数理解析研究所講究録
第 1671 巻 2009 年 78-90

Concentration and Stability of standing waves of nonlinear Schrödinger equation with inhomogeneous nonlinearity

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1 Introduction

In this paper, we consider the following nonlinear Schrödinger equation with inhomogeneous nonlinearity.

$$iu_t = -\Delta u - b(x)|u|^{p-1}u, \ (x,t) \in \mathbb{R}^{N+1}, \tag{1.1}$$

where $N \geq 1$, $u : \mathbb{R}^{N+1} \to \mathbb{C}$ is an unknown function, $p \in (1, 1+4/N)$ and $b(x)$ is a smooth function which satisfies

$$0 < \inf_{x\in\mathbb{R}^N} b(x) = \lim_{|x|\to\infty} b(x) \leq \sup_{x\in\mathbb{R}^N} b(x) = 1.$$

A standing wave is a solution of equation (1.1) with the form $u(x,t) = e^{i\omega t}\phi(x)$. In this case, ϕ satisfies the following partial differential equation.

$$-\Delta\phi + \omega\phi - b(x)|\phi|^{p-1}\phi = 0, \ x \in \mathbb{R}^N. \tag{1.2}$$

The flow of equation (1.1) conserves the L^2-norm and the following functional, which we call the energy.

$$\mathcal{E}(u) := \frac{1}{2}\int_{\mathbb{R}} |\nabla u|^2 \, dx - \frac{1}{p+1}\int_{\mathbb{R}} b(x)|u|^{p+1} \, dx.$$

The well-posedness of equation (1.1) is well known. See for example [2].

Proposition 1. *For every $u_0 \in H^1(\mathbb{R}^N)$, there exists a solution $u \in C(\mathbb{R}; H^1(\mathbb{R}^N))$ of* (1.1) *such that*

(a) $u(x,0) = u_0(x)$ for $x \in \mathbb{R}^N$.

(b) $\mathcal{E}(u(t)) = \mathcal{E}(u_0)$, $||u(t)||_{L^2} = ||u_0||_{L^2}$ for $t \in \mathbb{R}$.

Equation (1.1) appears in various regions of physics such as nonlinear optics, plasma physics and Bose-Einstein condensation (BEC). In the context of BEC, the ground states are considered to describe the physical properties of Bose gas in low temperature. Here, a ground state is a standing wave which minimizes the energy functional $\mathcal{E}$ under the constraint of the L^2-norm. Note that by the

Lagrange multiplier method, the ground state satisfies (1.2) for some $\omega \in \mathbb{R}$. For the case $b \equiv 1$, it is known that the ground state is unique ([5, 9]), and if $1 < p < 1 + 4/N$, it is stable ([1]). For the case $b = |x|^{-\beta}$, $\beta \in (0, 2)$, $N \geq 3$, it is proved that the ground state is stable ([4]).

We now state prepare the notations.

Definition 1. *Set*

$$\mathcal{G}_\alpha := \left\{ u \in H^1(\mathbb{R}^N) \mid ||u||_{L^2} = \alpha, \ \mathcal{E}(u) = E_\alpha \right\},$$

where

$$E_\alpha = \inf \left\{ \mathcal{E}(v) \mid v \in H^1(\mathbb{R}^N), \ ||v||_{L^2} = \alpha \right\}.$$

In this paper, we call the elements of $\mathcal{G}_\alpha$, the ground states.

For the case, b is a radial symmetric function, we can consider a minimizer of $\mathcal{E}$ under the constraint $u \in H^1_r(\mathbb{R}^N)$ and $\|u\|_{L^2} = \alpha$, where

$$H^1_r(\mathbb{R}^N) := \left\{ u \in H^1(\mathbb{R}^N) \mid u \text{ is radially symmetric} \right\}.$$

Definition 2. *Set*

$$\mathcal{G}_{\alpha,r} := \left\{ u \in H^1_r(\mathbb{R}^N) \mid ||u||_{L^2} = \alpha, \ \mathcal{E}(u) = E_{\alpha,r} \right\},$$

where

$$E_{\alpha,r} = \inf \left\{ \mathcal{E}(v) \mid v \in H^1_r(\mathbb{R}^N), \ ||v||_{L^2} = \alpha \right\}.$$

In this paper, we call the elements of $\mathcal{G}_{\alpha,r}$, the radial minimizers.

We investigate the concentration and stability of ground states and radial minimizers.

Definition 3. *We say that the $\mathcal{G}_\alpha$ (resp. $\mathcal{G}_{\alpha,r}$) concentrates for sufficiently large α if the elements of $\mathcal{G}_\alpha$ ($\mathcal{G}_{\alpha,r}$) satisfies the following: For arbitrary $\varepsilon > 0$, there exists an $\alpha_\varepsilon > 0$ such that for every $\alpha > \alpha_\varepsilon$ and every $\phi \in \mathcal{G}_\alpha$ ($\mathcal{G}_{\alpha,r}$), there exists $y_{\alpha,\phi} \in \mathbb{R}^N$ such that*

$$\int_{|x - y_{\alpha,\phi}| > \varepsilon} |\phi|^2 \, dx < \varepsilon \int_{\mathbb{R}^N} |\phi|^2 \, dx = \varepsilon \alpha^2.$$

We call $y_{\alpha,\phi} \in \mathbb{R}^N$, the concentration center.

