Markov processes on $\mathbf{R}^2$ with ultrametric jumps

Kumi Yasuda

Faculty of Business and Commerce, Keio University

1   Introduction

This is a report for a joint work with W. Karwowski on a regularity condition for Dirichlet forms on $L^2\left(\mathbf{R}^2\right)$ with ultrametric jump parts. The work is initially motivated by the paper by W. Karwowski ([K]) where Dirichlet forms on $L^2(\mathbf{R})$ with jump measures supported on the product of the ordinary Cantor sets are discussed. The jump measures were constructed as the image measures of the jump measures associated with Markov processes on the 2-adic numbers, through an identification of the Cantor set with 2-adic integers.

In this report we consider selfsimilar sets in $\mathbf{R}^2$ such as e.g. Sierpinski Gasket and Carpet, and give a regularity condition for Dirichlet forms having ultrametric jump parts supported on the products of such selfsimilar sets.

2 p-adic fields and l-adic trees

Let p be a fixed prime number. The p-adic field $\mathbf{Q}_p$ is the set of formal power series $a = \sum_{k=M}^{\infty} \alpha_k p^k$, with integers M and coefficients α_k in $\{0, 1, \ldots, p-1\}$.

The p-adic norm $|\ |_p$ on $\mathbf{Q}_p$ is defined by $|0|_p = 0$ and $\left|\sum_{k=M}^{\infty} \alpha_k p^k\right|_p = p^{-M}$ if $\alpha_M \neq 0$. It is not hard to see the p-adic norm enjoys the ultrametric property ; $|a + b|_p \leq \max\{|a|_p, |b|_p\}$ for all $a, b \in \mathbf{Q}_p$. Let us denote by $d_p(\ ,\ )$ the distance function on $\mathbf{Q}_p$ induced by the p-adic norm. It follows from the ultrametric property that, for a pair of disjoint balls B and B' in $\mathbf{Q}_p$, the distance from $a \in B$ to $b \in B'$ is independent of the choice of a and b, and hence we can define the p-adic distance between B and B' by $d_p(B, B') := d_p(a, b)$ for any $a \in B$ and $b \in B'$.

In [AK], S. Albeverio and W. Karwowski constructed additive processes on the p-adic field, and the author proved in [Y] that the processes given in [AK] make the class of all the additive processes X_t on $\mathbf{Q}_p$ such that, for any $u \in \mathbf{Q}_p$ with $|u|_p = 1$, uX_t has the same law as X_t for any $t \geq 0$. Since the p-adic field is totally disconnected, the processes are of pure jump type, and then determined by their Lévy measures F. If we denote by $B\left(a, p^M\right)$ the p-adic ball of radius p^M centered at $a \in \mathbf{Q}_p$ and put $a(M) = F\left(B\left(a, p^M\right)^c\right)$ for integers M, then it follows that

$$a(M) \geq a(M+1) \text{ for } M \in \mathbf{Z}, \text{ and } \lim_{M \to \infty} a(M) = 0. \tag{1}$$

Since the law of X_t is spherically symmetric, we can see that its Lévy measure F is completely determined by the values of $a(M)$, $M \in \mathbf{Z}$. Consequently, a process X_t corresponds in one to one way to a sequence $\{a(M)\}_{M \in \mathbf{Z}}$ of real numbers which satisfies (1). It should be remarked that the ultrametric property implies that the process X_t starting at the origin remains in the integer ring $\mathbf{Z}_p := B(0,1)$ for all $t \geq 0$ provided $a(M) = 0$ for all $M \geq 0$.

It is stated in [AK] that the Dirichlet form on $L^2(\mathbf{Q}_p)$ corresponding to the process X_t is given by

$$E(f,g) = \int_{\mathbf{Q}_p \times \mathbf{Q}_p \setminus d} (f(x) - f(y))(g(x) - g(y))j(dx, dy),$$

where d is the diagonal in $\mathbf{Q}_p \times \mathbf{Q}_p$, and j is the measure on $\mathbf{Q}_p \times \mathbf{Q}_p \setminus d$ given by

$$j\left(B\left(a, p^M\right), B\left(b, p^N\right)\right)$$
$$= \frac{1}{2} p^{N+M-n+1}(p-1)^{-1}(a(n-1) - a(n)),$$

if $B\left(a, p^M\right) \cap B\left(b, p^N\right) = \phi$ and if $d_p\left(B\left(a, p^M\right), B\left(b, p^N\right)\right) = p^n$.

