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On hitting times of time inhomogeneous diffusion processes to some moving domains

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Abstract

We introduce the concept of non-favorite recurrent set of time inhomogeneous diffusion processes on a space-time domain and give some conditions under which the space-time domain given by its t-section $B(0, r(t)) = \{x \in \mathbb{R}^d : |x| < r(t)\}$ being a non-favorite recurrent set of the diffusions in the framework of recurrent Dirichlet forms. Some related examples are presented.

1 Introduction

Consider the family of time dependent symmetric forms

$$\mathcal{E}^{(t)}(u, v) = \sum_{i,j=1}^{d} \int_{\mathbb{R}^d} a_{ij}^{(t)}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} \, dx, \quad u, v \in C_0^1(\mathbb{R}^d) \tag{1.1}$$

corresponding to a symmetric positive definite family $\{a_{ij}^{(t)}(x), t \geq 0\}_{1 \leq i,j \leq d}$ satisfying

$$\underline{a}(t) \sum_{i,j=1}^{d} a_{ij}^{(0)}(x)\xi_i\xi_j \leq \sum_{i,j=1}^{d} a_{ij}^{(t)}(x)\xi_i\xi_j \leq \frac{1}{\underline{a}(t)} \sum_{i,j=1}^{d} a_{ij}^{(0)}(x)\xi_i\xi_j \tag{1.2}$$

for some positive non-increasing function $\underline{a}(t)$. Here $C_0^1(\mathbb{R}^d)$ is the space of continuously differentiable functions with compact support in $\mathbb{R}^d$ and $(\xi_1, \xi_2, \cdots, \xi_d) \in \mathbb{R}^d$. Assuming $(\mathcal{E}^{(0)}, C_0^1(\mathbb{R}^d))$ is closable, the regular Dirichlet form $(\mathcal{E}^{(0)}, H^1(\mathbb{R}^d))$ on $L^2(\mathbb{R}^d)$ is defined by the smallest closed extension of (1.1) (cf.[2]). Then $(\mathcal{E}^{(t)}, H^1(\mathbb{R}^d))$ defines a family of time dependent regular Dirichlet forms on $L^2(\mathbb{R}^d)$.

A time inhomogeneous diffusion process $\mathbf{M} = \{X_t, \mathbf{P}_{(s,x)}, (s, x) \in [0, \infty) \times \mathbb{R}^d\}$ is said to be associated with $(\mathcal{E}^{(t)}, H^1(\mathbb{R}^d))$ if the transition function $u_t(s, x) = \mathbf{E}_{(s,x)}(f(X_{t-s}))$ of $\mathbf{M}$ satisfies the following terminal value problem

$$-\int_{\mathbb{R}^d} \frac{\partial u_t(s, x)}{\partial s} v(x) \, dx + \mathcal{E}^{(s)}(u_t(s, \cdot), v) = 0, \quad u_t(t, x) = f(x) \tag{1.3}$$

for any $s < t$ and $v \in C_0^1(\mathbb{R}^d)$. By making use of $\mathbf{M}$, we define the associated space-time diffusion process $\mathbf{Z} = \{Z_t, \mathbf{P}_{(s,x)}\}$ by $Z_t = (\tau(t), X_t)$, where $\tau(t) = \tau(0) + t$ is the

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uniform motion to the right. We especially denote by $\mathbf{M}^{(0)} = \{X_t, \mathbf{P}_x^{(0)}, x \in \mathbb{R}^d\}$ the time homogeneous diffusion process associated with $(\mathcal{E}^{(0)}, H^1(\mathbb{R}^d))$. Let Γ be a space-time domain of $[0,\infty) \times \mathbb{R}^d$ and denotes $\Gamma_t = \{x \in \mathbb{R}^d : (t,x) \in \Gamma\}$ the t-section of Γ. Let $\sigma_\Gamma = \inf\{t > 0 : X_t \in \Gamma_{\tau(t)}\}$ (resp. $\sigma_\Gamma = \inf\{t > 0 : Z_t \in \Gamma\}$) the first hitting time of X_t (resp. Z_t) to $\Gamma_{\tau(t)}$ (resp. Γ). In particular, we simply write σ_Γ as σ_C if $\Gamma = [0,\infty) \times C$ for a set $C \subset \mathbb{R}^d$.

Let us introduce the following definition: Γ is said to be a *non-favorite recurrent set* of $\mathbf{M}$ (or $\mathbf{Z}$) if Γ is a recurrent set of $\mathbf{M}$ (or $\mathbf{Z}$) (i.e., $\mathbf{P}_{(s,\varphi)}(\sigma_\Gamma < \infty) = 1$ for all $s \geq 0$ and for a measurable function φ on $\mathbb{R}^d$, where $\mathbf{P}_{(s,\varphi)}(\cdot) := \int_{\mathbb{R}^d} \mathbf{P}_{(s,x)}(\cdot)\, dx$), and for any $C \subset \mathbb{R}^d$ such that $C \cap \Gamma_{\tau(t)} = \emptyset$ for all $t \geq 0$,

$$\int_s^S \mathbf{P}_{(\tau,\varphi)}(\sigma_\Gamma < \sigma_C)\, d\tau = o(S) \qquad (S \to \infty) \tag{1.4}$$

for a measurable function φ having the support on $\Gamma^c_{\tau(t)}$.

Note that any compact subset K of $\mathbb{R}^d$ is a recurrent set of $\mathbf{M}^{(0)}$. However if the set K varies depending on time, the matters are not so simple (excepting the case of Brownian motion). Therefore it is a natural question that under what conditions on the time (homogeneous) inhomogeneous diffusion ($\mathbf{M}^{(0)}$) $\mathbf{M}$, a space-time domain Γ is to be a recurrent set. By applying a quite general answer for this question we are obtained, our another question is that under what conditions on the diffusion $\mathbf{M}$, Γ is to be a non-favorite recurrent set.

The purpose of this article is to suggest some partial answers for these problems under the framework of recurrent Dirichlet forms. In particular, we shall give some conditions under which a space-time domain Γ_B given by its t-section $B(0,r(t))^c = \{x \in \mathbb{R}^d : |x| > r(t)\}$ with a positive non-decreasing sphere function $r(t)$ being a non-favorite recurrent set of $\mathbf{M}$.