Definition 4. *We say that $\mathcal{G}_\alpha$ (resp. $\mathcal{G}_{\alpha,r}$) is stable if the following property is satisfied: For arbitrary $\varepsilon > 0$, there exists an $\delta_\varepsilon > 0$ such that for every $u_0 \in H^1$ with*

$$\inf_{v \in \mathcal{G}_\alpha (\mathcal{G}_{\alpha,r})} \|u_0 - v\|_{H^1} < \delta_\varepsilon,$$

the solution of equation (1.1) with $u(0) = u_0$ satisfies

$$\sup_{t>0} \inf_{v \in \mathcal{G}_\alpha (\mathcal{G}_{\alpha,r})} \|u(t) - v\|_{H^1} < \varepsilon.$$

If $\mathcal{G}_\alpha$ ($\mathcal{G}_{\alpha,r}$) is not stable, we say $\mathcal{G}_\alpha$ ($\mathcal{G}_{\alpha,r}$) is unstable.

The existence, concentration and stability of $\mathcal{G}_\alpha$ is well known.

Proposition 2. *For $\alpha > 0$, $\mathcal{G}_\alpha \neq \emptyset$ and $\mathcal{G}_\alpha$ is stable. Further, $\mathcal{G}_\alpha$ concentrates for sufficiently large α and the concentration center converges to some maximum point of b.*

Remark 1. For the existence of ground states, see Proposition 8.3.6 of [2]. For the stability result, see [1] and for the concentration result, see [13].

The purpose of this paper is to investigate the stability and concentration for the elements of $\mathcal{G}_{\alpha,r}$.

Proposition 3. *Let b radially symmetric. Then for $\alpha > 0$, we have $\mathcal{G}_\alpha \neq \emptyset$.*

Remark 2. Proposition 3 can be proved as the existence of ground states.

We first study the case $N \geq 2$.

Theorem 1. *Let $N \geq 2$. Then $\mathcal{G}_\alpha$ concentrates for sufficiently large α and the concentration center is 0. Further, if 0 is a nondegenerate minimum point (resp. maximum point), then for sufficiently large $\alpha > 0$, $\mathcal{G}_{\alpha,r}$ is stable (unstable).*

Thus, we see that the concentration result holds but the stability result some times fails for the case of radial mininizers. For the case $N = 1$, we see that also the concentration result sometimes fails.

Theorem 2. *Let $N = 1$.*

(i) *If $1 \geq b(0) > 2^{-(p-1)/2}$, then $\mathcal{G}_{\alpha,r}$ concentrates for sufficiently large α and the concentration center is 0. Further, if 0 is a nondegenerate minimum point (resp. maximum point), then for sufficiently large $\alpha > 0$, $\mathcal{G}_{\alpha,r}$ is stable (unstable).*

(ii) *If $0 < b(0) < 2^{-(p-1)/2}$, then $\mathcal{G}_\alpha$ is unstable and does not concentrate for sufficiently large α.*

The plan of this paper is as follows. In section 2, we rescale our problem. In section 3 and 4, we prove Theorems 1 and 2 respectively. The proof of the concentration result of Theorem 1 relies on the radial lemma due to Strauss [14]. For the proof of the concentration result of Theorem 2, we use the concentration compactness method due to Lions [10, 11]. For the stability result, we use the abstract theory developed by Grillakis, Shatah and Strauss [7] and for the instability result, we use the result of [12] for $N \geq 2$ and [6] for the case $N = 1$.

2 Preliminary

We rescale our problem. Take $\phi \in H_r^1(\mathbb{R}^N)$ with $||\phi||_{L^2} = 1$. Then, we have

$$\mathcal{E}(\alpha\phi) = \alpha^2 \left(\frac{1}{2} \int_{\mathbb{R}} |\nabla\phi|^2 \, dx - \frac{\alpha^{p-1}}{p+1} \int_{\mathbb{R}} b(x)|\phi|^{p+1} \, dx \right).$$

Next, set $\phi_\alpha(x) = \alpha^{AN/2}\phi(\alpha^A x)$, where $A = \frac{2(p-1)}{4-N(p-1)}$. Then, we have

$$\mathcal{E}_\alpha(\alpha\phi_\alpha) = \alpha^{2+2A} \left(\frac{1}{2} \int_{\mathbb{R}} |\nabla\phi|^2 \, dx - \frac{1}{p+1} \int_{\mathbb{R}} b(\alpha^{-A}x)|\phi|^{p+1} \, dx \right).$$

Therefore, we set

$$I_\alpha(\phi) := \frac{1}{2}\int_{\mathbb{R}} |\nabla\phi|^2\,dx - \frac{1}{p+1}\int_{\mathbb{R}} b(\alpha^{-A}x)|\phi|^{p+1}\,dx,$$

and

$$\mathcal{I}_{\alpha,r} := \left\{\phi \in H_r^1(\mathbb{R}^N) \mid ||\phi||_{L^2} = 1,\ I_\alpha(\phi) = \inf_{||\psi||_{L^2}=1, \psi\in H_r^1(\mathbb{R}^N)} I_\alpha(\psi)\right\}.$$

