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Logarithmic derivatives of densities for jump processes

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Abstract

The purpose of this paper is to study the sensitivity analysis for jump-type stochastic differential equations under the condition on the Lévy measure, and the Hörmander type condition on the coefficients. Our approach is based on the martingale property via the Kolmogorov backward equation for the integro-differential operator associated with the equation.

1 Introduction

Malliavin introduced the stochastic calculus of variations in order to exhibit the probabilistic proof of the hypoelliptic problem for differential operators (cf. [18]). The integration-by-parts formula plays a key role in the Malliavin calculus, and the formula over a probability space can be also established via the Girsanov transform on Brownian motions (cf. [3]). In [12], the Malliavin calculus on the Wiener space was applied to the Greeks computations for an asset price dynamics.

Recently, various types on the Malliavin calculus for jump processes have been introduced by many authors over the Poisson space, or the Wiener-Poisson space. See [1, 2, 4, 5, 9, 10, 17, 19] for details. The measure change technique for jump processes first found by [4] enables us to obtain that the uniformly elliptic condition, or the Hörmander type condition on the coefficients of stochastic differential equations yields the existence of smooth densities for the solution. Here the Hörmander type condition is the condition on the linear subspace generated by the coefficients, the Lie brackets of them, and the integrals of the jump term effects. See [16, 20] for details. Furthermore, there are a lot of works in which the Malliavin calculus for jump processes are applied to the sensitivity analysis in mathematical finance (cf. [1, 7, 8, 10]). This can be also regarded as the logarithmic derivatives of the density with respect to various parameters in certain sense. Although the process discussed in those works has jumps, most of them are focused on the effect only from the diffusion terms. In [21], the sensitivities for jump processes determined by stochastic differential equations are studied under the uniformly elliptic condition on the diffusion and the jump terms. Then, it is a natural question whether a similar problem can be studied in the hypoelliptic situation.

In this paper, we shall study the sensitivity analysis for the solution to the stochastic differential equation with jumps via the martingale approach based upon the Kolmogorov backward equation for the associated infinitesimal generator, in the hypoelliptic situation, that is, the case where the coefficients of the equation satisfy the Hörmander type condition. The result obtained in the present paper includes the effects from not only the diffusion terms, but also the jump terms. Moreover, the equation can be of a pure-jump type, and an infinite activity type.

The paper is organized as follows: Section 2 is devoted to the introduction of our framework, and the criterion on the existence of smooth densities as stated in [16, 20]. In Section 3, the sensitivity analysis with respect to the initial point of the equation is investigated in the hypoelliptic situation, which will be proved in Section 5. Some key lemmas in order to prove the main result are given in Section 4, and the example is exhibited in Section 6.

In the whole sequel, we shal denote the $\alpha \times \beta$-zero matrix by $\mathbf{0}_{\alpha,\beta} \in \mathbb{R}^{\beta} \otimes \mathbb{R}^{\alpha}$, and the identity by $I_{\gamma} \in \mathbb{R}^{\gamma} \otimes \mathbb{R}^{\gamma}$. C_b^k denotes the family of k-times differentiable functions with bounded derivatives of any orders more than 1. The symbols ∇, ∇_x, ∇_X and ∂_θ indicate the gradient operators. Define the mapping $\pi : \mathbb{R}^{\ell} \otimes \mathbb{R}^{d} \longrightarrow \mathbb{R}^{d\ell}$ by

$$\pi(A) = \begin{pmatrix} \pi_1(A) \\ \vdots \\ \pi_\ell(A) \end{pmatrix}, \quad \pi_k(A) = \begin{pmatrix} A_{1,k} \\ \vdots \\ A_{d,k} \end{pmatrix} \ (1 \le k \le \ell)$$

for $A = (A_{j,k})_{1 \le j \le d, 1 \le k \le \ell} \in \mathbb{R}^{\ell} \otimes \mathbb{R}^{d}$.

2 Preliminaries

Fix $T > 0$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and $d\nu$ the Lévy measure over $\mathbb{R}_0 := \mathbb{R} \setminus \{0\}$ such that

Assumption 1. *(a) for any $p \ge 1$,*

$$\int_{|\theta| \le 1} |\theta| \, d\nu + \int_{|\theta| > 1} |\theta|^p \, d\nu < +\infty,$$

(b) there exists a constant $\delta > 0$ such that

$$\liminf_{\rho \searrow 0} \rho^{\delta} \int_{\mathbb{R}_0} \left(|\theta/\rho|^2 \wedge 1 \right) d\nu > 0,$$

(c) there exists a C^1-density $g(\theta)$ with respect to the Lebesgue measure over $\mathbb{R}_0$ such that

$$\lim_{|\theta|\to+\infty} |g(\theta)| = 0.$$

Example 1. The Lévy measures of tempered stable processes, inverse Gaussian processes and CGMY processes (cf. [6]) satisfy Assumption 1. □

Remark 1. In order to study the existence of (smooth) densities, the following condition is assumed in [14, 19].

(d) there exists $0 < \alpha < 2$ such that

$$\liminf_{\rho\searrow 0} \rho^{-\alpha} \int_{|\theta|\le\rho} |\theta|^2 \, d\nu > 0.$$

We can check that the condition (d) implies (b) in Assumption 1. □

Let $\{W_t\}_{t\in[0,T]}$ be a 1-dimensional Brownian motion with $W_0 = 0$, and dJ a Poisson random measure over $[0,T] \times \mathbb{R}_0$ with the intensity $d\hat{J} = dt\, d\nu$. Denote by $\{\mathcal{F}_t\}_{t\in[0,T]}$ the augmented filtration generated by $\{W_t\}_{t\in[0,T]}$ and dJ. For simplicity of notations, write $d\tilde{J} = dJ - d\hat{J}$ and $d\overline{J} = I_{(|\theta|\le 1)}\, d\tilde{J} + I_{(|\theta|>1)}\, dJ$. Let $a_0(x)$, $a_1(x) \in C_b^{1,\infty}(\mathbb{R}^d;\mathbb{R}^d)$, and $b_\theta(x) \in C_b^{1,\infty}(\mathbb{R}^d \times \mathbb{R}_0;\mathbb{R}^d)$ such that

$$\inf_{x\in\mathbb{R}^d} \inf_{\theta\in\mathbb{R}_0} \left|\det \nabla \bar{b}_\theta(x)\right| > 0, \quad \lim_{|\theta|\searrow 0} b_\theta(x) = 0,$$

where $\bar{b}_\theta(x) = x + b_\theta(x)$. For $x \in \mathbb{R}^d$, consider the $\mathbb{R}^d$-valued process $\{x_t\,(\equiv x_t^x)\}_{t\in[0,T]}$ determined by the stochastic differential equation (SDE):

$$dx_t = a_0(x_t)\, dt + a_1(x_t) \circ dW_t + \int_{\mathbb{R}_0} b_\theta(x_{t-})\, d\overline{J}, \quad x_0 = x. \tag{2.1}$$