In section 2, we give a general criterion for Γ_B being a recurrent set of $\mathbf{M}$ by using the dual transition function of the part of the time inhomogeneous transformed process by a diffeomorphism. Some inequalities concerning parabolic harmonic functions of the space-time diffusion $\mathbf{Z}$ are also considered. In section 3, we shall show under certain conditions on $\underline{a}(t)$ and $r(t)$ that Γ_B is to be a non-favorite recurrent set of $\mathbf{M}$. Some related examples are considered in section 4. We use k_i to denote appropriate constants, and refer readers to [2] ([7]) for understanding the general theory of Dirichlet (time dependent Dirichlet) forms.

2 General criterion for recurrent sets

Let $\widehat{\mathbf{M}} = \{\widehat{X}_t, \widehat{\mathbf{P}}_{(t,y)}, (t,y) \in [0,\infty) \times \mathbb{R}^d\}$ be the dual process of $\mathbf{M}$ and $\widehat{\mathbf{Z}} = \{\widehat{Z}_t, \widehat{\mathbf{P}}_{(t,y)}\}$ with $\widehat{Z}_t = (\widehat{\tau}(t), \widehat{X}_t)$ the associated dual space-time process, where $\widehat{\tau}(t) = \widehat{\tau}(0) - t$ is the uniform motion to the left. Let $\widehat{\sigma}_\Gamma$ be the first hitting time of $\widehat{X}_t$ (or $\widehat{Z}_t$) to a space-time domain $\Gamma \subset [0,\infty) \times \mathbb{R}^d$. Consider the dual transition function of the part process of $\widehat{\mathbf{M}}$

on Γ^c, $\widehat{u}_s^\Gamma(t,y) = \widehat{\mathbf{E}}_{(t,y)}(\varphi(\widehat{X}_{t-s}) : t-s < \widehat{\sigma}_\Gamma)$ for a measurable function φ. Then by the duality, $\widehat{u}_s^\Gamma(t,\cdot)$ satisfies

$$\int_{\mathbb{R}^d} u(t,y)\frac{\partial \widehat{u}_s^\Gamma(t,y)}{\partial t}\,dy + \mathcal{E}^{(t)}(u(t,\cdot),\widehat{u}_s^\Gamma(t,\cdot)) = 0, \quad \widehat{u}_s^\Gamma(s,y) = \varphi(y) \tag{2.5}$$

for any $s < t$ and $u(t,\cdot) \in H^1(\mathbb{R}^d)$.

Let $\Phi(t,\cdot)$ be a diffeomorphism from $\mathbb{R}^d$ onto itself such that $\Phi(0,y) = y$ and smooth relative to t. Put $f(t,y) = u(t,\Phi(t,y))$ and $g(t,y) = v(t,\Phi(t,y))$. Then

$$\mathcal{E}^{(t)}(u(t,\Phi(t,\cdot)),v(t,\Phi(t,\cdot))) = \sum_{i,j=1}^d \int_{\mathbb{R}^d} \alpha_{ij}^{(t)}(y)\frac{\partial f}{\partial y_i}(t,y)\frac{\partial g}{\partial y_j}(t,y)\rho(t,y)\,dy \tag{2.6}$$

and

$$\int_{\mathbb{R}^d}\frac{\partial u}{\partial t}(t,\Phi(t,y))v(t,\Phi(t,y))\,dy = \int_{\mathbb{R}^d}\left(\frac{\partial f}{\partial t}(t,y) - \sum_{i=1}^d \beta_i(t,y)\frac{\partial f}{\partial y_i}(t,y)\right) g(t,y)\rho(t,y)\,dy, \tag{2.7}$$

where $\gamma_{ij}(t,y) = \partial\Phi_j(t,y)/\partial y_i$, $\rho(t,y) = det(\gamma_{ij}(t,y))$, $(\gamma_{ij}^{-1}) = (\gamma_{ij})^{-1}$,

$$\alpha_{ij}^{(t)}(y) = \sum_{k,l=1}^d \gamma_{ki}^{-1}a_{kl}^{(t)}(\Phi(t,y))\gamma_{lj}^{-1} \quad\text{and}\quad \beta_i(t,y) = \sum_{k=1}^d \gamma_{ki}^{-1}\frac{\partial\Phi_k}{\partial t}(t,y).$$

Let $Y_t = \Phi^{-1}(t,X_t)$ be the process determined by the inverse image of X_t by the inverse function $\Phi^{-1}(t,\cdot)$. Then $\mathbf{M}^\Phi := \{Y_t, \mathbf{P}_{(s,x)}^\Phi\}$ is the time inhomogeneous diffusion process corresponding to the family of time dependent Dirichlet form $(\mathcal{E}^{(t,\Phi)}, H^1(\mathbb{R}^d))$ on $L^2(\mathbb{R}^d;\mu_t)$ given by

$$\mathcal{E}^{(t,\Phi)}(f(t,\cdot),g(t,\cdot)) = \mathcal{E}^{(t)}\left(u(t,\Phi(t,\cdot)),v(t,\Phi(t,\cdot))\right), \quad \mu_t(dy) = \rho(t,y)dy.$$

By $\widehat{\mathbf{M}}^\Phi = \{\widehat{Y}_t, \widehat{\mathbf{P}}_{(t,y)}^\Phi\}$, we denote the dual process of $\mathbf{M}^\Phi$ relative to μ_t. We make the following assumption on the family $\{\alpha_{ij}^{(t)}(y)\}_{1\le i,j\le d}$:

(A) There exist a symmetric positive definite family $\{\alpha_{ij}^{(0)}(y)\}_{1\le i,j\le d}$ and a non-increasing positive function $\underline{\alpha}(t)$ such that

$$\underline{\alpha}(t)\sum_{i,j=1}^d \alpha_{ij}^{(0)}(y)\xi_i\xi_j \le \sum_{i,j=1}^d \alpha_{ij}^{(t)}(y)\xi_i\xi_j \tag{2.8}$$

for any $(\xi_1,\xi_2,\cdots,\xi_d) \in \mathbb{R}^d$.

