proved the property for the minimum quadrature domains. We improve the result [11, Proposition 3.10] as follows.

Lemma 5.2. *Let μ, ν be finite measures with compact support such that there exist the bounded minimum quadrature domains $\Omega(\mu)$, $\Omega(\mu+\nu)$ and $\Omega(\chi_{\Omega(\mu)}+\nu)$ of the measures μ, $\mu+\nu$ and $\chi_{\Omega(\mu)}+\nu$ for subharmonic functions, respectively. In addition, we assume that ν is of the form $\nu = f + \sum_{j=1}^{l}\alpha_j\delta_{c_j}$, where $f \in L^\infty(\mathbb{C})$, $\alpha_j > 0$ and $c_j \in \mathbb{C}$. Then it holds that*

$$\Omega(\mu+\nu) = \Omega\left(\chi_{\Omega(\mu)}+\nu\right).$$

With the above lemmas and Theorem 4.3, we give the following estimate for the distances from the barycenter w_l defined by (1.7) to the boundary points of quadrature domains of a linear combination of the Dirac measures.

Theorem 5.3. *Let $\alpha_1, \ldots, \alpha_l$ be positive numbers and $c_1, \ldots, c_l \in \mathbb{C}$ with $l \geq 2$, and define $w_1, \ldots, w_l$ by (1.7). Then, there exists a non-negative function $\varepsilon_l(t)$ such that for any quadrature domain $\Omega_\Delta(t)$ of the measure $t\sum_{j=1}^{l}\alpha_j\delta_{c_j}$ for subharmonic functions the inequality*

$$\sqrt{\frac{t}{\pi}\sum_{j=1}^{l}\alpha_j} - \varepsilon_l(t) \leq |z-w_l| \leq \sqrt{\frac{t}{\pi}\sum_{j=1}^{l}\alpha_j} + \varepsilon_l(t)$$

holds for all $z \in \partial\Omega_\Delta(t)$, $t>0$, and it has the following asymptotic behavior:

$$\varepsilon_l(t) = \sqrt{\frac{\pi}{\sum_{j=1}^{l}\alpha_j}}\left(\sum_{k=2}^{l}\frac{\alpha_k\sum_{j=1}^{k-1}\alpha_j}{\left(\sum_{j=1}^{k}\alpha_j\right)^2}|w_{k-1}-c_k|^2\right)t^{-1/2} + O\left(t^{-1}\right) \quad \text{as } t \to \infty.$$

The proof is based on induction on l. The case $l=2$ can be proved by combining Theorem 4.3 and Lemma 5.1. In the case $l \geq 3$, we apply Lemma 5.2 and reduce the estimate for $\Omega_\Delta(t)$ to the one for $\Omega(t\sum_{j=1}^{l-1}\alpha_j\delta_{c_j})$. To see this, we note that

$$\begin{aligned}\Omega_\Delta(t) &= \Omega\left(\chi_{\Omega\left(t\sum_{j=1}^{l-1}\alpha_j\delta_{c_j}\right)} + t\alpha_l\delta_{c_l}\right) \\ &\subset \Omega\left(\chi_{D\left(w_{l-1},\, \sqrt{t\pi^{-1}\sum_{j=1}^{l-1}\alpha_j}+\varepsilon_{l-1}(t)\right)} + t\alpha_l\delta_{c_l}\right) = \Omega\left(t\hat{\alpha}(t)\delta_{w_{l-1}} + t\alpha_l\delta_{c_l}\right)\end{aligned}$$

with an appropriate number $\hat{\alpha}(t)$. Then, by the result of the case $l=2$ we can estimate the domain $\Omega(t\hat{\alpha}(t)\delta_{w_{l-1}} + t\alpha_l\delta_{c_l})$ and finally we obtain the desired estimate from above. The estimate from below is similarly obtained.

6 Proof of Theorem 1.1

We are now in a position to prove Theorem 1.1 by combining Theorem 5.3 with Theorem 4.4.