Thus, we obtain

$$\mathcal{G}_{\alpha,r} = \left\{\alpha\phi_\alpha \mid \phi \in \mathcal{I}_{\alpha,r}\right\}.$$

We also define the following functional:

$$I_{\infty,b}(\phi) := \frac{1}{2}\int_{\mathbb{R}} |\nabla\phi|^2\,dx - \frac{b}{p+1}\int_{\mathbb{R}} |\phi|^{p+1}\,dx.$$

Then, it is well known that there exists a unique positive radial minimizer $\psi_{b,\beta}$ of $I_{\infty,b}$ under the constraint $||\phi||_{L^2}^2 = \beta$. That is

$$\begin{aligned}\mathcal{I}_{\infty,r,b,\beta} &:= \left\{\phi \in H_r^1(\mathbb{R}^N) \mid ||\phi||_{L^2}^2 = \beta,\ I_{\infty,b}(\phi) = \inf_{||\varphi||_{L^2}^2=\beta, \varphi\in H_r^1} I_{\infty,b}(\varphi)\right\} \\ &= \left\{c\psi_{b,\beta} \mid c \in \mathbb{C},\ |c| = 1\right\}.\end{aligned}$$

Remark 3. The uniqueness of positive radial solution of equation (1.2) in the case $b(x) \equiv b > 0$ is proved by Kwong [9]. Further, letting $\phi_{b,\omega}$ be the unique positive radial solution of equation (1.2) in the case $b(x) \equiv b > 0$, we have $\phi_{b,\omega}(x) = \omega^{\frac{1}{p-1}}\phi_b(\omega^{1/2}x)$, where ϕ_b is the unique positive radial solution of

$$-\Delta\phi_b + \phi_b - b\phi_b^p = 0,\ x \in \mathbb{R}^N.$$

Therefore, we see $\frac{d}{d\omega}||\phi_{b,\omega}||_{L^2}^2 > 0$ for $1 < p < 1 + 4/N$. This implies the uniqueness of the radial minimizer up to constant phase.

We now calculate the value

$$I_{\infty,b}(\psi_{b,\beta}) = \inf\left\{I_{\infty,b}(\phi) \mid \phi \in H_r^1(\mathbb{R}^N),\ ||\phi||_{L^2}^2 = \beta\right\}.$$

Lemma 1. *Let*

$$J_\infty = \inf_{||u||_{L^2}=1} I_{\infty,1}(u) = I_{\infty,1}(\psi_{1,1}) < 0.$$

Then

$$I_{\infty,b}(\psi_{b,\beta}) = b^{\frac{2A}{p-1}}\beta^{1+A}J_\infty,$$

where $A = \frac{2(p-1)}{4-N(p-1)} > 0$.

Proof.

$$\begin{aligned} I_{\infty,b}(\psi_{b,\beta}) &= \inf_{\phi\in H_r^1, ||\phi||_{L^2}^2=\beta} \left(\frac{1}{2}\int_{\mathbb{R}} |\nabla\phi|^2\,dx - \frac{b}{p+1}\int_{\mathbb{R}} |\phi|^{p+1}\,dx\right) \\ &= \beta \inf_{||\phi||_{L^2}=1} \left(\frac{1}{2}\int_{\mathbb{R}} |\nabla\phi|^2\,dx - \frac{b\beta^{\frac{p-1}{2}}}{p+1}\int_{\mathbb{R}} |\phi|^{p+1}\,dx\right). \end{aligned}$$

Now, setting $\phi(x) = (b\beta^{\frac{p-1}{2}})^{\frac{N}{4-N(p-1)}}\varphi((b\beta^{\frac{p-1}{2}})^{\frac{2}{4-N(p-1)}}x)$, we have $||\varphi||_{L^2} = ||\phi||_{L^2}$ and

$$\frac{1}{2}\int_{\mathbb{R}} |\nabla\phi|^2\,dx - \frac{b\beta^{\frac{p-1}{2}}}{p+1}\int_{\mathbb{R}} |\phi|^{p+1}\,dx = (b\beta^{\frac{p-1}{2}})^{\frac{4}{4-N(p-1)}} I_{\infty,1}(\varphi).$$

Thus, we have

$$\inf_{||u||_{L^2}^2=\beta} I_{\infty,b}(u) = b^{\frac{4}{4-N(p-1)}}\beta^{1+\frac{2(p-1)}{4-N(p-1)}} J_\infty.$$

□

We further prepare some compactness results. To show the concentration result of Theorem 1, we use the following lemma due to Strauss [14].

Lemma 2. *Let $N \geq 2$. Then every $u \in H_r^1$ is almost everywhere equal to a function U, continuous for $x \neq 0$, such that*

$$|U(x)| \leq C_N |x|^{-\frac{(N-1)}{2}} ||u||_{H^1} \text{ for } |x| \geq C_N,$$

where C_N depends only on the dimension N.

To show Theorem 2, we prepare two concentration compactness lemmas, which are slight modifications of the concentration conpactness lemma due to Lions [10, 11] (See also [2]).