Fix a connected open set D of $\mathbb{R}^d$ and set

$$\eta_D(t) = \sup_{y\in D}\frac{1}{\rho(t,y)}\left(\sum_{i=1}^d \frac{\partial(\beta_i\rho)}{\partial y_i} - \frac{\partial\rho}{\partial t}\right)(t,y)$$
$$\lambda_D(t) = k_0\underline{\alpha}(t)\frac{\inf_{y\in D}\rho(t,y)}{\sup_{y\in D}\rho(t,y)},$$

where k_0 is a constant satisfying the inequality

$$k_0 \int_D \psi(y)^2 dy \leq \mathcal{E}^{(0,\Phi)}(\psi, \psi) := \sum_{i=1}^{d} \int_D \alpha_{ij}^{(0)}(y) \frac{\partial \psi}{\partial y_i} \frac{\partial \psi}{\partial y_j} dy \tag{2.9}$$

for any $\psi \in C_0^1(D)$. Such a constant k_0 can be taken as $\|R^0 1\|_\infty^{-1}$ (if $\|R^0 1\|_\infty < \infty$) by using the potential R^0 of the diffusion process on D associated with $(\mathcal{E}^{(0,\Phi)}, H_0^1(D))$ ([9]).

Now we give a general criterion for a space-time domain Γ corresponding to a fixed domain $F = \mathbb{R}^d \setminus D$ by a diffeomorphism Φ being a recurrent set of $\mathbf{M}$.

Theorem 2.1. *Assume* (**A**) *and set* $\Gamma := \{(t, \Phi(t, y)) : s \leq t, \ y \in F\}$. *If*

$$\lim_{T \to \infty} \sqrt{\mu_T(D)} \exp\left(\int_s^T \left(\frac{1}{2}\eta_D(t) - \lambda_D(t) \right) dt \right) = 0, \tag{2.10}$$

then $\mathbf{P}_{(s,\varphi)}(\sigma_\Gamma < \infty) = 1$ *for any* $s \geq 0$ *and an initial distribution* φ *on* $\mathbb{R}^d$. *In other words,* Γ *is a recurrent set of* $\mathbf{M}$.

Proof. The idea of the proof is essentially to due [4]. Let $\widehat{u}_s^F(t, y) = \widehat{\mathbf{E}}_{(t,y)}^{\Phi}(\varphi(\widehat{Y}_{t-s}) : t - s < \widehat{\sigma}_F^{\Phi})$ be the dual transition function of the part process of $\widehat{\mathbf{M}}^{\Phi}$ on D, where $\widehat{\sigma}_F^{\Phi}$ denotes the first hitting time of $\widehat{Y}_t$ to F. Then since $\widehat{u}_s^F(t, \cdot) = \widehat{u}_s^{\Gamma}(t, \Phi(t, \cdot))$, it satisfies for $s < t$

$$\begin{aligned} &\mathcal{E}^{(t,\Phi)}\left(f(t, \cdot), \widehat{u}_s^F(t, \cdot)\right) \\ &= - \int_D f(t, y) \frac{\partial(\widehat{u}_s^F \rho)}{\partial t}(t, y)\, dy - \sum_{i=1}^{d} \int_D \beta_i(t, y) \frac{\partial f}{\partial y_i}(t, y) \widehat{u}_s^F(t, y)\, d\mu_t(y) \end{aligned} \tag{2.11}$$

in view of (2.5), (2.6) and (2.7). In particular, by taking $f = \widehat{u}_s^F$ in (2.11), we have

$$\begin{aligned} &\mathcal{E}^{(t,\Phi)}\left(\widehat{u}_s^F(t, \cdot), \widehat{u}_s^F(t, \cdot)\right) \\ &= - \frac{1}{2}\frac{d}{dt} \int_D \widehat{u}_s^F(t, y)^2 d\mu_t(y) - \frac{1}{2} \int_D \frac{\partial \rho}{\partial t}(t, y) \widehat{u}_s^F(t, y)^2 dy \\ &\quad - \frac{1}{2} \sum_{i=1}^{d} \int_D \beta_i(t, y) \frac{\partial(\widehat{u}_s^F(t, y)^2)}{\partial y_i} d\mu_t(y) \\ &= - \frac{1}{2}\frac{d}{dt} \widehat{H}_D(t) + \frac{1}{2} \int_D \left(\sum_{i=1}^{d} \frac{\partial(\beta_i \rho)}{\partial y_i} - \frac{\partial \rho}{\partial t} \right)(t, y) \widehat{u}_s^F(t, y)^2 dy \\ &\leq - \frac{1}{2}\frac{d}{dt} \widehat{H}_D(t) + \frac{1}{2}\eta_D(t) \widehat{H}_D(t), \end{aligned} \tag{2.12}$$

where

$$\widehat{H}_D(t) = \int_D \widehat{u}_s^F(t, y)^2 d\mu_t(y).$$

On the other hand, by the assumption (**A**) and (2.9),

$$\lambda_D(t) \widehat{H}_D(t) \leq \underline{\alpha}(t) \mathcal{E}^{(0,\Phi)}\left(\widehat{u}_s^F(t, \cdot), \widehat{u}_s^F(t, \cdot)\right) \leq \mathcal{E}^{(t,\Phi)}\left(\widehat{u}_s^F(t, \cdot), \widehat{u}_s^F(t, \cdot)\right). \tag{2.13}$$

Now, combining (2.12) and (2.13), we see that $\widehat{H}_D(t)$ satisfies

$$\frac{d}{dt}\widehat{H}_D(t) \leq (\eta_D(t) - 2\lambda_D(t))\,\widehat{H}_D(t)$$

and hence

$$\widehat{H}_D(\tau) \leq \|\varphi\|^2_{\mu_s} \exp\left(\int_s^\tau (\eta_D(t) - 2\lambda_D(t))\,dt\right). \tag{2.14}$$

Note that $\widehat{H}_D(\cdot)$ depends only on Γ and is independent on the choice of Φ whenever F and Γ are common. So we have

$$\begin{aligned} \mathbf{P}_{(s,\varphi)}(\tau - s < \sigma_\Gamma) &= \mathbf{P}^\Phi_{(s,\varphi\cdot\mu_s)}(\tau - s < \sigma^\Phi_F) \\ &= \int_D \widehat{u}^F_s(\tau, y)d\mu_\tau(y) \\ &\leq \sqrt{\mu_\tau(D)}\sqrt{\widehat{H}_D(\tau)} \\ &\leq \|\varphi\|_{\mu_s}\sqrt{\mu_\tau(D)}\exp\left(\int_s^\tau \left(\frac{1}{2}\eta_D(t) - \lambda_D(t)\right)dt\right) \\ &\longrightarrow 0 \quad \text{as } \tau \longrightarrow \infty \end{aligned}$$

by virtue of the duality and (2.14). □

For a space-time domain $\Gamma \subset [0,\infty) \times \mathbb{R}^d$, let $h(t,x)$ be an α-excessive function of the space-time diffusion $\mathbf{Z} = \{Z_t, \mathbf{P}_{(s,x)}\}$ such that $h \cdot I_\Gamma \in L^2([0,\infty) \times \mathbb{R}^d; dtdx)$. Put

$$H^\alpha_\Gamma h(t,x) = \mathbf{E}_{(t,x)}\left(e^{-\alpha\sigma_\Gamma} h\left(Z_{\sigma_\Gamma}\right)\right), \quad \alpha > 0. \tag{2.15}$$