It is sufficient to prove the estimate (1.10) for the minimum quadrature domain $\Omega(t) = \Omega(\chi_{\Omega(0)} + t\sum_{j=1}^{l} \alpha_j \delta_{c_j})$. Let $\varepsilon_-(t) := \varepsilon_l(t)$, where $\varepsilon_l(t)$ is obtained by Theorem 5.3. Then, by the inclusion relation $\Omega(t\sum_{j=1}^{l} \alpha_j\delta_{c_j}) \subset \Omega(t)$ we see that

$$\sqrt{\frac{t}{\pi}\sum_{j=1}^{l}\alpha_j} - \varepsilon_-(t) \leq |z - w_l| \quad \text{for all } z \in \partial\Omega(t),\ t > 0. \tag{6.1}$$

Next we estimate $|z - w_l|$ from above. In the definition (1.8) of r_0, we can take minimum instead of infimum. To show this, we take sequences $\{c^{(k)}\}$, $\{r^{(k)}\}$ such that $r^{(k)} \to r_0$ and $\Omega(0) \subset D(c^{(k)}, r^{(k)})$. Then, $\{c^{(k)}\}$ is bounded since $\{r^{(k)}\}$ is bounded. Hence, there exists a subsequence $\{c^{(k_p)}\}$ of $\{c^{(k)}\}$ which converges to a point $c_0 \in \mathbb{C}$. Therefore,

$$\Omega(0) \subset \bigcap_{p=1}^{\infty} D\left(c^{(k_p)}, r^{(k_p)}\right) \subset \bigcap_{p=1}^{\infty} D\left(c_0, r^{(k_p)} + \left|c^{(k_p)} - c_0\right|\right) \subset \overline{D(c_0, r_0)},$$

so that $\Omega(0) \subset D(c_0, r_0)$. By Lemma 5.2 and Theorem 5.3, observe that

$$\begin{aligned}\Omega(t) &\subset \Omega\left(\chi_{D(c_0,r_0)} + t\sum_{j=1}^{l}\alpha_j\delta_{c_j}\right) = \Omega\left(\chi_{D(c_0,r_0)} + \chi_{\Omega(t\sum_{j=1}^{l}\alpha_j\delta_{c_j})}\right) \\ &\subset \Omega\left(\chi_{D(c_0,r_0)} + \chi_{D(w_l,R(t))}\right) = \Omega\left(\pi {r_0}^2\delta_{c_0} + \pi R(t)^2\delta_{w_l}\right),\end{aligned} \tag{6.2}$$

where $R(t) := \sqrt{t\pi^{-1}\sum_{j=1}^{l}\alpha_j} + \varepsilon_l(t)$. Therefore, applying Theorem 4.4 to the right hand side of (6.2) yields the estimate for $|z - w_l|$ from above as follows:

$$|z - w_l| \leq \sqrt{\frac{t}{\pi}\sum_{j=1}^{l}\alpha_j} + \varepsilon_+(t) \quad \text{for all } z \in \partial\Omega(t),\ t > 0. \tag{6.3}$$

Here $\varepsilon_+(t)$ satisfies

$$\varepsilon_+(t) = \sqrt{\frac{t}{\pi}\sum_{j=1}^{l}\alpha_j} + \sqrt{\frac{\pi}{\sum_{j=1}^{l}\alpha_j}}\left(\sum_{k=2}^{l}\frac{\alpha_k\sum_{j=1}^{k-1}\alpha_j}{\left(\sum_{j=1}^{k}\alpha_j\right)^2}|w_{k-1} - c_k|^2 + \frac{{r_0}^2}{2}\right)\frac{1}{\sqrt{t}} + O\left(t^{-1}\right)$$

as $t \to \infty$.

For any $\sigma \in \mathfrak{S}_l$, the above argument to obtain the estimates (6.1) and (6.3) can be applied to the case where j is replaced by $\sigma(j)$. Therefore, by taking the minima of

$\varepsilon_-(t)$ and $\varepsilon_+(t)$ over $\sigma \in \mathfrak{S}_l$ and writing them as $\varepsilon_-(t)$ and $\varepsilon_+(t)$ again, we obtain the desired estimate (1.10) with (1.11), since $\Omega(t)$ is irrelevant to the way of numbering the injection points. This completes the proof.

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