Under the conditions on the coefficients, there exists a unique solution to (2.1), and the associated infinitesimal generator $\mathcal{L}$ is

$$\mathcal{L}f = \mathcal{A}_0 f + \frac{1}{2}\mathcal{A}_1\mathcal{A}_1 f + \int_{\mathbb{R}_0} \left\{(f \circ \bar{b}_\theta) - f - I_{(|\theta|\le 1)}\, \mathcal{B}_\theta f\right\} d\nu,$$

where $\mathcal{A}_i = a_i(x) \cdot \nabla$ and $\mathcal{B}_\theta = b_\theta(x) \cdot \nabla$ are vector fields over $\mathbb{R}^d$.

For $y, z \in \mathbb{R}^d \otimes \mathbb{R}^d$ with $\det y \neq 0$, $\det z \neq 0$, let $\{y_t\,(\equiv y_t^{x,y})\}_{t\in[0,T]}$ and $\{z_t\,(\equiv z_t^{x,z})\}_{t\in[0,T]}$ be the $\mathbb{R}^d \otimes \mathbb{R}^d$-valued processes determined by the linear SDEs: $y_0 = y$, $z_0 = z$ and

$$dy_t = \nabla a_0(x_t)\, y_t\, dt + \nabla a_1(x_t)\, y_t \circ dW_t + \int_{\mathbb{R}_0} \nabla b_\theta(x_{t-})\, y_{t-}\, d\overline{J}, \tag{2.2}$$

$$
\begin{aligned}
dz_t = &-z_t \left\{ \nabla a_0(x_t) - \int_{|\theta|\le 1} \left(\nabla \bar{b}_\theta(x_t)\right)^{-1} \left(\nabla b_\theta(x_t)\right)^2 d\nu \right\} dt \\
&- z_t \, \nabla a_1(x_t) \circ dW_t - \int_{\mathbb{R}_0} z_{t-} \left(\nabla \bar{b}_\theta(x_{t-})\right)^{-1} \nabla b_\theta(x_{t-}) \, d\bar{J}.
\end{aligned} \tag{2.3}
$$

Then, for each $t \in [0,T]$, the mapping $\mathbb{R}^d \ni x \longmapsto x_t^x \in \mathbb{R}^d$ has a C^1-modification such that $\nabla_x x_t^x = y_t^{x,I_d}$ and $y_t^{x,z^{-1}} z_t^{x,z} = I_d$. For $v \in \mathbb{R}^d \otimes \mathbb{R}^d$, let $\{v_t \, (\equiv v_t^{x,z,v})\}_{t\in[0,T]}$ be the $\mathbb{R}^d \otimes \mathbb{R}^d$-valued process defined by the following SDE: $v_0 = v$ and

$$
\begin{aligned}
dv_t = &\nabla a_0(x_t) \, v_t \, dt + \nabla a_1(x_t) \, v_t \circ dW_t + \int_{\mathbb{R}_0} \nabla b_\theta(x_{t-}) \, v_{t-} \, d\bar{J} \\
&+ a_1(x_t) \, a_1(x_t)^* \, z_t^* dt + \int_{\mathbb{R}_0} \nabla \bar{b}_\theta(x_{t-}) \, \tilde{b}_\theta(x_{t-}) \, \tilde{b}_\theta(x_{t-})^* \, z_{t-}^* dJ,
\end{aligned} \tag{2.4}
$$

where $\tilde{b}_\theta(x) = \left(\nabla \bar{b}_\theta(x)\right)^{-1} \partial_\theta b_\theta(x) \, \theta$. Then, it can be easily checked that

$$
\begin{aligned}
v_t^{x,z,v} = y_t^{x,z^{-1}} \Big\{ zv &+ \int_0^t z_s^{x,z} \, a_1(x_s^x) \, a_1(x_s^x)^* \, (z_s^{x,z})^* ds \\
&+ \int_0^t \int_{\mathbb{R}_0} z_{s-}^{x,z} \, \tilde{b}_\theta(x_{s-}^x) \, \tilde{b}_\theta(x_{s-}^x)^* \, (z_{s-}^{x,z})^* dJ \Big\}.
\end{aligned}
$$

We shall introduce the criterion on the existence of smooth densities for solutions to SDEs (cf. [16, 20]). Write $\Theta = \{0,1\} \cup \mathbb{R}_0$, and define

$$
\begin{gathered}
\tilde{a}_0(x) = a_0(x) + \frac{1}{2} \mathcal{A}_1 a_1(x), \quad d\mu = d\delta_{\{0\}} + d\delta_{\{1\}} + d\nu, \\
[\varphi, \psi](x) = \nabla\psi(x) \, \varphi(x) - \nabla\varphi(x) \, \psi(x)
\end{gathered}
$$

for $\varphi, \psi \in C^1(\mathbb{R}^d; \mathbb{R}^d)$. Denote the families of $\mathbb{R}^d$-valued functions on $\mathbb{R}^d$ by

$$
\mathcal{V}_0 = \left\{ a_1, \, \tilde{b}_\theta \, ; \, \theta \in \mathbb{R}_0 \right\}, \quad \mathcal{V}_k = \left\{ \wp_\theta \varphi \, ; \, \varphi \in \mathcal{V}_{k-1}, \; \theta \in \Theta \right\} \; (k \ge 1),
$$

and define

$$
\wp_\theta \varphi = I_{(\theta=0)}[\tilde{a}_0, \varphi] + I_{(\theta=1)}[a_1, \varphi] + I_{(\theta\in\mathbb{R}_0)} \left\{ \left(\nabla \bar{b}_\theta\right)^{-1} (\varphi \circ \bar{b}_\theta) - \varphi \right\}.
$$