Lemma 3. *Let $\{u_n\} \subset H_r^1(\mathbb{R})$ be such that*

$$||u_n||_{L^2} = 1, \ \sup_{n\in\mathbb{N}} ||\nabla u_n||_{L^2} < \infty. \tag{2.1}$$

Set

$$\widetilde{\mu} = \lim_{t\to\infty} \liminf_{n\to\infty} \int_{|x|<t} |u_n|^2\,dx. \tag{2.2}$$

Then, there exists a subsequence $\{u_{n_k}\}$ that satisfies the following.

(i) *If $\widetilde{\mu} = 1$, then there exists a $u \in H_r^1(\mathbb{R})$ such that $u_{n_k} \to u$ in $L^p(\mathbb{R})$ for $p \in [2, \infty]$.*

(ii) *There exist* $\{v_k\}$, $\{w_{k,+}\}$ *and* $\{w_{k,-}\} \subset H^1_r(\mathbb{R})$ *such that*

$$\begin{aligned} \mathrm{supp} w_{k,+} \subset (0,\infty),\ \mathrm{supp} w_{k,-} \subset (-\infty,0), \\ \mathrm{supp} v_k \cap \mathrm{supp} w_{k,+} = \mathrm{supp} v_k \cap \mathrm{supp} w_{k,-} = \emptyset, \\ |v_k| + |w_{k,+}| + |w_{k,-}| \le |u_{n_k}| \\ ||v_k||_{H^1} + ||w_{k,+}||_{H^1} + ||w_{k,-}||_{H^1} \le ||u_{n_k}||_{H^1} \\ ||v_k||^2_{L^2} \to \widetilde{\mu},\ ||w_{k,+}||^2_{L^2} \to \frac{1}{2}(1-\widetilde{\mu})\ ||w_{k,-}||^2_{L^2} \to \frac{1}{2}(1-\widetilde{\mu}) \\ \liminf_{k\to\infty} \int (|\nabla u_{n_k}|^2 - |\nabla v_k|^2 - |\nabla w_{k,+}|^2 - |\nabla w_{k,-}|^2) \ge 0 \\ \left| \int (|u_{n_k}|^p - |v_k|^p - |w_{k,+}|^p - |w_{k,-}|^p) \right| \to 0,\ (k\to\infty) \end{aligned}$$

for all $2 \le p \le \infty$.

Lemma 4. *Let* $\{u_n\}$ *satisfy (2.1). Define* $\widetilde{\mu}$ *as (2.2) and*

$$\mu := \lim_{t\to\infty} \liminf_{n\to\infty} \sup_{y\in\mathbb{R}} \int_{|x-y|<t} |u_n|^2\, dx.$$

Assume $\widetilde{\mu} = 0$. *Then,* $0 \le \mu \le 1/2$ *and there exists a subsequence* $\{u_{n_k}\}$ *that satisfies the following.*

(i) *If* $\mu = 1/2$, *then there exist* $u \in H^1_r(\mathbb{R})$ *and* $y_k > 0$ *such that* $y_k \to \infty$ *and* $\chi_+(\cdot - y_k)u_{n_k}(\cdot - y_k) \to u$ *in* $L^p(\mathbb{R})$ *for* $p \in [2,\infty]$, *where* $\chi_+ \in C^\infty$ *satisfies* $0 \le \chi_+ \le 1$, $\mathrm{supp}\chi_+ \subset [0,\infty)$ *and* $\chi_+(x) = 1$ *for* $x \ge 1$.

(ii) *If* $\mu = 0$, *then* $u_{n_k} \to 0$ *in* L^p *for* $p \in (2,\infty]$.

(iii) *There exist* $\{v_{k,+}\}$, $\{v_{k,-}\}$, $\{w_{k,+}\}$ *and* $\{w_{k,-}\} \subset H^1_r(\mathbb{R})$ *such that*

$$\begin{aligned} \mathrm{supp} v_{k,+},\ \mathrm{supp} w_{k,+} \subset (0,\infty),\ \mathrm{supp} v_{k,-},\ \mathrm{supp} w_{k,-} \subset (-\infty,0), \\ \mathrm{supp} v_{k,+} \cap \mathrm{supp} w_{k,+} = \mathrm{supp} v_{k,-} \cap \mathrm{supp} w_{k,-} = \emptyset, \\ |v_{k,+}| + |v_{k,-}| + |w_{k,+}| + |w_{k,-}| \le |u_{n_k}| \\ ||v_{k,+}||_{H^1} + ||v_{k,-}||_{H^1} + ||w_{k,+}||_{H^1} + ||w_{k,-}||_{H^1} \le ||u_{n_k}||_{H^1} \\ ||v_{k,\pm}||^2_{L^2} \to \widetilde{\mu},\ ||w_{k,\pm}||^2_{L^2} \to \frac{1}{2}(1-\widetilde{\mu}) \\ \liminf_{k\to\infty} \int (|\nabla u_{n_k}|^2 - |\nabla v_{k,+}|^2 - |\nabla v_{k,-}|^2 - |\nabla w_{k,+}|^2 - |\nabla w_{k,-}|^2) \ge 0 \\ \left| \int (|u_{n_k}|^p - |v_{k,+}|^p - |v_{k,-}|^p - |w_{k,+}|^p - |w_{k,-}|^p) \right| \to 0,\ (k\to\infty) \end{aligned}$$

for all $2 \le p \le \infty$.