Then it is a quasi-version of $e^{(\alpha)}_{h\cdot I_\Gamma}(t,x) = \lim_{\varepsilon\to 0} h_\varepsilon(t,x)$, where $h_\varepsilon(t,x)$ is the unique α-excessive function of $\mathbf{Z}$ satisfying

$$-\int_{\mathbb{R}^d} \frac{\partial h_\varepsilon}{\partial t}(t,x)v(x)\,dx + \mathcal{E}^{(t)}_\alpha\left(h_\varepsilon(t,\cdot), v(\cdot)\right) = \frac{1}{\varepsilon}\int_{\mathbb{R}^d} (h_\varepsilon - h\cdot I_\Gamma)^-(t,x)v(x)\,dx \tag{2.16}$$

for any $\varepsilon > 0$ and $v \in H^1(\mathbb{R}^d)$ (cf. Lemma 3.1 and Theorem 3.1 in [7]). Here $\mathcal{E}^{(t)}_\alpha(\varphi,\psi) := \mathcal{E}^{(t)}(\varphi,\psi) + \alpha\int_{\mathbb{R}^d}\varphi(x)\psi(x)\,dx$. Let $\phi(t,x)$ be a non-negative function such that $\phi(t,x) = h(t,x)$ on Γ. Then since

$$\int_{\mathbb{R}^d} (h_\varepsilon - h\cdot I_\Gamma)^-(t,x)(h_\varepsilon - \phi)(t,x)\,dx \leq 0,$$

we have from (2.16) that

$$\begin{aligned} &\mathcal{E}^{(t)}_\alpha\left(h_\varepsilon(t,\cdot), (h_\varepsilon - \phi)(t,\cdot)\right) \\ &\leq \int_{\mathbb{R}^d} \frac{\partial h_\varepsilon}{\partial t}(t,x)(h_\varepsilon - \phi)(t,x)\,dx \\ &= \frac{1}{2}\int_{\mathbb{R}^d} \frac{\partial(h_\varepsilon - \phi)^2}{\partial t}(t,x)\,dx + \int_{\mathbb{R}^d}\frac{\partial\phi}{\partial t}(t,x)(h_\varepsilon - \phi)(t,x)\,dx. \end{aligned} \tag{2.17}$$

By multiplying $\underline{a}(t)^{-1}$ and integrating both sides of (2.17) over $[t_1, t_2]$ $(0 \le t_1 < t_2 < \infty)$, it yields that

$$\begin{aligned}
&\int_{t_1}^{t_2} \frac{1}{\underline{a}(t)} \mathcal{E}_\alpha^{(t)} \left(h_\varepsilon(t,\cdot), (h_\varepsilon - \phi)(t,\cdot)\right) dt \\
&\le \frac{1}{2\underline{a}(t_2)} \|(h_\varepsilon - \phi)(t_2,\cdot)\|_2^2 - \frac{1}{2\underline{a}(t_1)} \|(h_\varepsilon - \phi)(t_1,\cdot)\|_2^2 \\
&\quad - \frac{1}{2} \int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{1}{\underline{a}(t)}\right) \|(h_\varepsilon - \phi)(t,\cdot)\|_2^2 \, dt \\
&\quad + \int_{t_1}^{t_2} \frac{1}{\underline{a}(t)} \left(\int_{\mathbb{R}^d} \frac{\partial \phi}{\partial t}(t,x)(h_\varepsilon - \phi)(t,x)\, dx\right) dt, \qquad (2.18)
\end{aligned}$$

where $\|\cdot\|_2$ denotes the L^2-norm in $\mathbb{R}^d$. Suppose $\phi(t,\cdot)$ is non-increasing relative to t on $[t_1, t_2]$. By letting $\varepsilon \to 0$ and $\alpha \to 0$, we then see that $\lim_{\alpha \to 0} H_\Gamma^\alpha h(t,\cdot) \equiv H_\Gamma h(t,\cdot) = \mathbf{E}_{(t,x)}(h(Z_{\sigma_\Gamma})) \in H^1(\mathbb{R}^d)$ and the inequality (2.18) also holds by replacing h_ε to $H_\Gamma h$ in view of the remark mentioned right after (2.15). Noting $(a_2 - b_2)^2 - (a_1 - b_1)^2 + b_1^2 - b_2^2 \le a_2^2 + 2a_1 b_1$,

$$\int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{1}{\underline{a}(t)}\right) \|(H_\Gamma h - \phi)(t,\cdot)\|_2^2 \, dt \ge 0 \quad \text{and} \quad \int_{t_1}^{t_2} \int_{\mathbb{R}^d} \frac{\partial \phi}{\partial t}(t,x) H_\Gamma h(t,x)\, dx dt \le 0,$$

we have the following lemma.

Theorem 2.2. *Let $\phi(t,x)$ be a non-negative function on $[0,\infty) \times \mathbb{R}^d$ such that $\phi(t,x) = h(t,x)$ on Γ and non-increasing relative to $t \in [t_0, \infty)$ for some $t_0 \ge 0$. Then for $t_0 \le t_1 < t_2 < \infty$,*

$$\begin{aligned}
&\int_{t_1}^{t_2} \frac{1}{\underline{a}(t)} \mathcal{E}^{(t)} \left(H_\Gamma h(t,\cdot), H_\Gamma h(t,\cdot)\right) dt \\
&\le \frac{1}{2\underline{a}(t_2)} \|H_\Gamma h(t_2,\cdot)\|_{L^2(\Gamma^c_{\tau(t_2)})}^2 + \frac{1}{\underline{a}(t_1)} \|(\phi \cdot H_\Gamma h)(t_1,\cdot)\|_{L^2(\Gamma^c_{\tau(t_1)})}^2 \\
&\quad + \frac{1}{2} \int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{1}{\underline{a}(t)}\right) \|\phi(t,\cdot)\|_{L^2(\Gamma^c_{\tau(t)})}^2 \, dt + \int_{t_1}^{t_2} \frac{1}{\underline{a}(t)} \mathcal{E}^{(t)} \left(H_\Gamma h(t,\cdot), \phi(t,\cdot)\right) dt. \qquad (2.19)
\end{aligned}$$

3 Non-favorite recurrent sets

In this section, we apply general criterion of the previous section for recurrent sets to the space-time domain Γ_B given by its t-section $B(0, r(t))^c = \{x \in \mathbb{R}^d : |x| > r(t)\}$ with a non-decreasing smooth sphere function $r(t)$ such that $r(0) = 1$, and give some conditions on $\underline{a}(t)$ and $r(t)$ under which Γ_B being a non-favorite recurrent set of $\mathbf{M}$.