Fact 1 (cf. [16, 20]). *Suppose that the measure $d\nu$ satisfies Assumption 1. If the coefficients of* (2.1) *satisfy the Hörmander type conditions, that is, there exist a positive*

constant ι and a non-negative integer n such that

$$\liminf_{\rho \searrow 0} \rho^{\iota} \sum_{k=0}^{n} \sum_{\varphi \in \mathcal{V}_k} \int_{\Theta^k} \left\{ (v \cdot \wp_{\theta_k} \cdots \wp_{\theta_1} \varphi(x)/\rho)^2 \wedge 1 \right\} \, d\mu^{\otimes k} > 0 \tag{2.5}$$

for any $x \in \mathbb{R}^d$ and $v \in \mathbb{S}^{d-1}$, then the probability law of x_T^x has a density $p_T(x, \tilde{x})$ with respect to the Lebesgue measure over $\mathbb{R}^d$ such that the function $\mathbb{R}^d \ni \tilde{x} \longmapsto p_T(x, \tilde{x})$ is smooth.

3 Main result

In this section, we shall present the sensitivity formula with respect to $x \in \mathbb{R}^d$. For $z, v \in \mathbb{R}^d \otimes \mathbb{R}^d$ with $\det z \neq 0$, write

$$X = \begin{pmatrix} x \\ \pi(z^*) \\ \pi(v) \end{pmatrix}, \quad \overline{X} = \begin{pmatrix} x \\ \pi(I_d) \\ \pi(\mathbf{0}_{d,d}) \end{pmatrix}, \quad X_t (\equiv X_t^X) = \begin{pmatrix} x_t^x \\ \pi\big((z_t^{x,z})^*\big) \\ \pi(v_t^{x,z,v}) \end{pmatrix}.$$

Define

$$\tilde{A}_0(X) = \begin{pmatrix} \mathbf{0}_{d,1} \\ \pi\Big(\Big(\int_{|\theta| \le 1} z \, (\nabla \bar{b}_\theta(x))^{-1} (\nabla b_\theta(x))^2 \, d\nu \Big)^* \Big) \\ \pi\Big(a_1(x) \, a_1(x)^* \, z^* + \int_{|\theta| \le 1} (\nabla \bar{b}_\theta(x)) \, \tilde{b}_\theta(x) \, \tilde{b}_\theta(x)^* \, d\nu \, z^* \Big) \end{pmatrix},$$

$$A_i(X) = \begin{pmatrix} a_i(x) \\ -\pi\Big(\big(z \, \nabla a_i(x) \big)^* \Big) \\ \pi\big(\nabla a_i(x) \, v \big) \end{pmatrix} \quad (i = 0, 1),$$

$$B_\theta(X) = \begin{pmatrix} b_\theta(x) \\ -\pi\Big(\big(z \, (\nabla \bar{b}_\theta(x))^{-1} \nabla b_\theta(x) \big)^* \Big) \\ \pi\Big(\nabla b_\theta(x) \, v + (\nabla \bar{b}_\theta(x)) \, \tilde{b}_\theta(x) \, \tilde{b}_\theta(x)^* \, z^* \Big) \end{pmatrix}.$$

Let $N = d + 2d^2$. Then, the $\mathbb{R}^N$-valued process $\{ X_t \, (\equiv X_t^X) \}_{t \in [0,T]}$ satisfies the equation of the form: $X_0 = X$ and

$$dX_t = \big(A_0(X_t) + \tilde{A}_0(X_t) \big) dt + A_1(X_t) \circ dW_t + \int_{\mathbb{R}_0} B_\theta(X_{t-}) \, d\overline{J}.$$

Moreover, for each $t \in [0,T]$, the mapping $\mathbb{R}^N \ni X \longmapsto X_t^X \in \mathbb{R}^N$ has a C^1-modification, and its Jacobi matrix

$$Y_t := \nabla_X X_t^X = \begin{pmatrix} y_t^{x,z^{-1}} & \mathbf{0}_{d,d^2} & \mathbf{0}_{d,d^2} \\ \nabla_x \pi\big((z_t^{x,z})^*\big) & \partial_{\pi(z^*)} \pi\big((z_t^{x,z})^*\big) & \mathbf{0}_{d^2,d^2} \\ \nabla_x \pi(v_t^{x,z,v}) & \partial_{\pi(z^*)} \pi(v_t^{x,z,v}) & \partial_{\pi(v)} \pi(v_t^{x,z,v}) \end{pmatrix}$$

is invertible, because of the condition on b: $\inf_{x,\theta} \left|\det \nabla \bar{b}_\theta(x)\right| > 0$. Denote the inverse matrix of Y_t by Z_t, which can be computed as follows:

$$Z_t = \begin{pmatrix} z_t^{x,z} & \mathbf{0}_{d,d^2} & \mathbf{0}_{d,d^2} \\ -\nabla_x \pi\big((z_t^{x,z})^*\big) & \big\{\partial_{\pi(z^*)} \pi\big((z_t^{x,z})^*\big)\big\}^{-1} & \mathbf{0}_{d^2,d^2} \\ -\nabla_x \pi(v_t^{x,z,v}) & -\partial_{\pi(z^*)} \pi(v_t^{x,z,v}) & \big\{\partial_{\pi(v)} \pi(v_t^{x,z,v})\big\}^{-1} \end{pmatrix}$$
$$+ \begin{pmatrix} \mathbf{0}_{d,d} & \mathbf{0}_{d,d^2} & \mathbf{0}_{d,d^2} \\ \mathbf{0}_{d^2,d} & \mathbf{0}_{d^2,d^2} & \mathbf{0}_{d^2,d^2} \\ \partial_{\pi(z^*)} \pi(v_t^{x,z,v})\, \nabla_x \pi\big((z_t^{x,z})^*\big) & \mathbf{0}_{d^2,d^2} & \mathbf{0}_{d^2,d^2} \end{pmatrix}.$$