3 Proof of Theorem 1

Let $\psi_{b(0),1} \in \mathcal{I}_{\infty,r,b(0),1}$, $\psi_{b(0),1} > 0$. We show that the rescaled radial minimizers converge to $\psi_{b(0),1}$.

Lemma 5. *Let $N \geq 2$ and b radially symmetric. Let $\phi_n \in \mathcal{I}_{\alpha_n}$ with $\phi_n > 0$, where $\alpha_n \to \infty$ as $n \to \infty$. Then $\{\phi_n\}$ is a minimizing sequence of $I_{\infty,b(0)}$ under the constraint $||\phi||_{L^2} = 1$. In particular, $\phi_n \to \psi_{b(0),1}$.*

Proof. We calculate $I_{\infty,b(0)}(\phi_n)$.

$$\begin{aligned}
I_{\infty,b(0)}(\phi_n) &= \frac{1}{2}\int_{\mathbb{R}} |\nabla\phi_n|^2\,dx - \frac{b(0)}{p+1}\int_{\mathbb{R}} |\phi_n|^{p+1}\,dx \\
&\leq I_{\alpha_n}(\phi_n) + \frac{1}{p+1}\int_{\mathbb{R}} |b(\alpha_n^{-A}x) - b(0)||\phi_n|^{p+1}\,dx \\
&\leq I_{\alpha_n}(\psi_{b(0),1}) + \frac{1}{p+1}\int_{\mathbb{R}} |b(\alpha_n^{-A}x) - b(0)||\phi_n|^{p+1}\,dx \\
&\leq I_{\infty,b(0)}(\psi_{b(0),1}) \\
&\quad + \frac{1}{p+1}\int_{\mathbb{R}} |b(\alpha_n^{-A}x) - b(0)|\left(|\phi_n|^{p+1} + |\psi_{b(0),1}|^{p+1}\right)\,dx,
\end{aligned}$$

where $A = \frac{2(p-1)}{4-N(p-1)} > 0$. Now, for arbitrary $\varepsilon > 0$, there exists $R_\varepsilon > 0$ such that $|b(x) - b(0)| < \varepsilon$ for $|x| < R_\varepsilon$. Therefore, we have

$$\int_{\mathbb{R}} |b(\alpha_n^{-A}x) - b(0)||\psi_{b(0),1}|^{p+1}\,dx \leq \varepsilon\int_{\mathbb{R}} |\psi_{b(0),1}|^{p+1}\,dx + \int_{|x|>\alpha_n^A R_\varepsilon} |\psi_{b(0),1}|^{p+1}.$$

Further, for sufficiently large α_n, we have

$$\frac{1}{p+1}\int_{|x|>\alpha_n^A R_\varepsilon} |\psi_{b(0),1}|^{p+1} \leq \varepsilon.$$

Thus, we obtain

$$\frac{1}{p+1}\int_{\mathbb{R}} |b(\alpha_n^{-A}x) - b(0)||\psi_{b(0),1}|^{p+1}\,dx \to 0,\ n \to \infty$$

Next, using the fact that ϕ_n is a radial minimizer of I_{α_n}, we see that $I_{\alpha_n}(\phi_n) < 0$. Combining this to Gagliardo-Nirenberg's inequality, we see that $||\phi_n||_{H^1}$ is uniformly bounded. Therefore, by Lemma 2, we have

$$\begin{aligned}
\int_{\mathbb{R}} |b(\alpha_n^{-A}x) - b(0)||\phi_n|^{p+1}\,dx &\leq \varepsilon\int_{\mathbb{R}} |\phi_n|^{p+1}\,dx + C\int_{|x|>\alpha_n^A R_\varepsilon} |x|^{-\frac{(N-1)(p+1)}{2}}\,dx \\
&\leq C\varepsilon + C(\alpha_n R_\varepsilon)^{1-\frac{(N-1)(p+1)}{2}}.
\end{aligned}$$

Since $1 - \frac{(N-1)(p+1)}{2} < 0$, we see that

$$\frac{1}{p+1}\int_{\mathbb{R}} |b(\alpha_n^{-A}x) - b(0)||\phi_n|^{p+1}\,dx \to 0,\ n \to \infty.$$

Therefore, we see that ϕ_n is a minimizing sequence of $I_{\infty,b(0)}$. □

We now prove Theorem 1.

Proof of Theorem 1. Let $u_n \in \mathcal{G}_{\alpha_n}$ with $\alpha_n \to \infty$ as $n \to \infty$. Then, there exists $\phi_n \in \mathcal{I}_{\alpha_n}$ such that

$$\alpha_n^{1+NA/2}\phi_n(\alpha^A x) = u_n(x),$$

where $A = \frac{2(p-1)}{4-N(p-1)}$. We compute $\left(\int_{|x|>\varepsilon} |u_n|^2\, dx\right)^{1/2}$.