Let us consider the diffeomorphism of the form $\Phi(t,y) = r(t)y$. Then Φ maps $F = \{y \in \mathbb{R}^d : |y| \ge 1\}$ to $B(0, r(t))^c$ and for any $i, j = 1, 2, \cdots, d$,

$$\left\{\begin{array}{l} \gamma_{ij}(t,y) = r(t)\delta_{ij} \\ \rho(t,y) = r(t)^d \\ \alpha_{ij}^{(t)}(y) = \frac{1}{r(t)^2} a_{ij}^{(t)}(r(t)y) \end{array}\right. \qquad \left\{\begin{array}{l} \beta_i(t,y) = \frac{r'(t)}{r(t)} y_i \\ \eta(t) = \mathrm{div}\beta - d\frac{r'(t)}{r(t)} = 0, \end{array}\right.$$

where $\beta = (\beta_1, \beta_2, \cdots, \beta_d)$ and divβ stands for the divergence of β. In this case, the inequality (2.12) becomes

$$\mathcal{E}^{(t,\Phi)}\left(\widehat{u}_s^F(t,\cdot), \widehat{u}_s^F(t,\cdot)\right) = -\frac{1}{2}\frac{d}{dt}\widehat{H}_D(t)$$

and moreover

$$\sum_{i,j=1}^{d} \alpha_{ij}^{(t)}(y)\xi_i\xi_j = \frac{1}{r(t)^2}\sum_{i,j=1}^{d} a_{ij}^{(t)}(r(t)y)\xi_i\xi_j \geq \frac{\underline{a}(t)}{r(t)^2}\sum_{i,j=1}^{d} a_{ij}^{(0)}(r(t)y)\xi_i\xi_j$$

by virtue of (1.2). Therefore, if

$$\sum_{i,j=1}^{d} a_{ij}^{(0)}(r(t)y)\xi_i\xi_j \geq b(t)\sum_{i,j=1}^{d} \alpha_{ij}^{(0)}(y)\xi_i\xi_j \tag{3.20}$$

for some positive non-increasing function $b(t)$, then (2.8) holds for $\underline{\alpha}(t) = \underline{a}(t)b(t)r(t)^{-2}$. Hence we have from (2.14) that

$$\widehat{H}_D(\tau) \leq \|\varphi\|_{\mu_s}\exp\left(-2k_0\int_s^\tau \frac{\underline{a}(t)b(t)}{r(t)^2}dt\right).$$

Now, the following theorem will be immediately obtained by the same procedures as in the rest of the proof of Theorem 2.1.

Theorem 3.1. *Suppose that there exists a positive non-increasing function $b(t)$ satisfying (3.20) and*

$$\lim_{\tau\to\infty} r(\tau)^{d/2}\exp\left(-k_0\int_s^\tau \frac{\underline{a}(t)b(t)}{r(t)^2}dt\right) = 0. \tag{3.21}$$

Then $\mathbf{P}_{(s,\varphi)}(\sigma_\Gamma < \infty) = 1$ for any $s \geq 0$ and an initial distribution φ on $\mathbb{R}^d$. In other words, Γ_B is a recurrent set of $\mathbf{M}$.

Now let us consider a criterion for non-favorite recurrent set. We shall do this under the framework of recurrent Dirichlet forms: Let assume that $(\mathcal{E}^{(t)}, H^1(\mathbb{R}^d))$ is a recurrent Dirichlet form for each fixed $t \geq 0$.

Let C be a relatively compact neighbourhood of the origin given by $C = \{x \in \mathbb{R}^d : |x| \leq \ell\}$ for $0 < \ell < 1$. Let $B(0,R) = \{x \in \mathbb{R}^d : |x| \leq R\}$. For fixed $\tau > 0$, put

$$\Lambda \equiv \Lambda(\tau) = \{(t,x) : t \leq \tau,\ r(t) < |x| \leq r(\tau) + 1\}$$

and denote by $\Lambda_t \equiv \Lambda_t(\tau)$ the t-section of Λ. Note that $\Lambda(\tau) \nearrow \Gamma_B$ and $\Lambda_t(\tau) \nearrow B(0, r(t))$ as $\tau \nearrow \infty$. Let $\mathbf{M}^{B,C} = \{X_t, \mathbf{P}_{(s,x)}^{B,C}\}$ be the time inhomogeneous diffusion process on $B(0, r(\tau)+1) \setminus C$ corresponding to the time dependent Dirichlet form $\mathcal{E}^{(t)}$ with the reflecting boundary $\partial B_{r(\tau)+1}$ and the absorbing boundary ∂C (see [8] for the construction of such a process). Let $\xi_\tau(t)$ be a decreasing function relative to t such that $\xi_\tau(t) = 1$ for

$t \le \tau - 1$ and $\xi_\tau(t) = 0$ for $t \ge \tau$. Then $h(t,x) := \xi_\tau(t)$ is an excessive function relative to the associated space-time process $\mathbf{Z}^{B,C} = \{Z_t, \mathbf{P}^{B,C}_{(s,x)}\}$ of $\mathbf{M}^{B,C}$, that is,

$$\mathbf{E}^{B,C}_{(s,x)}(h(Z_t)) = \mathbf{E}^{B,C}_{(s,x)}(\xi_\tau(s+t)) \le \xi_\tau(s) = h(s,x),$$

where $\mathbf{E}^{B,C}$ denotes the expectation of $\mathbf{Z}^{B,C}$. Set

$$H^{B,C}_\Lambda h(s,x) = \mathbf{E}^{B,C}_{(s,x)}\left(h\left(Z_{\sigma_\Lambda}\right) : \sigma_\Lambda < \sigma_C\right).$$

