数理解析研究所講究録
第 1671 巻 2009 年 91-95



# Non-homogeneous semilinear elliptic equations involving critical Sobolev exponent

Yūki Naito[a] and Tokushi Sato[b]

[a] Department of Mathematics, Ehime University, Matsuyama 790-8577, Japan
[b] Mathematical Institute, Tohoku University, Sendai 980-8578, Japan

Let $\Omega \subset \mathbf{R}^N$ be a bounded domain with smooth boundary $\partial\Omega$ with $N \geq 3$. We consider the existence of multiple positive solutions of the following semilinear elliptic equations

$$(1.1) \qquad \begin{cases} -\Delta u + \kappa u = u^p + \lambda f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\kappa \in \mathbf{R}$, $\lambda > 0$ are parameters, $p$ is the critical Sobolev exponent $p = (N+2)/(N-2)$, and $f(x)$ is a non-homogeneous perturbation satisfying

$$(1.2) \qquad f \in H^{-1}(\Omega), \quad f \geq 0, \ f \not\equiv 0 \quad \text{a.e. in } \Omega.$$

Since $p$ is a critical Sobolev exponent for which the embedding $W^{1,2}(\Omega) \subset L^{2N/(N-2)}(\Omega)$ is not compact, we encounter serious difficulties in applying variational methods to the problem (1.1).

Let us recall the results for the case $f \equiv 0$;

$$(1.3) \qquad \begin{cases} -\Delta u + \kappa u = u^p & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

In this case, by using the Pohozaev identity, it can be shown that (1.3) admits no nontrivial solutions for each $\kappa \geq 0$, provided that $\Omega$ is star-shaped. On the other hand, Brezis and Nirenberg [1] obtained the following results when $\kappa < 0$: let $\kappa_1$ be the first eigenvalue of $-\Delta$ with zero Dirichlet condition on $\Omega$; then

(i) if $N \geq 4$, then for every $\kappa \in (-\kappa_1, 0)$, there exists a positive solution;

(ii) if $N = 3$ and $\Omega$ is a ball, then there exists a positive solution if and only if $\kappa \in (-\kappa_1, -\kappa_1/4)$.

Let us consider the case where $f$ satisfies (1.2). Tarantello [6] considered the problem with $\kappa = 0$;

$$
(1.4) \qquad \begin{cases} -\Delta u = u^p + \lambda f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}
$$

and showed that (1.4) has at least two positive solution if $\lambda$ is small enough. The main idea is to divide the Nehari manifold $\Lambda = \{u \in H_0^1(\Omega) : \langle I'(u), u \rangle = 0\}$ into three parts $\Lambda^+, \Lambda^-$ and $\Lambda_0$, and to use the Ekeland principle to get one solution for $\Lambda^+$ and another solution for $\Lambda^-$. We note here that no positive solution exists if $\lambda$ is sufficiently large.

The existence of two nontrivial solutions for more general problem

$$
\begin{cases} -\Delta u = u^p + g(x, u) + \lambda f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}
$$

where $g(x, u)$ is a suitable lower-order perturbation of $u^p$, was proved by Cao and Zhou [2]. These achievements have been extended to the $p$-Laplace equation by Chabrouski [3] and Zhou [7], and to more general problems by Squassina [5].

In this paper we will consider the problem (1.1) with $\kappa \in \mathbf{R}$ in the case where $f$ satisfies (1.2), and show that, when $\kappa > 0$, the situation is drastically different between the cases $N = 3, 4, 5$ and $N \geq 6$.

We call a positive minimal solution $\underline{u}_\lambda$ of $(1.1)_\lambda$, if $\underline{u}_\lambda$ satisfies $\underline{u}_\lambda \leq u$ in $\Omega$ for any positive solution $u$ of $(1.1)_\lambda$. Our main results are stated as following theorems.

**Theorem 1.** *Assume that $\kappa > -\kappa_1$. Then there exists $\overline{\lambda} \in (0, \infty)$ such that*

(i) *if $0 < \lambda < \overline{\lambda}$ then the problem $(1.1)_\lambda$ has a positive minimal solution $\underline{u}_\lambda \in H_0^1(\Omega)$. Furthermore, if $0 < \lambda < \dot{\lambda} < \overline{\lambda}$ then $\underline{u}_\lambda < \underline{u}_{\dot{\lambda}}$ a.e. in $\Omega$;*

(ii) *if $\lambda > \overline{\lambda}$ then the problem (1.1) has no positive solution $u \in H_0^1(\Omega)$.*

**Remark.** There is no positive solution of (1.1) with $\kappa \leq -\kappa_1$. Assume to the contrary that there exists a positive solution $u$ of (1.1) with $\kappa \leq -\kappa_1$. Let $\phi_1$ be the eigenfucntion of $-\Delta$ corresponding to $\kappa_1$ with $\phi_1 > 0$ on $\Omega$. Then we have

$$
0 = \int_\Omega \nabla u \cdot \nabla \phi_1 - \kappa_1 u \phi_1 dx \geq \int_\Omega \nabla u \cdot \nabla \phi_1 + \kappa u \phi_1 dx = \int_\Omega u^p \phi_1 + \lambda f \phi_1 dx > 0.
$$

This is a contradiction.