$$\begin{aligned}\left(\int_{|x|>\varepsilon} |u_n|^2\, dx\right)^{\frac{1}{2}} &= \alpha \left(\int_{|x|>\varepsilon\alpha^A} |\phi_n|^2\, dx\right)^{\frac{1}{2}} \\ &\leq \alpha \left(\int_{\mathbb{R}^N} |\psi_{b(0),1} - \phi_n|^2\, dx\right)^{\frac{1}{2}} + \alpha \left(\int_{|x|>\varepsilon\alpha^A} |\psi_{b(0),1}|^2\, dx\right)^{\frac{1}{2}},\end{aligned}$$

where $\psi_{b(0),1}$ is the positive radial minimizer of $I_{\infty,b(0)}$ under the constraint $||\phi||_{L^2} = 1$. Since $\phi_n \to \psi_{b(0),1}$ in $L^2(\mathbb{R}^N)$, we have

$$\left(\int_{\mathbb{R}} |\psi - \phi_n|^2\, dx\right)^{1/2} < \frac{1}{2}\varepsilon^{1/2}$$

for sufficiently large n. Further, since $\frac{2(p-1)}{4-N(p-1)} > 0$ and $\alpha_n \to \infty$, we see

$$\left(\int_{|x|>\varepsilon\alpha_n^A} |\psi|^2\, dx\right)^{1/2} < \frac{1}{2}\varepsilon^{1/2},$$

for sufficiently large n. Therefore, we have the concentration result.

We next show the stability for the case 0 is a nondegenerate minimum point of b. For this case, modifying the result of Grossi [8], we see that for large $\alpha > 0$, the radial minimizer is unique up to constant phase. Therefore, the radial minimizer must correspond to the ground state with a penalizer which was introduced in [3]. Since this ground state is stable, we see that also the radial minimizer is stable.

Finally for the proof of the instability for the case 0 is a nondegenerate maximum point of b, see [12]. □

4 Proof of Theorem 2

Proof of Theorem 2 (i). Let $u_n \in \mathcal{G}_{\alpha_n}$ with $u_n > 0$ and $\alpha_n \to \infty$ as $n \to \infty$. Then, there exists $\phi_n \in \mathcal{I}_{\alpha_n}$ such that

$$\alpha_n^{1+\frac{p-1}{5-p}} \phi_n(\alpha_n^{\frac{2(p-1)}{5-p}} x) = u_n(x).$$

Since $||\phi_n||_{L^2} = 1$ and $\sup_n ||\nabla\phi_n||_{L^2} < \infty$, we apply Lemma 3 to $\{\phi_n\}$. As in the proof of Theorem 1, if we can show $\phi_n \to \psi_{b(0),1}$ in $H^1(\mathbb{R})$, where $\psi_{b(0),1}$ is the minimizer of $I_{\infty,b(0)} = I_\infty$ under the constraint $||u||_{L^2} = 1$, we have the concentration result. Further, the stability and instability follows as in the proof of Theorem 1.

Therefore, it suffices to show $\phi_n \to \psi_{b(0),1}$ in $H^1(\mathbb{R})$. Now, let

$$\tilde{\mu} = \lim_{t\to\infty} \liminf_{n\to\infty} \int_{|x|<t} |\phi_n|^2\, dx.$$

We show $\widetilde{\mu} = 1$. If $\widetilde{\mu} = 1$, we have a subsequence ϕ_{n_k} and ϕ such that $\phi_{n_k} \to \phi$ in L^p, $p \in [2, \infty]$. Thus, we have $||\phi||_{L^2} = 1$ and

$$\begin{aligned}
I_{\infty,b(0)}(\phi) &\leq \liminf_{k\to\infty} I_{\infty,b(0)}(\phi_{n_k}) \\
&\leq \liminf_{k\to\infty} \left(I_{\alpha_{n_k}}(\phi_{n_k}) + \int |b(0) - b(\alpha_n^{-A}x)||\phi_{n_k}|^{p+1}\, dx \right) \\
&\leq \liminf_{k\to\infty} \left(I_{\alpha_n}(\psi_{b(0),1}) + \int |b(0) - b(\alpha_n^{-A}x)||\phi_{n_k}|^{p+1}\, dx \right) \\
&\leq I_{\infty,b(0)}(\psi_{b(0)}) + \liminf_{k\to\infty} \int |b(0) - b(\alpha_n^{-A}x)|(|\phi_{n_k}|^{p+1} + |\psi_{b(0),1}|^{p+1})\, dx \\
&= I_{\infty,b(0)}(\psi_{b(0),1}),
\end{aligned}$$

where $A = \frac{2(p-1)}{5-p}$. Therefore, from the definition of $\psi_{b(0),1}$ and the uniqueness of the radial minimizer of $I_{\infty,b(0)}$, we see that $\phi_{n_k} \to \psi_{b(0),1}$ in $H^1(\mathbb{R})$.