Let $\phi_\tau(t,x)$ be a non-increasing function relative to t such that $\phi_\tau(t,x) = \xi_\tau(t)$ on Λ and $\phi_\tau(t,x) = 0$ on $x \in C$. Then by applying Theorem 2.2 to $H^{B,C}_\Lambda h(t,x)$ and $\phi_\tau(t,x)$,

$$\begin{aligned} &\int_{t_1}^{t_2} \frac{1}{\underline{a}(t)} \mathcal{E}^{(t)}\left(H^{B,C}_\Lambda \xi_\tau(t,\cdot), H^{B,C}_\Lambda \xi_\tau(t,\cdot)\right) dt \\ &\le \frac{1}{2\underline{a}(t_2)} \left\| H^{B,C}_\Lambda \xi_\tau(t_2,\cdot)\right\|^2_{L^2(\Lambda^c_{\tau(t_2)})} + \frac{1}{\underline{a}(t_1)} \left\| (\phi_\tau \cdot H^{B,C}_\Lambda \xi_\tau)(t_1,\cdot)\right\|^2_{L^2(\Lambda^c_{\tau(t_1)})} \\ &\quad + \frac{1}{2}\int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{1}{\underline{a}(t)}\right) \|\phi_\tau(t,\cdot)\|^2_{L^2(\Lambda^c_{\tau(t)})} dt + \int_{t_1}^{t_2} \frac{1}{\underline{a}(t)} \mathcal{E}^{(t)}\left(H^{B,C}_\Lambda \xi_\tau(t,\cdot), \phi_\tau(t,\cdot)\right) dt. \end{aligned} \tag{3.22}$$

Put $\phi(t,x) = \lim_{\tau\to\infty} \phi_\tau(t,x)$. Then the inequality (3.22) also holds for u_{Γ_B} and ϕ instead of $H^{B,C}_\Lambda \xi_\tau$ and ϕ_τ respectively because

$$u_{\Gamma_B}(t,x) \equiv \lim_{\tau\to\infty} H^{B,C}_\Lambda \xi_\tau(t,x) = \mathbf{P}^C_{(t,x)}\left(\sigma_{\Gamma_B} < \sigma_C\right) = \mathbf{P}_{(t,x)}\left(\sigma_{\Gamma_B} < \sigma_C\right).$$

Put $t_1 = s$, $t_2 = S$ and divide $S - s$ on both side of (3.22). Now we see by letting $S \to \infty$,

$$\begin{aligned} &\lim_{S\to\infty} \frac{1}{S-s}\int_s^S \frac{1}{\underline{a}(t)} \mathcal{E}^{(t)}\left(u_{\Gamma_B}(t,\cdot), u_{\Gamma_B}(t,\cdot)\right) dt \\ &\le \lim_{S\to\infty} \frac{1}{\underline{a}(S)(S-s)} \|u_{\Gamma_B}(S,\cdot)\|^2_{L^2(B(0,r(S)))} \\ &\quad + \lim_{S\to\infty} \frac{1}{S-s}\int_s^S \frac{d}{dt}\left(\frac{1}{\underline{a}(t)}\right) \|\phi(t,\cdot)\|^2_{L^2(B(0,r(t)))} dt \\ &\quad + \lim_{S\to\infty} \frac{1}{S-s}\int_s^S \frac{1}{\underline{a}(t)} \mathcal{E}^{(t)}\left(u_{\Gamma_B}(t,\cdot), \phi(t,\cdot)\right) dt. \end{aligned} \tag{3.23}$$

Therefore, if

$$\lim_{S\to\infty} \frac{1}{\underline{a}(S)(S-s)} \int_{B(0,r(S))} u_{\Gamma_B}(S,x)^2 dx = 0, \tag{3.24}$$

$$\lim_{S\to\infty} \frac{1}{S-s}\int_s^S \frac{d}{dt}\left(\frac{1}{\underline{a}(t)}\right)\left(\int_{B(0,r(t))} \phi^2(t,x)\, dx\right) dt = 0 \tag{3.25}$$

and

$$\lim_{S\to\infty} \frac{1}{S-s}\int_s^S \frac{1}{\underline{a}(t)} \mathcal{E}^{(t)}\left(\phi(t,\cdot), \phi(t,\cdot)\right) dt = 0 \tag{3.26}$$

are fulfilled, then the lefthand side of the inequality (3.23) vanishes.

Note that since $\int_{B(0,r(S))} u_{\Gamma_B}(S,x)^2 dx \le k_1 r(S)^d$, (3.24) is satisfied if

$$\lim_{S\to\infty} \frac{r(S)^d}{S\underline{a}(S)} = 0. \tag{3.27}$$

To obtain a function ϕ satisfying the conditions (3.25) and (3.26), put

$$\overline{a}^{(t)}(r) = \sum_{i,j=1}^{d} \int_{\partial B(0,1)} a_{ij}^{(t)}(\theta r)\theta_i\theta_j d\sigma(\theta),$$

where σ is the surface measure on $\partial B(0,1)$. For each $t \ge 0$, define the function $\phi(t,x)$ by

$$\phi(t,x) = \begin{cases} 0 & (|x| \le \ell) \\ A(t)^{-1}\int_\ell^{|x|} \overline{a}^{(t)}(r)^{-1} r^{1-d} dr & (\ell < |x| < r(t)) \\ 1 & (r(t) \le |x|) \end{cases} \tag{3.28}$$

where $A(t) = \int_\ell^{r(t)} \overline{a}^{(t)}(r)^{-1} r^{1-d} dr$. Then

$$\begin{aligned} &\phi(s,x) - \phi(t,x) \\ &= \left(A(s)^{-1} - A(t)^{-1}\right) \int_\ell^{|x|} \overline{a}^{(s)}(r)^{-1} r^{1-d} dr + A(t)^{-1} \int_\ell^{|x|} \left(\overline{a}^{(s)}(r)^{-1} - \overline{a}^{(t)}(r)^{-1}\right) r^{1-d} dr \end{aligned}$$

for $\ell < |x| < r(s)$ $(s < t)$ and $\phi(t,x) \le 1 = \phi(s,x)$ for $r(s) \le |x|$. Hence, if we choose a non-decreasing function $r(t)$ so that $A(t)$ being non-decreasing relative to t, then $\phi(t,x)$ is non-increasing relative to t. For such a function $\phi(t,x)$, we see $\int_{B(0,r(t))} \phi^2(t,x)\,dx \le k_2 r(t)^d$ and thus (3.25) holds if

$$\lim_{t\to\infty} \frac{d}{dt}\left(\frac{1}{\underline{a}(t)}\right) r(t)^d = 0. \tag{3.29}$$