As is presented in [AKY], the p-adic field is identified with limit points of the p-adic tree, and the processes appearing above are interporated as those on the limits points of the tree. Even for a non-prime integer l, we can define the l-adic distance on limit points of the l-adic tree, and construct l-adic valued additive processes associated with real sequences $\{a(M)\}_{M \in \mathbf{Z}}$ satisfying (1). In particular, we can define the set of "l-adic integers"

$$\mathbf{Z}_l := \left\{ a = \sum_{k=0}^{\infty} \alpha_k l^k \mid \alpha_k \in \{0, 1, \ldots, l-1\} \right\}$$

equipped with the l-adic distance $d_l \left(\sum_{k=0}^{\infty} \alpha_k l^k, \sum_{k=0}^{\infty} \beta_k l^k \right) := l^{-\min\{M : \alpha_M \neq \beta_M\}}$, and we have Markov processes on $\mathbf{Z}_l$ associated with non-increasing and non-negative real sequences $\{a(M)\}_{M<0}$ (by considering $a(M) = 0$ for all $M \geq 0$), corresponding to the Dirichlet form

$$E(f,g) = \int_{\mathbf{Z}_l \times \mathbf{Z}_l \setminus d} (f(x) - f(y))(g(x) - g(y)) j(dx, dy), \qquad (2)$$

where d is the diagonal in $\mathbf{Z}_l \times \mathbf{Z}_l$, and j is the measure on $\mathbf{Z}_l \times \mathbf{Z}_l \setminus d$ given by

$$j \left(B \left(a, l^M \right), B \left(b, l^N \right) \right)$$
$$= \frac{1}{2} l^{N+M-n+1} (p-1)^{-1} (a(n-1) - a(n)), \qquad (3)$$

if $B \left(a, l^M \right) \cap B \left(b, l^N \right) = \phi$ and $d_l \left(B \left(a, l^M \right), B \left(b, l^N \right) \right) = l^n$.

3 Selfsimilar sets in $\mathbf{R}^2$

Let $n \geq 3$ be a natural number, consider a regular n-polygon $G_n \subset \mathbf{R}^2$, and put v_i, $i = 1, 2, \ldots, n$, for its vertices. Fix a natural number $l \geq n$ and contraction maps ψ_i, $i = 1, 2, \ldots, l$, of G_n with common rate r such that

- $\psi_i(G_n) \subset G_n$ for all i,

- for $1 \leq i \leq n$, one of the vertices of $\psi_i(G_n)$ coincides with v_i,

- r is small and $\psi_i(G_n)$ are placed so that $\psi_i(G_n) \cap \psi_j(G_n)$ has Lebesgue measure 0 if $i \neq j$.

Then for $k \geq 1$, define the set S_k by the union of the images of the k-th iterations : $S_k := \cup_{i_1, i_2, \ldots, i_k = 1}^{l} \psi_{i_k} \circ \psi_{i_{k-1}} \circ \cdots \circ \psi_1(G_n)$, and put $S = \cap_{k=1}^{\infty} S_k$.
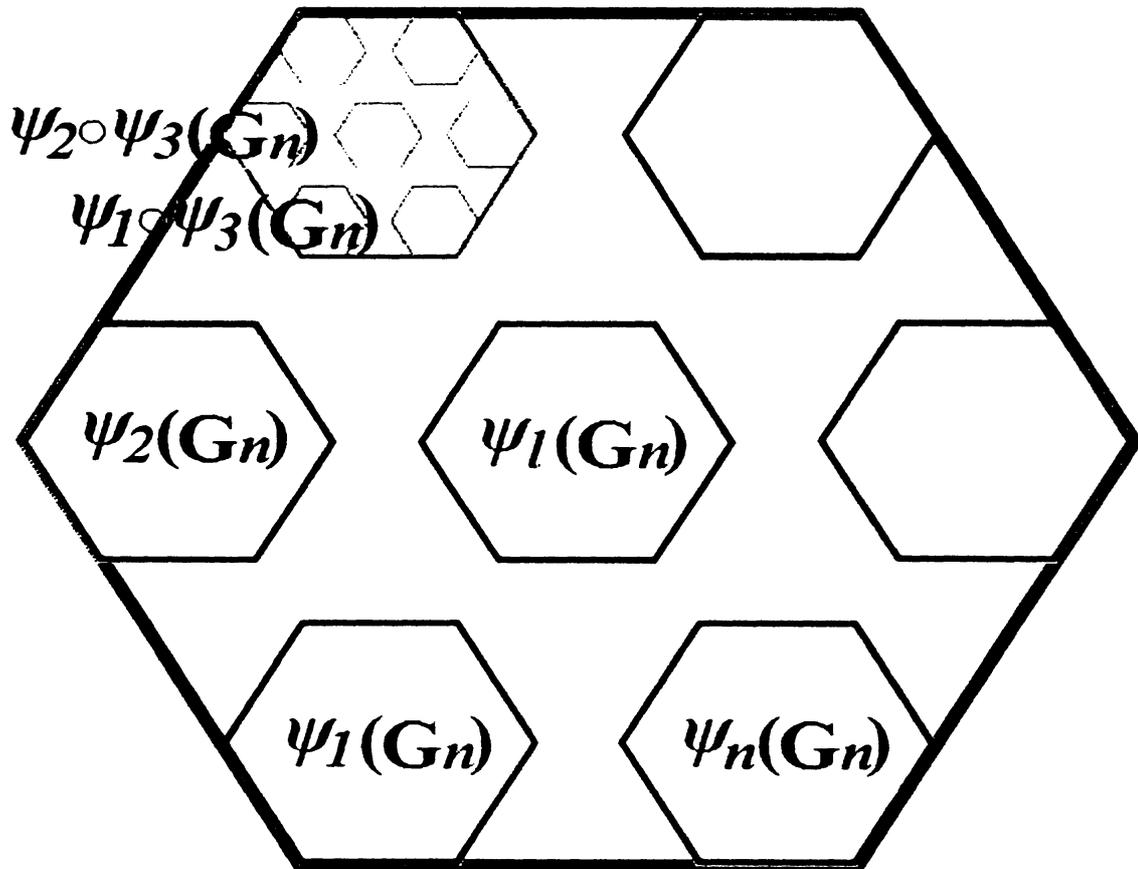
examples.