Therefore, it suffices to show $\widetilde{\mu} = 1$. Suppose $\widetilde{\mu} < 1$. Then, by Lemma 3, there exist $\{v_k\}$, $\{w_{k,+}\}$ and $\{w_{k,-}\}$ and we have

$$\liminf_{k\to\infty} I_{\alpha_{n_k}}(\phi_{n_k}) \geq \limsup_{k\to\infty} \left(I_{\alpha_{n_k}}(v_k) + I_{\infty,1}(w_{k,+}) + I_{\infty,1}(w_{k,-}) \right).$$

We claim $\limsup_{k\to\infty} I_{\alpha_{n_k}}(v_k) \geq b(0)^{\frac{2A}{p-1}} \widetilde{\mu}^{1+A} J_\infty$, where $A = \frac{2(p-1)}{5-p}$. Indeed, since $|v_k| \leq |u_{n_k}|$, taking arbitrary $\varepsilon > 0$, there exists $R_\varepsilon > 0$ such that

$$\limsup_{k\to\infty} \int_{|x|>R_\varepsilon} |v_k|^2\, dx < \varepsilon.$$

Therefore, we have

$$\begin{aligned}
\limsup_{k\to\infty} I_{\alpha_{n_k}}(v_k) \geq \limsup_{k\to\infty} \Bigg(& I_{\infty,b(0)}(v_k) - \int_{|x|<R_\varepsilon} |b(\alpha_{n_k}^{-A}x) - b(0)||v_k|^{p+1}\, dx \\
& - \int_{|x|>R_\varepsilon} |b(\alpha^{-A}x) - b(0)||v_k|^{p+1}\, dx \Bigg).
\end{aligned}$$

Furhter, since $\sup_k ||v_k||_{L^\infty} \leq C_1 \sup_k ||v_k||_{H^1} \leq C_2 \sup_k ||\phi_{n_k}||_{H^1} < C_3$, we have

$$\int_{|x|>R_\varepsilon} |b(\alpha^{-A}x) - b(0)||v_k|^{p+1}\, dx \leq 2C_3^{p-1}\varepsilon,$$

and taking α_{n_k} sufficiently large, we have

$$\int_{|x|<R_\varepsilon} |b(\alpha_{n_k}^{-A}x) - b(0)||v_k|^{p+1}\, dx \leq \varepsilon \int_{\mathbb{R}} |v_k|^{p+1}\, dx \leq C\varepsilon.$$

Therefore, we obtain

$$\liminf_{k\to\infty} I_{\alpha_{n_k}}(\phi_{n_k}) \geq \left(b(0)^{\frac{2A}{p-1}} \widetilde{\mu}^{1+A} + 2\left(\frac{1-\widetilde{\mu}}{2}\right)^{1+A} \right) J_\infty.$$

On the other hand, we have

$$\liminf_{k\to\infty} I_{\alpha_{n_k}}(\phi_{n_k}) \leq \liminf_{k\to\infty} I_{\alpha_{n_k}}(\psi_{b(0)}) = b(0)^{\frac{2A}{p-1}} J_\infty.$$

Therefore, since $J_\infty < 0$, we have

$$b(0)^{\frac{2A}{p-1}} \leq \frac{(1-\widetilde{\mu})^{1+A}}{2^A(1-\widetilde{\mu}^{1+A})}.$$

Since, $\frac{(1-\widetilde{\mu})^{1+A}}{1-\widetilde{\mu}^{1+A}} \leq 1$, we obtain

$$b(0) \leq 2^{-\frac{p-1}{2}}.$$

However we have assumed $b(0) > 2^{-\frac{p-1}{2}}$. Therefore, this is a contradiction. □

Proof of Theorem 2 (ii). Let $u_n \in \mathcal{G}_{\alpha_n}$ with $u_n > 0$ and $\alpha_n \to \infty$ as $n \to \infty$. Then, there exists $\phi_n \in \mathcal{I}_{\alpha_n,r}$ such that

$$\alpha_n^{1+\frac{p-1}{5-p}} \phi_n(\alpha_n^{\frac{2(p-1)}{5-p}} x) = u_n(x).$$

We first show $\widetilde{\mu} = 0$. Suppose $\widetilde{\mu} > 0$. Then as in the proof of Theorem 2 (i), using Lemma 3, we have

$$\lim_{k\to\infty} I_{\alpha_{n_k}}(\phi_n) \geq \left(b(0)^{\frac{2A}{p-1}} \widetilde{\mu}^{1+A} + 2\left(\frac{1-\widetilde{\mu}}{2}\right)^{1+A} \right) J_\infty,$$

where $A = \frac{2(p-1)}{5-p}$. On the other hand, take $x_0 > 0$ to satisfy $b(x_0) = 1$ and set

$$\varphi_k(x) = t_k\left(\psi_{1,1/2}(x - \alpha_{n_k}^A x_0) + \psi_{1,1/2}(x + \alpha_{n_k}^A x_0)\right),$$

where ψ is the minimizer of $I_{\infty,1}$ under the constraint $||u||_{L^2}^2 = 1/2$ and $t_k > 1$, $t_k \to 1$ as $k \to \infty$ is taken so that $||\varphi_k||_{L^2} = 1$. By a simple calculation, we have

$$\lim_{k\to\infty} I_{\alpha_{n_k}}(\varphi_k) = 2^{-A} J_\infty. \tag{4.1}$$

Since $I_{\alpha_{n_k}}(\phi_{n_k}) \leq I_{\alpha_{n_k}}(\varphi_k)$ and $J_\infty < 0$, we have

$$b(0)^{\frac{2A}{p-1}} \widetilde{\mu}^{1+A} + 2\left(\frac{1-\widetilde{\mu}}{2}\right)^{1+A} \geq 2^{-A}. \tag{4.2}$$

However, (4.2) implies

$$b(0) \geq 2^{-\frac{p-1}{2}}.$$

Thus, we have contradiction since we are assuming $b(0) < 2^{-\frac{p-1}{2}}$.