Let $\{V_t\}_{t\in[0,T]}$ be the $\mathbb{R}^N \otimes \mathbb{R}^N$-valued process determined by the SDE: $V_0 = \mathbf{0}_{N,N}$ and

$$\begin{aligned} dV_t = {} & \big(\nabla A_0(X_t) + \nabla \tilde{A}_0(X_t)\big)\, V_t\, dt + \nabla A_1(X_t)\, V_t \circ dW_t \\ & + \int_{\mathbb{R}_0} \nabla B_\theta(X_{t-})\, V_{t-}\, d\bar{J} + A_1(X_t)\, A_1(X_t)^*\, Z_t^* dt \\ & + \int_{\mathbb{R}_0} \nabla \overline{B}_\theta(X_{t-})\, \tilde{B}_\theta(X_{t-})\, \tilde{B}_\theta(X_{t-})^*\, Z_{t-}^* dJ, \end{aligned} \tag{3.1}$$

where $\overline{B}_\theta(X) = X + B_\theta(X)$, and $\tilde{B}_\theta(X) = \big(\nabla \overline{B}_\theta(X)\big)^{-1} \partial_\theta B_\theta(X)\, \theta$. Then, the Itô formula enables us to see that

$$\begin{aligned} V_t = Y_t \Big\{ & \int_0^t Z_s A_1(X_s) A_1(X_s)^* Z_s^* ds \\ & + \int_0^t \int_{\mathbb{R}_0} Z_{s-} \tilde{B}_\theta(X_{s-}) \tilde{B}_\theta(X_{s-})^* Z_{s-}^* dJ \Big\}. \end{aligned}$$

Define V_t^{11}, V_t^{21} and V_t^{31} by

$$V_t\big|_{X=\overline{X}} \begin{pmatrix} I_d \\ \mathbf{0}_{2d^2,d} \end{pmatrix} = \begin{pmatrix} V_t^{11} \\ V_t^{21} \\ V_t^{31} \end{pmatrix}.$$

Denote by $\mathcal{U}$ the family of bounded domains and their complements in $\mathbb{R}^d$. Define the classes $C_{LG}(\mathbb{R}^d)$ and $\mathfrak{F}(\mathbb{R}^d)$ of $\mathbb{R}$-valued functions by

$$C_{LG}(\mathbb{R}^d) = \left\{ f \in C(\mathbb{R}^d) \, ; \, |f(x)| \leq const. \, (1 + |x|) \right\},$$

$$\mathfrak{F}(\mathbb{R}^d) = \left\{ f = \sum_{k=1}^{n} \alpha_k f_k I_{A_k} \, ; \, n \in \mathbb{N}, \, \alpha_k \in \mathbb{R}, \, f_k \in C_{LG}(\mathbb{R}^d), \, A_k \in \mathcal{U} \right\}.$$

Theorem 1. *Suppose that the Hörmander type condition* (2.5) *stated in Fact 1 is satisfied. Then, for* $\varphi \in \mathfrak{F}(\mathbb{R}^d)$, *it holds that*

$$\nabla_z \left(\mathbb{E} \left[\varphi(x_T) \right] \right) = \mathbb{E} \left[\varphi(x_T) \, \Gamma_T \big|_{X=\overline{X}} \right],$$

where $\Gamma_T = (\Gamma_T^1, \ldots, \Gamma_T^d)$, $\mathfrak{m}_T = (\mathfrak{m}_T^1, \ldots, \mathfrak{m}_T^d)$ *and*

$$\mathfrak{m}_T = \left(\int_0^T z_t \, a_1(x_t) \, dW_t \right)^* - \int_0^T \int_{\mathbb{R}_0} \frac{\partial_\theta [g(\theta) \, \theta \, \tilde{b}_\theta(x_{t-})^* \, z_{t-}^*]}{g(\theta)} \, d\tilde{J}$$

$$\Gamma_T^k = [\mathfrak{m}_T \, v_T^{-1}]_\kappa + \sum_{\alpha,\beta=1}^{d} [V_T^{21} v_T^{-1} y_T]_{(\beta-1)d+\alpha,\beta} \, [y_T]_{\alpha,k}$$

$$+ \sum_{\alpha,\beta=1}^{d} [v_T^{-1}]_{\beta,\alpha} \, [V_T^{31}]_{(\beta-1)d+\alpha,\beta} \, [v_T^{-1} y_T]_{\beta,k}.$$

Remark 2. (1) Although a similar result to Theorem 1 can be also obtained via the Girsanov transform approach (cf. [4, 16, 20]), or the Malliavin calculus on the Wiener space (cf. [18]), most of them are paid attention to only the diffusion term. Our formula in Theorem 1 is stated in terms of not only the diffusion term, but also the jump term.

(2) Similarly to Theorem 1, the sensitivities in the other parameters which govern the process can be studied. This will be discussed elsewhere.

4 Key lemmas

In this section, we shall prepare some key lemmas, which will play an important role in the proof of Theorem 1. Write

$$X_t = X_t^X, \quad Y_t = Y_t^{X, I_N}, \quad Z_t = Z_t^{X, I_N}, \quad V_t = V_t^{X, I_N, 0_{N,N}}.$$

Let $\Phi \in C^2(\mathbb{R}^N)$ with compact support. For $t \in [0,T]$ and $\tilde{X} \in \mathbb{R}^N$, write $U(t,\tilde{X}) = \mathbb{E}[\Phi(X_{T-t})|X_0 = \tilde{X}]$. Define the operator $\mathfrak{B}_\theta$ by

$$\mathfrak{B}_\theta \Psi(X) := \Psi(\overline{B}_\theta(X)) - \Psi(X).$$

Then, the Kolmogorov backward equation (cf. [13]) implies that

Lemma 4.1 (cf. [21], Lemma 4.1). *For $\Phi \in C^2(\mathbb{R}^N)$ with compact support, the following equality holds.*

$$\begin{aligned}\Phi(X_T) = \mathbb{E}[\Phi(X_T)] &+ \int_0^T \nabla U(s, X_s) A_1(X_s) dW_s \\ &+ \int_0^T \int_{\mathbb{R}_0} \mathfrak{B}_\theta U(s, X_{s-}) d\tilde{J}.\end{aligned} \tag{4.1}$$

Lemma 4.1 helps us to obtain the key equalities as stated below, which can be regarded as the integration by parts formula.