- S = Sierpinski Gasket : $n = l = 3$, $r = \frac{1}{2}$.

- S = Sierpinski Carpet : $n = 4$, $l = 8$, $r = \frac{1}{3}$.

- S = Pentaflake : $n = 5$, $l = 6$, $r = \frac{1}{1+\phi}$, where $\phi = \frac{1+\sqrt{5}}{2}$ is the golden ratio.

For every point $x \in S$, there exists a sequence $\{i_k(x)\}_{k=1,2,\ldots} \in \{1, 2, \ldots, l\}^{\mathbf{N}}$ such that $x \in \cap_{k=1}^{\infty} \psi_{i_k} \circ \psi_{i_{k-1}} \circ \cdots \circ \psi_{i_1}(G_n)$. (In case $\psi_i(G_n) \cap \psi_j(G_n) \neq \phi$ for some $i \neq j$, such sequence is not unique for those x which is in the intersection of some polygons. However, by the third assumption to the contraction maps ψ_i, the set of such x has Lebesgue measure 0.) Choose and fix such $\{i_k(x)\}_{k=1,2,\ldots}$ for each $x \in S$, and define a map $F : S \to \mathbf{Z}_l$ by

$$F(x) := \sum_{k=0}^{\infty} (i_k(x) - 1) l^k.$$

Lemma 3.1. *The map* $F : S \subset \mathbf{R}^2 \to \mathbf{Z}_l$ *is measurable.*

4 Dirichlet forms with ultrametric jump parts

Let $\{a(M)\}_{M<0}$ be a non-increasing and non-negative real sequence, and j be the measure on $\mathbf{Z}_l \times \mathbf{Z}_l \setminus d$ given in (3). Then put J for the image masure of j with respect to the map $F \times F : S \times S \setminus d_S \to \mathbf{Z}_l \times \mathbf{Z}_l \setminus d$, d_S being the diagonal of $S \times S$, and consider the symmetric form $\mathcal{E}$ on $L^2\left(\mathbf{R}^2\right)$ defined by

$$\mathcal{E}(u, v) := \int_{\mathbf{R}^2} \nabla u(x) \cdot \nabla v(x) dx$$
$$+ \int_{S \times S \setminus d_S} (u(x) - u(y))(v(x) - v(y)) J(dx, dy),$$
$$\mathcal{D}(\mathcal{E}) = \{ u \in C_0^\infty\left(\mathbf{R}^2\right)$$
$$: \int_{S \times S \setminus d_S} (u(x) - u(y))^2 J(dx, dy) < \infty \} . \tag{4}$$

Theorem 4.1. *We have*

$$\int_{S \times S \setminus d_S} |x - y|^2 J(dx, dy) < \infty, \tag{5}$$

if and only if

$$\sum_{M=-\infty}^{-1} r^{-2M} a(M) < \infty.$$

Under this condition, the form (4) is a symmetric closable Markovian form on $L^2\left(\mathbf{R}^2\right)$, and its closure is a regular Dirichlet form.

Here let us discuss on a particular case where the sequence $\{a(M)\}_{M<0}$ is given by

$$a(M) = \text{const. } l^{-\alpha M}, \tag{6}$$

with some $\alpha > 0$. It is shown in [Y] that, if l is a prime and if $\{a(M)\}_{M \in \mathbf{Z}}$ is a geometric sequence with ratio $l^{-\alpha}$, then the associate process on the l-adic field is semistable of index α. Therefore the process on $\mathbf{Z}_l$ corresponding to the sequence (6) is viewed as given from a semistable process by cutting off all the jumps larger than 1. In this case the regularity condition in Theorem 4.1 is represented in terms of the Hausdorff dimension $\dim_H S$ of the set S.

Corollary 4.2. *In case $\{a(M)\}_{M<0}$ is given by (6), the condition (5) is equivalent to*

$$\alpha < \frac{2 \log r^{-1}}{\log l} = 2(\dim_H S)^{-1}.$$

References

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