Therefore, we have $\widetilde{\mu} = 0$. We use Lemma 4. Suppose, $\mu = 0$. Then, by Lemma 4 (ii), we have $\liminf_{k\to\infty} I_{\alpha_{n_k}}(\phi_{n_k}) \geq 0$, so it contradicts to

$$\liminf_{k\to\infty} I_{\alpha_{n_k}}(\phi_{n_k}) \leq \liminf_{k\to\infty} I_{\alpha_{n_k}}(\varphi_k) < 0.$$

Suppose $0 < \mu < 1/2$. Then calculating as the proof of Theorem 2 (i) and using Lemma 4 instead of Lemma 3, we obtain

$$\liminf_{k\to\infty} I_{\alpha_{n_k}}(\phi_{n_k}) \geq \left(2\mu^{1+A} + 2\left(\frac{1-2\mu}{2}\right)^{1+A}\right) J_\infty.$$

However, this implies $\liminf_{k\to\infty} I_{\alpha_{n_k}}(\phi_{n_k}) > \lim_{k\to\infty} I_{\alpha_{n_k}}(\varphi_k)$ and we have a contradiction. Therefore, we have $\mu = 1/2$.

By Lemma 4, there exist ϕ and $y_k > 0$ such that $\chi_+(\cdot - y_k)\phi_{n_k}(\cdot - y_k) \to \phi$ in $L^p(\mathbb{R})$ for $p \in [2, \infty]$. Thus, we see that $\|\chi_+(\cdot - y_k)\phi_{n_k}(\cdot - y_k)\|_{L^2}^2 \to 1/2$. We claim $\chi_+(\cdot - y_k)\phi_{n_k}(\cdot - y_k) \to \psi_{1,1/2}$ in $H^1(\mathbb{R})$, where $\psi_{1,1/2}$ is the positive radial minimizer of $I_{\infty,1}$ under the constraint $\|\phi\|_{L^2}^2 = 1/2$. To show this, it suffices to show

$$I_{\infty,1}(\chi_+(\cdot - y_k)\phi_{n_k}(\cdot - y_k)) \to I_{\infty,1}(\psi_{1,1/2}) = 2^{-(1+A)} J_\infty.$$

Now, suppose there exists $\varepsilon_0 > 0$ such that

$$\frac{1}{p+1}\int_{\mathbb{R}} (1 - b(\alpha_{n_k}^{-A} x))\phi_{n_k}^{p+1}\, dx \geq \varepsilon_0.$$

Then, we have

$$\begin{aligned}
\lim_{k\to\infty} I_{\infty,1}(\varphi_k) &= \lim_{k\to\infty} I_{\alpha_{n_k}}(\varphi_k) \\
&\geq \liminf_{k\to\infty} I_{\alpha_{n_k}}(\phi_{n_k}) \\
&= \liminf_{k\to\infty}\left(I_{\infty,1}(\phi_{n_k}) + \frac{1}{p+1}\int_{\mathbb{R}}(1 - b(x/\alpha_{n_k}^A))\phi_{n_k}\, dx\right) \\
&\geq 2I_{\infty,1}(\psi_{1,1/2}) + \varepsilon_0 \\
&= \lim_{k\to\infty} I_{\infty,1}(\varphi_k) + \varepsilon_0.
\end{aligned}$$

Therefore, we have

$$\lim_{k\to\infty}\frac{1}{p+1}\int_{\mathbb{R}}(1 - b(x/\alpha_{n_k}^A))\phi_{n_k}^{p+1}\, dx = 0.$$

Thus, since $\widetilde{\mu} = 0$, we have

$$\begin{aligned}
&\liminf_{k\to\infty} I_{\infty,1}(\chi_+(\cdot - y_k)\phi_{n_k}(\cdot - y_k)) = \liminf_{k\to\infty} I_{\infty,1}(\chi_+\phi_{n_k}) \\
&= \liminf_{k\to\infty}\frac{1}{2} I_{\infty,1}(\phi_{n_k}) \\
&= \liminf_{k\to\infty}\frac{1}{2}\left(I_{\alpha_{n_k}}(\phi_{n_k}) + \frac{1}{p+1}\int_{\mathbb{R}}(1 - b(x/\alpha_{n_k}^A))\phi_{n_k}^{p+1}\, dx\right) \\
&\leq \lim_{k\to\infty}\frac{1}{2} I_{\alpha_{n_k}}(\varphi_k) \\
&= I_{\infty,1}(\psi_{1,1/2})
\end{aligned}$$

Therefore, we see that $\chi_+(\cdot - y_k)\phi_{n_k}(\cdot - y_k) \to \phi$ in H^1. Since $y_k \to \infty$, we see that ϕ_{n_k} cannot concentrate around some point.

The instability follows from the fact that $\phi_{n_k} \sim \psi_{1,1/2}(\cdot - y_k) + \psi_{1,1/2}(\cdot + y_k)$. We see that there exists two directions which is tangent to the hypersurface $\{\phi \in H^1(\mathbb{R}) \mid ||\phi||_{L^2} = \alpha\}$ and decreases the energy. Using this fact, by [6], we can show the linear instability of u_n and the instability follows from the linear instability. □

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