Lemma 4.2. *For $\Phi \in C^2(\mathbb{R}^N)$ with compact support, it holds that*

$$\begin{aligned}&\mathbb{E}\left[\nabla_X(\Phi(X_T)) \int_0^T Z_s A_1(X_s) A_1(X_s)^* Z_s^* ds\right] \\ &= \mathbb{E}\left[\Phi(X_T) \left(\int_0^T Z_s\, A_1(X_s)\, dW_s\right)^*\right].\end{aligned} \tag{4.2}$$

Proof. Multiplying both sides of the equality (4.1) in Lemma 4.1 by $\left(\int_0^T Z_s\, A_1(X_s)\, dW_s\right)^*$, we see that

$$\begin{aligned}&\mathbb{E}\left[\Phi(X_T) \left(\int_0^T Z_s\, A_1(X_s)\, dW_s\right)^*\right] \\ &= \mathbb{E}\left[\int_0^T \nabla U(s, X_s)\, A_1(X_s)\, dW_s \left(\int_0^T Z_s\, A_1(X_s)\, dW_s\right)^*\right] \\ &= \int_0^T \mathbb{E}\left[\nabla_X(U(s, X_s))\; Z_s\, A_1(X_s)\, A_1(X_s)^*\, Z_s^*\right] ds \\ &= \mathbb{E}\left[\nabla_X(\Phi(X_T)) \int_0^T Z_s\, A_1(X_s)\, A_1(X_s)^*\, Z_s^*\, ds\right].\end{aligned}$$

□

Lemma 4.3. *For $\Phi \in C^2(\mathbb{R}^N)$ with compact support, it holds that*

$$\begin{aligned} &\mathbb{E}\left[\nabla_X \Phi(X_T) \int_0^T \int_{\mathbb{R}_0} Z_{s-} \tilde{B}_\theta(X_{s-}) \tilde{B}_\theta(X_{s-})^* Z_{s-}^* dJ\right] \\ &= -\mathbb{E}\left[\Phi(X_T) \int_0^T \int_{\mathbb{R}_0} \frac{\partial_\theta [g(\theta)\, \theta\, \tilde{B}_\theta(X_{s-})^*\, Z_{s-}^*]}{g(\theta)} d\tilde{J}\right]. \end{aligned} \tag{4.3}$$

Proof. Write

$$\begin{aligned} M_T &= \int_0^T \int_{\mathbb{R}_0} Z_{s-}\, \tilde{B}_\theta(X_{s-})\, \tilde{B}_\theta(X_{s-})^*\, Z_{s-}^*\, dJ, \\ \hat{M}_T &= \int_0^T \int_{\mathbb{R}_0} Z_s\, \tilde{B}_\theta(X_s)\, \tilde{B}_\theta(X_s)^*\, Z_s^*\, d\hat{J}. \end{aligned}$$

Since

$$\mathbb{E}\left[\Phi(X_T)\, \hat{M}_T\right] = \mathbb{E}\left[\int_0^T \int_{\mathbb{R}_0} U(s, X_s)\, Z_s\, \tilde{B}_\theta(X_s)\, \tilde{B}_\theta(X_s)^*\, Z_s^*\, d\hat{J}\right],$$

multiplying both sides of the equality (4.1) in Lemma 4.1 by $M_T - \hat{M}_T$ enables us to see that

$$\begin{aligned} \mathbb{E}\left[\Phi(X_T)\, M_T\right] &= \mathbb{E}\left[\Phi(X_T)\, (M_T - \hat{M}_T + \hat{M}_T)\right] \\ &= \mathbb{E}\left[\int_0^T \int_{\mathbb{R}_0} \mathfrak{B}_\theta U(s, X_s)\, Z_s\, \tilde{B}_\theta(X_s)\, \tilde{B}_\theta(X_s)^*\, Z_s^*\, d\hat{J}\right] \\ &\quad + \mathbb{E}\left[\int_0^T \int_{\mathbb{R}_0} U(s, X_s)\, Z_s\, \tilde{B}_\theta(X_s)\, \tilde{B}_\theta(X_s)^*\, Z_s^*\, d\hat{J}\right] \\ &= \mathbb{E}\left[\int_0^T \int_{\mathbb{R}_0} U(s, \overline{B}_\theta(X_s))\, Z_s\, \tilde{B}_\theta(X_s)\, \tilde{B}_\theta(X_s)^*\, Z_s^*\, d\hat{J}\right]. \end{aligned}$$

Taking the derivative in $X \in \mathbb{R}^N$ yields that the right hand side is equal to

$$\begin{aligned} &\nabla_X \left(\mathbb{E}\left[\int_0^T \int_{\mathbb{R}_0} U(s, \overline{B}_\theta(X_s))\, Z_s\, \tilde{B}_\theta(X_s)\, \tilde{B}_\theta(X_s)^*\, Z_s^*\, d\hat{J}\right]\right) \\ &= \mathbb{E}\left[\int_0^T \int_{\mathbb{R}_0} \nabla U(s, \overline{B}_\theta(X_s))\, \nabla \overline{B}_\theta(X_s)\, \tilde{B}_\theta(X_s)\, \tilde{B}_\theta(X_s)^*\, Z_s^*\, d\hat{J}\right] \\ &\quad + \mathbb{E}\left[\int_0^T \int_{\mathbb{R}_0} U(s, \overline{B}_\theta(X_s))\, \nabla_X \left(Z_s\, \tilde{B}_\theta(X_s)\, \tilde{B}_\theta(X_s)^*\, Z_s^*\right) d\hat{J}\right] \\ &= -\mathbb{E}\left[\int_0^T \int_{\mathbb{R}_0} \mathfrak{B}_\theta U(s, X_s)\, \partial_\theta [g(\theta)\, \theta\, \tilde{B}_\theta(X_s)^*\, Z_s^*]\, d\theta\, ds\right] \end{aligned}$$

$$+\mathbb{E}\left[\int_0^T\int_{\mathbb{R}_0}\{\mathfrak{B}_\theta U(s,X_s)+U(s,X_s)\}\,\nabla_X\left(Z_s\,\tilde{B}_\theta(X_s)\,\tilde{B}_\theta(X_s)^*\,Z_s^*\right)d\hat{J}\right]$$
$$=-\mathbb{E}\left[\Phi(X_T)\int_0^T\int_{\mathbb{R}_0}\frac{\partial_\theta[g(\theta)\,\theta\,\tilde{B}_\theta(X_{s-})^*\,Z_{s-}^*]}{g(\theta)}\,d\tilde{J}\right]+\mathbb{E}\left[\Phi(X_T)\,\nabla_X M_T\right].$$