We consider the existence of the solutions of (1.1) at the extremal value $\lambda = \overline{\lambda}$, so called extremal solutions.

**Theorem 2.** *Let $\kappa > -\kappa_1$. If $\lambda = \overline{\lambda}$ then the problem* (1.1) *has a unique positive solution in $H_0^1(\Omega)$.*

Next, let us consider the existence and nonexistence of second positive solutions to (1.1) for $0 < \lambda < \overline{\lambda}$.

**Theorem 3.** *Assume that either* (i) *or* (ii) *holds.*

$$\text{(i) } \kappa \in (-\kappa_1, 0] \text{ and } N \geq 3; \qquad \text{(ii) } \kappa > 0 \text{ and } N = 3, 4, 5.$$

*If $0 < \lambda < \overline{\lambda}$ then* (1.1) *has a positive solution $\overline{u}_\lambda \in H_0^1(\Omega)$ satisfying $\overline{u}_\lambda > \underline{u}_\lambda$.*

**Theorem 4.** *Assume that $\kappa > 0$ and $N \geq 6$.*

(i) *There exists $\lambda^* = \lambda^*(\kappa) \in (0, \overline{\lambda})$ such that if $\lambda^* < \lambda < \overline{\lambda}$ then the problem* (1.1) *has a positive solution $\overline{u}_\lambda \in H_0^1(\Omega)$ satisfying $\overline{u}_\lambda > \underline{u}_\lambda$.*

(i) *Let $\Omega = \{x \in \mathbf{R}^N : |x| < R\}$ with some $R > 0$, and let $f = f(|x|)$ be radially symmetric about the origin. Assume that $f \in C^\alpha([0, R])$ with some $0 < \alpha < 1$, and $f(r)$ is nonincreasing in $r \in (0, R)$. Then there exists $\lambda_* \in (0, \lambda^*)$ such that $(1.1)_\lambda$ has a unique positive solution $\underline{u}_\lambda$ for $\lambda \in (0, \lambda_*]$.*

In the proof of Theorem 1, we will employ the bifurcation results and the comparison argument for solutions of (1.1) to obtain the minimal solutions. We will prove Theorem 2 by establishing a priori bound for the solutions of (1.1) at $\lambda = \overline{\lambda}$.

In order to find a second positive solution of (1.1), we introduce the problem

$$-\Delta v + \kappa v = (v + \underline{u}_\lambda)^p - \underline{u}_\lambda^p \quad \text{in } \Omega, \quad v \in H_0^1(\Omega), \tag{1.5}$$

where $\underline{u}_\lambda$ is the minimal positive solution of (1.1) for $\lambda \in (0, \overline{\lambda})$ obtained in Theorem 1. In fact, assume that (1.5) has a positive solution $v$, and put $\overline{u}_\lambda = v + \underline{u}_\lambda$. Then $\overline{u}_\lambda \in H_0^1(\Omega)$ and solves (1.1) and satisfies $\overline{u}_\lambda > \underline{u}_\lambda$ in $\Omega$. In the proof of Theorem 3, we will show the existence of solutions of (1.5) by using a variational method. To this end we define the corresponding variational functional of (1.5) by

$$I_\kappa(v) = \frac{1}{2} \int_\Omega \left( |\nabla v|^2 + \kappa v^2 \right) dx - \int_{\mathbf{R}^N} G(v, \underline{u}_\lambda) dx$$

for $v \in H_0^1(\Omega)$, where

$$G(t, s) = \frac{1}{p+1} (t_+ + s)^{p+1} - \frac{1}{p+1} s^{p+1} - s^p t_+.$$

It is easy to see that $I_\kappa : H_0^1(\Omega) \to \mathbf{R}$ is $C^1$ and the critical point $v_0 \in H_0^1(\Omega)$ satisfies

$$\int_\Omega \left( \nabla v_0 \cdot \nabla \psi + \kappa v_0 \psi + g(v_0, \underline{u}_\lambda) \psi \right) dx = 0$$

for any $\psi \in H_0^1(\Omega)$, where

$$g(t,s) = (t_+ + s)^p - s^p.$$

Denote by $S$ the best Sobolev constant of the embedding $H_0^1(\Omega) \subset L^{p+1}(\Omega)$, which is given by

$$S = \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2 dx}{\left(\int_\Omega |u|^{p+1} dx\right)^{2/(p+1)}}.$$

We will obtain Theorem 3 as a consequence of the following two propositions.