Furthermore, since $\mathcal{E}^{(t)}(\phi(t,\cdot),\phi(t,\cdot)) = A(t)^{-1}$, (3.26) is satisfied if

$$\lim_{t\to\infty} \underline{a}(t)A(t) = \infty. \tag{3.30}$$

Note that for the existence of the function $r(t)$ satisfying (3.30), it is necessary that $\overline{a}^{(t)}(r)$ satisfies

$$\int_\ell^\infty \overline{a}^{(t)}(r)^{-1} r^{1-d} dr = \infty \tag{3.31}$$

for each fixed $t \ge 0$. Indeed, from Ichihara's test, (3.31) implies that $\mathcal{E}^{(t)}$ and hence $\mathcal{E}^{(0)}$ is a recurrent Dirichlet form (Theorem 1.6.7 in [2]).

Theorem 3.2. *Suppose that $\mathbf{M}^{(0)}$ is Harris recurrent. If a positive non-decreasing function $r(t)$ such that $r(0) = 1$ satisfies the conditions (3.27), (3.29) and (3.30), then*

$$\int_s^S \mathbf{P}_{(t,\varphi)}(\sigma_{\Gamma_B} < \sigma_C)\,dt = o(S) \qquad (S\to\infty) \tag{3.32}$$

for any initial distribution φ having the support on $B(0,r(t))$. In particular, if $\mathcal{E}^{(t)} = \mathcal{E}^{(0)}$ for all $t \ge 0$, then $\lim_{t\to\infty} \mathbf{P}_{(t,\varphi)}(\sigma_{\Gamma_B} < \sigma_C) = 0$.

Proof. Recall that the lefthand side of (3.23) vanishes under the conditions (3.27), (3.29) and (3.30). Therefore, under the hypotheses,

$$\lim_{S\to\infty} \frac{1}{S-s} \int_s^S \mathcal{E}^{(0)}\left(u_{\Gamma_B}(t,\cdot), u_{\Gamma_B}(t,\cdot)\right) dt = 0 \tag{3.33}$$

by virtue of (1.2). On the other hand, it is known under the Harris recurrence of $\mathbf{M}^{(0)}$ that there exist a strictly positive function $\varphi \in L^1(\mathbb{R}^d)$, a positive constant $K(\varphi)$ and a non-null set G satisfying

$$\int_{\mathbb{R}^d} |u(t,x) - \langle u(t,\cdot)\rangle_G|\, \varphi(x)\, dx \le K(\varphi)\sqrt{\mathcal{E}^{(0)}(u(t,\cdot), u(t,\cdot))}, \quad u(t,\cdot) \in H_e^1(\mathbb{R}^d) \tag{3.34}$$

([5]). Here $\langle u\rangle_G = \int_G u(x)\, dx / \int_G dx$ and $H_e^1(\mathbb{R}^d)$ is the extended Dirichlet space of $H^1(\mathbb{R}^d)$. Note that the set G can be taken as a subset of C. Thus, if $u(t,\cdot) = 0$ on C, we can remove the term $\langle u\rangle_G$ from the lefthand side of the inequality (3.34). Furthermore, (3.34) also holds for any strictly positive function dominated by φ, and thus we may assume that φ has a compact support. Now applying $u_{\Gamma_B}(t,\cdot)$ as $u(t,\cdot)$ to (3.34), we have

$$\left(\int_{\mathbb{R}^d} u_{\Gamma_B}(t,x)\varphi(x)\, dx\right)^2 \le K(\varphi)^2 \mathcal{E}^{(0)}(u_{\Gamma_B}(t,\cdot), u_{\Gamma_B}(t,\cdot))$$

and which implies

$$\begin{aligned}
&\lim_{S\to\infty} \frac{1}{S-s} \int_s^S \left\{\mathbf{P}_{(t,\varphi)}\left(\sigma_{\Gamma_B} < \sigma_C\right)\right\}^2 dt \\
&= \lim_{S\to\infty} \frac{1}{S-s} \int_s^S \left(\int_{B(0,r(t))} u_{\Gamma_B}(t,x)\varphi(x) dx\right)^2 dt \\
&= 0
\end{aligned}$$

in view of (3.33). The last assertion is clear from (3.32) because $\mathbf{P}_\varphi^{(0)}(\sigma_{\Gamma^{(t)}} < \sigma_C)$ is non-increasing relative to t and is equal to $\mathbf{P}_{(t,\varphi)}(\sigma_{\Gamma_B} < \sigma_C)$. Here $\Gamma^{(t)} := ([t,\infty) \times \mathbb{R}^d) \cap \Gamma_B$ □

It is clear that Theorem 3.2 holds for a space-time domain $\Gamma \subset [0,\infty) \times \mathbb{R}^d$, then it also holds for any subset of Γ.

Corollary 3.1. *Under the hypotheses in Theorem 3.1 and Theorem 3.2, the space-time domain Γ_B is a non-favorite recurrent set of* $\mathbf{M}$.

4 Examples

In this section, we discuss some examples for Corollary 3.1 under certain time inhomogeneous diffusion processes as well as time changes of Brownian motions.

Example 4.1. Suppose that $a_{ij}^{(t)}(x) = \frac{1}{2}\delta_{ij}$ for $1 \le i,j \le d$. The corresponding diffusion process of (1.1) is then a Brownian motion $\mathbf{M}^{(0)} = \{B_t, \mathbf{P}_x^{(0)}, x \in \mathbb{R}^d\}$. In this case, it is immediately to see that the condition (3.21) will be satisfied if

$$r(t) = (t+1)^\beta, \quad 0 < \beta < \frac{1}{2}. \tag{4.35}$$

Let us assume in addition that $d \ge 3$ (i.e., $\mathbf{M}^{(0)}$ is transient). Then the criterion on escape rates of Brownian motions on $B(0, r(t))$ implies that the sphere function $r(t)$ satisfying (4.35) is indeed to be a *lower radius* of $\mathbf{M}^{(0)}$, that is, $B_t \notin B(0, r(t))$ for all large enough t with probability 1 (see [1],[3]). On the other hand, assume that $d = 1$ or 2 (i.e., $\mathbf{M}^{(0)}$ is Harris recurrent). If $d = 1$, then $A(t) = k_3(r(t) - \ell)$. So all conditions of Theorem 3.2 will be satisfied if $r(t) = (t+1)^\beta, 0 < \beta < 1$. If $d = 2$, then $A(t) = k_4(\log r(t) - \log \ell)$ and thus, all conditions of Theorem 3.2 are satisfied when we choose $r(t)$ of (4.35).