Here we have used the integration-by-parts formula via Assumption 1 (iii) in the second equality, and (4.1) of Lemma 4.1 in the third equality. On the other hand, the left hand side is equal to

$$\nabla_X\mathbb{E}\left[\Phi(X_T)M_T\right]=\mathbb{E}\left[\nabla_X\Phi(X_T)\,M_T\right]+\mathbb{E}\left[\Phi(X_T)\,\nabla_X M_T\right].$$

Therefore, we can get the assertion. □

Combining Lemma 4.2 and Lemma 4.3, we have

Corollary 4.1. *For $\Phi\in C^2\left(\mathbb{R}^N\right)$ with compact support, it holds that*

$$\mathbb{E}\left[\nabla_X\Phi(X_T)\,Z_T\,V_T\right]=\mathbb{E}\left[\Phi(X_T)\,\mathfrak{M}_T\right]. \tag{4.4}$$

where

$$\mathfrak{M}_T:=\left(\int_0^T Z_t A_1(X_t)dW_t\right)^*-\int_0^T\int_{\mathbb{R}_0}\frac{\partial_\theta[g(\theta)\theta\tilde{B}_\theta(X_{t-})^*Z_{t-}^*]}{g(\theta)}d\tilde{J}.$$

5 Proof of Theorem 1

In this section, we shall prove Theorem 1. Since the probability law of x_T^x admits a smooth density with respect to the Lebesgue measure over $\mathbb{R}^d$ from Fact 1, it is sufficient to study the case of $\varphi\in C^2\left(\mathbb{R}^d\right)$ with compact support via the standard density argument as stated in [7, 15, 21], instead of $\varphi\in\mathfrak{F}\left(\mathbb{R}^d\right)$.

Let $\Phi\in C^2\left(\mathbb{R}^N\right)$ with compact support. Multiplying both sides of (4.4) in Corollary 4.1 by $(I_d,\,\mathbf{0}_{d,d^2},\,\mathbf{0}_{d,d^2})^*\in\mathbb{R}^d\otimes\mathbb{R}^N$, we have

$$\begin{aligned}\mathbb{E}\left[\nabla\Phi(X_T)\,(V_T^{11*},\,V_T^{21*},\,V_T^{31*})^*\right]&=\mathbb{E}\left[\Phi(X_T)\,\mathfrak{M}_T\,(I_d,\,\mathbf{0}_{d,2d^2})^*\big|_{X=\overline{X}}\right]\\&=\mathbb{E}\left[\Phi(X_T)\,\mathrm{m}_T\,v_T^{-1}\big|_{X=\overline{X}}\right].\end{aligned}$$

Take $\Phi(X)=\varphi(x)\,[v^{-1}z^{-1}]_{j,k}$ $(1\le j,k\le d)$. Remark that, for $1\le\alpha,\beta\le d$,

$$\begin{aligned}\nabla_{x_\alpha}\Phi(X)&=\nabla_\alpha\varphi(x)\,[v^{-1}z^{-1}]_{j,k},\\\partial_{[z^*]_{\alpha,\beta}}\Phi(X)&=-\varphi(x)\,[z^{-1}]_{\alpha,k}\,[v^{-1}z^{-1}]_{j,\beta},\end{aligned}$$

$$\partial_{[v]_{\alpha,\beta}} \Phi(X) = -\varphi(x) \, [v^{-1}]_{j,\alpha} \, \delta_j^\beta \, [v^{-1} z^{-1}]_{j,k}.$$

Then, we have

$$\begin{aligned}
&\sum_{j=1}^{d} \mathbb{E}\left[\left[\nabla \Phi(X_T) \, (V_T^{11*}, \, V_T^{21*}, \, V_T^{31*})^* \right]_j \right] \\
&= \sum_{j=1}^{d} \left\{ \mathbb{E}\left[\sum_{\alpha=1}^{d} \nabla_\alpha \varphi(x_T) \, [v_T^{-1} y_T]_{j,k} \, [V_T^{11}]_{\alpha j} \right] \right. \\
&\qquad - \mathbb{E}\left[\varphi(x_T) \sum_{\alpha,\beta=1}^{d} [y_T]_{\alpha,k} \, [v_T^{-1} y_T]_{j,\beta} \, [V_T^{21}]_{(\beta-1)d+\alpha,j} \right] \\
&\qquad \left. - \mathbb{E}\left[\varphi(x_T) \sum_{\alpha,\beta=1}^{d} [v_T^{-1}]_{j,\alpha} \, \delta_j^\beta \, [v_T^{-1} y_T]_{j,k} \, [V_T^{31}]_{(\beta-1)d+\alpha,j} \right] \right\} \Big|_{X=\overline{X}} \\
&= \left\{ \mathbb{E}\left[\left[\nabla \varphi(x_T) \, V_T^{11} \, v_T^{-1} \, y_T \right]_k \right] \right. \\
&\qquad - \mathbb{E}\left[\varphi(x_T) \sum_{\alpha,\beta=1}^{d} [y_T]_{\alpha,k} \, [V_T^{21} v_T^{-1} y_T]_{(\beta-1)d+\alpha,\beta} \right] \\
&\qquad \left. - \mathbb{E}\left[\varphi(x_T) \sum_{\alpha,\beta=1}^{d} [v_T^{-1}]_{\beta,\alpha} \, [v_T^{-1} y_T]_{\beta,k} \, [V_T^{31}]_{(\beta-1)d+\alpha,\beta} \right] \right\} \Big|_{X=\overline{X}}.
\end{aligned}$$