**Proposition 5.** *Let $\lambda \in (0, \lambda^*)$. Assume that there exists $v_0 \in H_0^1(\Omega)$ with $v_0 \geq 0$, $v_0 \not\equiv 0$ such that*

$$\sup_{t>0} I_\kappa(tv_0) < \frac{1}{N} S^{N/2}. \tag{1.6}$$

*Then there exists a positive solution $v \in H_0^1(\Omega)$ of* (1.5).

**Proposition 6.** *Assume that either* (i) *or* (ii) *holds.*

(i) $\kappa \in (-\kappa_1, 0]$ *and* $N \geq 3$; (ii) $\kappa > 0$ *and* $N = 3, 4, 5$.

*Then there exists a positive function $v_0 \in H_0^1(\Omega)$ such that* (1.6) *holds.*

In the proof of Proposition 5, we will derive some estimates to establish inequalities relating certain minimizing sequences. In order to prove Proposition 6, for $\varepsilon > 0$, we will set

$$u_\varepsilon(x) = \frac{\phi(x)}{(\varepsilon + |x|^2)^{(N-2)/2}},$$

where $\phi \in C_0^\infty(\mathbf{R}^N)$, $0 \leq \phi \leq 1$, is a cut off function, and will show that (1.6) holds with $v_0 = u_\varepsilon$ for sufficiently small $\varepsilon > 0$.

In the proof of Theorem 4 (ii), we will verify the nonexistence of positive solutions of (1.5) in the radial case by the Pohozaev type argument for the associated ODE. In fact, by [4], the solution $v$ of (1.5) must be radially symmetric, and $v = v(r)$, $r = |x|$, satisfies the problem of the following ordinary differential equation

$$\begin{cases} (r^{N-1}v_r)_r - \kappa r^{N-1} v + r^{N-1} g(v, \underline{u}_\lambda) = 0, & 0 < r < R, \\ v_r(0) = v(R) = 0. \end{cases} \tag{1.7}$$

For the solution $v$ to (1.7), we will obtain the following Pohozaev type identity:

$$\int_0^R r^{N-1}\left[\frac{2N}{N-2} G(u, \underline{u}_\lambda) - g(u, \underline{u}_\lambda)u\right] dr + \frac{2}{N-2}\int_0^R r^N G_s(u, \underline{u}_\lambda)\underline{u}_\lambda' dr$$
$$+ \frac{2\kappa}{N-2}\int_0^\infty r^{N-1} u^2 dr = \frac{1}{N-2} R^N v_r(R)^2.$$

In the proofs of Theorems 2, 3 and 4, the results concerning the eigenvalue problems to the linearized equations around the minimal solutions

$$-\Delta\phi + \phi = \mu p(\underline{u}_\lambda)^{p-1}\phi \quad \text{in } \Omega, \quad \phi \in H_0^1(\Omega).$$

play a crucial role.

## References


[1] H. Brezis and L. Nirenberg, Positive solutions of nonlinear elliptic equations involving critical Sobolev exponent, Comm. Pure Appl. Math. 36 (1983) 437–477.

[2] D.-M. Cao and H.-S. Zhou, On the existence of multiple solutions of nonhomogeneous elliptic equations involving critical Sobolev exponents, Z. Angew. Math. Phys. 47 (1996) 89–96.

[3] J. Chabrowski, On multiple solutions for the nonhomogeneous p-Laplacian with a critical Sobolev exponent, Differential Integral Equations 8 (1995) 705–716.

[4] B. Gidas, W.-M. Ni, and L. Nirenberg, Symmetry and related properties via the maximum principle, Comm. Math. Phys. 68 (1979) 209–243.

[5] M. Squassina, Two solutions for inhomogeneous nonlinear elliptic equations at critical growth, NoDEA Nonlinear Differential Equations Appl. 11 (2004) 53–71.

[6] G. Tarantello, On nonhomogeneous elliptic equations involving critical Sobolev exponent, Ann. Inst. H. Poincaré Anal. Non Linéeaire 9 (1992) 281–304.

[7] H. S. Zhou, Solutions for a quasilinear elliptic equation with critical sobolev exponent and perturbations on $R^n$, Differential Integral Equations 13 (2000) 595–612.