As a conclusion, a Brownian path B_t moving in $B(0, r(t))$ with $r(t) = (t+1)^\beta, (0 < \beta < 1/2)$, namely $B_1(0, r(t))$, leaves it within a finite time almost surely for all dimensions. In particular, B_t never returns into $B_1(0, r(t))$ for all large enough t almost surely when $d \ge 3$, and it may return to inside of $B_1(0, r(t))$ but it tends to enter a neighborhood of the origin before leaving $B_1(0, r(t))$ for large enough t almost surely when $d = 2$ or 1.

Note that the condition (3.21) is also satisfied when $\beta = 1/2$ providing $d < 2k_0$. So we have from Theorem 3.1 and (4.35) that $\mathbf{P}_0^{(0)}(|B_t| > r(t)) = 1$ if $r(t) = (t+1)^\beta$, $0 < \beta \le 1/2$, $d < 2k_0$ for large enough t. On the other hand, by a direct calculation,

$$\begin{aligned}
\mathbf{P}_0^{(0)}(|B_t| > r(t)) &= \frac{1}{(2\pi t)^{d/2}} \int_{\{|x| > r(t)\}} e^{-|x|^2/2t} dx \\
&\le \frac{k_5}{(2\pi)^{d/2}} \int_{r(t)/\sqrt{t}}^{\infty} y^{d-1} e^{-y^2/2} dy \\
&\le \left(k_6 + k_7 \left(\frac{r(t)}{\sqrt{t}} \right)^{d-2} \right) e^{-r(t)^2/(2t)}.
\end{aligned}$$

Therefore, we see that $\lim_{t\to\infty} \mathbf{P}_0^{(0)}(|B_t| > r(t)) = 0$ if

$$r(t) = (t+1)^\beta, \quad \beta > \frac{1}{2}. \tag{4.36}$$

Indeed, the sphere function satisfying (4.36) is an *upper radius* of $\mathbf{M}^{(0)}$, that is, $B_t \in B(0, r(t))$ for all large enough t with probability 1, in view of the law of the iterated logarithm ([3]).

Let $a(t)$ be a non-negative function such that $a(0) = 1$ and $\underline{a}(t) \le a(t) \le \underline{a}(t)^{-1}$ for some non-increasing positive function $\underline{a}(t)$, $(t \ge 0)$.

Example 4.2. Suppose that $a_{ij}^{(t)}(x) = \frac{1}{2}a(t)\delta_{ij}$ for $1 \le i,j \le d$. Then the corresponding diffusion process of (1.1) is the time changed Brownian motion $\mathbf{M} = \{B_{c(t)}, \mathbf{P}_{(s,x)}\}$ with

$c(t) = \int_0^t a(s)ds$ ([6]). Assume that $d = 1$ or 2 (i.e., $\mathbf{M}^{(0)}$ is Harris recurrent). If $d = 1$, then $A(t) \geq k_8\underline{a}(t)(r(t) - \ell)$ and the condition (3.30) is satisfied when $\underline{a}(t) = r(t)^{-\varepsilon}, 0 < \varepsilon < 1/2$. Take $r(t) = (t+1)^\beta, 0 < \beta < 1/(1+\varepsilon)$. Then the conditions (3.27) and (3.29) are satisfied. In this case, the condition (3.21) holds for $0 < \beta < 1/(2+\varepsilon)$ (also for $\beta = 1/(2+\varepsilon), 1 < (2+\varepsilon)k_0$). Therefore, the space-time domain Γ_B given by its section $B(0, r(t))$ with $r(t) = (t+1)^\beta, (0 < \beta < 1/(2+\varepsilon))$ is to be a non-favorite recurrent set of $\mathbf{M}$.

In a similar way, we can also see when $d = 2$ that if

$$\underline{a}(t) = (\log r(t))^{-\varepsilon}, \quad 0 < \varepsilon < \frac{1}{2} \quad \text{and} \quad r(t) = (t+1)^\beta, \quad 0 < \beta < \frac{1-\varepsilon}{2},$$

all conditions for Theorem 3.1 and Theorem 3.2 are fulfilled and thus the same behaviour of $B_{c(t)}$ to $B(0, r(t))$ with $r(t) = (t+1)^\beta, 0 < \beta < (1-\varepsilon)/2$ also holds.

Example 4.3. Suppose that $a_{ij}^{(t)}(x) = \frac{1}{2}a(t)\delta_{ij}|x|^{-p}$, $(|x| > \ell, p > 0, d \leq 2+p)$. Then the corresponding process of (1.1) is a diffusion process $\mathbf{M} = \{X_t, \mathbf{P}_{(s,x)}\}$ with polynomially decreasing potential. In this case, since $\overline{a}^{(t)}(r) = k_9 a(t) r^{-p}$,

$$\int_\ell^\infty \overline{a}^{(t)}(r)^{-1} r^{1-d} dr = a(t)^{-1} \int_\ell^\infty r^{1-d+p} dr = \infty \tag{4.37}$$

for each fixed $t \geq 0$. Therefore $\mathbf{M}^{(0)}$ is Harris recurrent from Ichihara's test. Similarly to (4.37), the condition (3.30) is satisfied by employing $\underline{a}(t) = r(t)^{-(2-d+p)\varepsilon}$ for $0 < \varepsilon < 1/2$. Thus if we choose $r(t) = (t+1)^\beta, (0 < \beta < 1/(d + (2-d-p)\varepsilon))$, then the conditions (3.27) and (3.29) are satisfied. In particular, the condition (3.21) holds for $0 < \beta < 1/(2 + (2-d-p)\varepsilon)$ (also for $\beta = 1/(2+(2-d-p)\varepsilon), d < (2+(2-d-p)\varepsilon)k_0$). Hence the same behaviour of X_t to $B(0, r(t))$ with

$$r(t) = (t+1)^\beta, \qquad 0 < \beta < \frac{1}{2 + (2-d-p)\varepsilon}$$

also holds as like in the previous examples.

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