Since $V_T^{11} = v_T$, we can get

$$\begin{aligned}
&\mathbb{E}\left[[\nabla \varphi(x_T) \, y_T]_k \right] \\
&= \mathbb{E}\left[\varphi(x_T) \, [\mathrm{m}_T \, v_T^{-1}]_k \right] \Big|_{X=\overline{X}} \\
&\qquad + \mathbb{E}\left[\varphi(x_T) \sum_{\alpha,\beta=1}^{d} [y_T]_{\alpha,k} \, [V_T^{21} v_T^{-1} y_T]_{(\beta-1)d+\alpha,\beta} \right] \Big|_{X=\overline{X}} \\
&\qquad + \mathbb{E}\left[\varphi(x_T) \sum_{\alpha,\beta=1}^{d} [v_T^{-1}]_{\beta,\alpha} \, [v_T^{-1} y_T]_{\beta,k} \, [V_T^{31}]_{(\beta-1)d+\alpha,\beta} \right] \Big|_{X=\overline{X}} \\
&= \mathbb{E}\left[\varphi(x_T) \, \Gamma_T^k \Big|_{X=\overline{X}} \right],
\end{aligned}$$

which completes the proof. □

6 Example

Let $m = 1$ and $(\gamma, \sigma_1, \sigma_2) \in \mathbb{R} \times (0, +\infty) \times (0, +\infty)$. Suppose that the measure $d\nu$ satisfies Assumption 1. For $\zeta \in \mathbb{R}$, consider the $\mathbb{R}$-valued Lévy process $\{\zeta_t (\equiv \zeta_t^\zeta)\}_{t \in [0,T]}$ given by

$$\zeta_t = \zeta + \gamma\, t + \sigma_1\, W_t + \sigma_2 \int_0^t \int_{\mathbb{R}_0} \theta d\overline{J}. \tag{6.1}$$

Let $f \in C^\infty(\mathbb{R})$ with $f' \neq 0$. For $\eta \in \mathbb{R}$, define the process $\{\eta_t\}_{t \in [0,T]}$ by

$$\eta_t = \eta + \int_0^t f(\zeta_s)\, ds. \tag{6.2}$$

Now, we are in position that

$$x = (\zeta, \eta)^*,\ a_0(x) = (\gamma, f(\zeta))^*,\ a_1(x) = (\sigma_1, 0)^*,\ b_\theta(x) = (\sigma_2 \theta, 0)^*.$$

Since $f' \neq 0$, $\sigma_1 > 0$ and

$$\wp_0 a_1(x) = [a_1, \tilde{a}_0](x) = (0, \sigma_1 f'(\zeta))^*,$$

the probability law of $x_T^x = (\zeta_T^\zeta, \eta_T^\eta)$ admits a smooth density $p_T(x, \tilde{x})$ with respect to the Lebesgue measure on $\mathbb{R}^2$ from Fact 1. Our interest is to study the sensitivity for η_T with respect to $\zeta \in \mathbb{R}$.

In order to get our desired result, we have to compute V_T^{11}, V_T^{21} and V_T^{31} explicitly. Define

$$\Phi[\Psi]_t = \int_0^t \Psi_s\, d\Phi_s, \quad F_t = \int_0^t f'(\zeta_s)\, ds,$$
$$G_t = \sigma_1^2\, t + \int_0^t \int_{\mathbb{R}_0} (\sigma_2\, \theta)^2\, dJ, \quad H = \sigma_1^2 + \int_{|\theta| \leq 1} (\sigma_2 \theta)^2\, d\nu,$$
$$K_t = \int_0^t \int_{\mathbb{R}_0} \sigma_2 \theta^2 dJ, \quad L_t = \int_0^t \int_{\mathbb{R}_0} 2(\sigma_2 \theta)^3 dJ.$$

Then we see that, for $z,\ v \in \mathbb{R}^2 \otimes \mathbb{R}^2$ with $\det z \neq 0$,

$$z_t^{x,z*} = \begin{pmatrix} 1 & -F_t \\ 0 & 1 \end{pmatrix} z^*,$$
$$v_t^{x,z,v} = \begin{pmatrix} 1 & 0 \\ F_t & 1 \end{pmatrix} v + \begin{pmatrix} G_t & -G[F]_t \\ F[G]_t & -F[G[F]]_t \end{pmatrix} z^*.$$

Write

$$X = \begin{pmatrix} x \\ \pi(z^*) \\ \pi(v) \end{pmatrix}, \ \overline{X} = \begin{pmatrix} x \\ \pi(I_2) \\ \pi(\mathbf{0}_{2,2}) \end{pmatrix}, \ X_t(\equiv X_t^X) = \begin{pmatrix} x_t^x \\ \pi(z_t^{x,z*}) \\ \pi(v_t^{x,z,v}) \end{pmatrix}.$$

Then, the process $\{X_t\}_{t\in[0,T]}$ satisfies the following SDE: $X_0 = X$ and

$$dX_t = \big(A_0(X_t) + \tilde{A}_0(X_t)\big)dt + A_1(X_t) \circ dW_t + \int_{\mathbb{R}_0} B_\theta(X_{t-})d\overline{J},$$

where $A_1(X) = (\sigma_1, \mathbf{0}_{1,9})^*$, $\tilde{A}_0(X) = (\mathbf{0}_{1,6}, z^{11}H, 0, z^{21}H, 0)^*$ and

$$\begin{aligned} A_0(X) &= \big(\gamma, f(\zeta), -z^{12}f'(\zeta), 0, -z^{22}f'(\zeta), 0, 0, v^{11}f'(\zeta), 0, v^{12}f'(\zeta)\big)^*, \\ B_\theta(X) &= \big(\sigma_2\theta, \mathbf{0}_{1,5}, z^{11}(\sigma_2\theta)^2, 0, z^{21}(\sigma_2\theta)^2, 0\big)^*. \end{aligned}$$

Then, we have

$$Y_t\big|_{X=\overline{X}} = \nabla_X X_t^X\big|_{X=\overline{X}} = \begin{pmatrix} Y_t^{11} & \mathbf{0}_{2,4} & \mathbf{0}_{2,4} \\ Y_t^{21} & Y_t^{22} & \mathbf{0}_{4,4} \\ Y_t^{31} & Y_t^{32} & Y_t^{33} \end{pmatrix},$$

where

$$Y_t^{11} = \begin{pmatrix} 1 & 0 \\ F_t & 1 \end{pmatrix}, \ Y_t^{21} = \begin{pmatrix} \mathbf{0}_{2,1} & \mathbf{0}_{2,1} \\ -\partial_\zeta F_t & 0 \\ 0 & 0 \end{pmatrix},$$

$$Y_t^{22} = \begin{pmatrix} (Y_t^{11*})^{-1} & \mathbf{0}_{2,2} \\ \mathbf{0}_{2,2} & (Y_t^{11*})^{-1} \end{pmatrix}, \ Y_t^{31} = \begin{pmatrix} 0 & 0 \\ (\partial_\zeta F)[G]_t & 0 \\ -G[\partial_\zeta F]_t & 0 \\ -\partial_\zeta\left(F[G[F]]_t\right) & 0 \end{pmatrix},$$

$$Y_t^{32} = \begin{pmatrix} G_t & -G[F]_t & 0 & 0 \\ F[G]_t & -F[G[F]]_t & 0 & 0 \\ 0 & 0 & G_t & -G[F]_t \\ 0 & 0 & F[G]_t & -F[G[F]]_t \end{pmatrix}, \ Y_t^{33} = \begin{pmatrix} Y_t^{11} & \mathbf{0}_{2,2} \\ \mathbf{0}_{2,2} & Y_t^{11} \end{pmatrix}.$$

Moreover, we see that

$$Z_t = Y_t^{-1}\big|_{X=\overline{X}} = \begin{pmatrix} Z_t^{11} & \mathbf{0}_{2,4} & \mathbf{0}_{2,4} \\ Z_t^{21} & Z_t^{22} & \mathbf{0}_{4,4} \\ Z_t^{31} & Z_t^{32} & Z_t^{33} \end{pmatrix},$$

where

$$Z_t^{11} = \begin{pmatrix} 1 & 0 \\ -F_t & 1 \end{pmatrix}, \quad Z_t^{21} = \begin{pmatrix} \mathbf{0}_{2,1} & \mathbf{0}_{2,1} \\ \partial_\zeta F_t & 0 \\ 0 & 0 \end{pmatrix},$$

$$Z_t^{22} = \begin{pmatrix} (Z_t^{11*})^{-1} & \mathbf{0}_{2,2} \\ \mathbf{0}_{2,2} & (Z_t^{11*})^{-1} \end{pmatrix}, \quad Z_t^{31} = \begin{pmatrix} 0 & 0 \\ -(\partial_\zeta F)[G]_t & 0 \\ -(\partial_\zeta F)[G]_t & 0 \\ 2(\partial_\zeta F)[G[F]]_t & 0 \end{pmatrix},$$

$$Z_t^{32} = \begin{pmatrix} -G_t & G[F]_t & 0 & 0 \\ -F[G]_t & F[G[F]]_t & 0 & 0 \\ 0 & 0 & -G_t & G[F]_t \\ 0 & 0 & -F[G]_t & F[G[F]]_t \end{pmatrix}, \quad Z_t^{33} = \begin{pmatrix} Z_t^{11} & \mathbf{0}_{2,2} \\ \mathbf{0}_{2,2} & Z_t^{11} \end{pmatrix}.$$

Hence, we can get

$$V_T^{11} = Y_T^{11}\int_0^T Z_{t-}^{11}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}(Z_{t-}^{11})^* dG_t = \begin{pmatrix} G_T & -G[F]_T \\ F[G]_T & -F[G[F]]_T \end{pmatrix},$$

$$V_T^{21} = Y_T^{21}\int_0^T Z_{t-}^{11}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}(Z_{t-}^{11})^* dG_t + Y_T^{22}\int_0^T Z_{t-}^{21}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}(Z_{t-}^{11})^* dG_t$$

$$= \begin{pmatrix} \mathbf{0}_{2,1} & \mathbf{0}_{2,1} \\ -(\partial_\zeta F)[G]_T & (\partial_\zeta F)[G[F]]_T \\ 0 & 0 \end{pmatrix},$$

$$V_T^{31} = Y_T^{31}\int_0^T Z_{t-}^{11}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}(Z_{t-}^{11})^* dG_t + Y_T^{32}\int_0^T Z_{t-}^{21}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}(Z_{t-}^{11})^* dG_t$$

$$+ Y_T^{33}\int_0^T Z_{t-}^{31}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}(Z_{t-}^{11})^* dG_t + Y_T^{33}\int_0^T Z_{t-}^{33}\begin{pmatrix} 1 & 0 \\ \mathbf{0}_{3,1} & \mathbf{0}_{3,1} \end{pmatrix}(Z_{t-}^{11})^* dL_t$$

$$= \begin{pmatrix} 0 & 0 \\ ((\partial_\zeta F)[G])\,[G]_T & -((\partial_\zeta F)[G])\,[G[F]]_T \\ 0 & 0 \\ -((\partial_\zeta F)[G])\,[G[F]]_T & ((\partial_\zeta F)[G[F]])\,[G[F]]_T \end{pmatrix}$$
$$+ \begin{pmatrix} L_T & -L[F]_T \\ F[L]_T & -F[L[F]]_T \\ -G[(\partial_\zeta F)[G]]_T & G[(\partial_\zeta F)[G[F]]]_T \\ -F[G[(\partial_\zeta F)[G]]]_T & F[(G[F])[(\partial_\zeta F)[G]]]_T \end{pmatrix}.$$

From Theorem 1, we can calculate the weight Γ_T concretely as follows.

$$\Gamma_T = \left(\sigma_1 W_T + \sigma_2 \int_0^T \int_{\mathbb{R}_0} \frac{\partial_\theta [g(\theta)\,\theta^2]}{g(\theta)} d\tilde{J},\ 0\right) v_T^{-1} + \left([V_T^{21} v_T^{-1} y_T]_{3,2},\ 0\right) y_T$$
$$+ \left([v_T^{-1} V_T^{31}]_{1,1},\ \sum_{j=1}^{2} [v_T^{-1}]_{2,j}\,[V_T^{31}]_{j+2,2}\right) v_T^{-1} y_T.$$

□

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