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Weighted Poincaré Inequality of Fractional Order

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Abstract

One of key tools to obtain heat kernel estimate for jump processes is a weighted Poincaré inequality of fractional order. The purpose of this note is to give the full proof of a variant but strengthened version of the weighted Poincaré inequality of fractional order that is established in [CKK].

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A weighted Poincaré inequality of fractional order was obtained in [CKK], which played an important role in obtaining sharp heat kernel estimate for finite range symmetric stable processes there. For weighted Poincaré inequality for Laplacian operators and its applications, we refer readers to [SC]. The purpose of this note is to present a variant but strengthened version of the weighted Poincaré inequality of fractional order established in [CKK]. The proof is very similar to that in [CKK]. But for reader's convenience, we give the full details here.

Throughout this paper, $r \geq 1$, $\sigma \in (0, \infty)$ and $\alpha \in (0,2)$. Recall that μ_d denotes the Lebesgue measure in $\mathbb{R}^d$. In this section, the exact values of the constants c's are always independent of r and they might change from one appearance to another. Let $\mathcal{M}(\sigma)$ be the set of all non-increasing function Ψ from $[0,1]$ to $[0,1]$ such that $\Psi(s) > \Psi(1) = 0$ for every $s \in [0,1)$ and

$$\Psi(s + \tfrac{1}{2}((1-s) \wedge \tfrac{1}{2})) \geq \sigma\, \Psi(s), \qquad s \in (0,1). \tag{1}$$

We will use $\mathcal{N}(\sigma)$ to denote all the functions Φ of the form $c\Psi(|x|)$ for some $\Psi \in \mathcal{M}(\sigma)$ having $\int_{\mathbb{R}^d} \Phi(x)dx = 1$. Note that, when $\beta \in (0,2)$, $c(1-|x|^2)^{12/(2-\beta)} \mathbf{1}_{B(0,1)}(x)$ is in $\mathcal{N}((1/8)^{12/(2-\beta)})$. Condition (1) says that for each $\Phi \in \mathcal{N}(\sigma)$, values of Φ at points with comparable distance from the unit sphere $\partial B(0,1)$ are comparable. This implies that values of Φ in balls in Whitney-type covering, which will be discussed below, are universally comparable to each other. This property will be used in many places below.

For $\Phi \in \mathcal{N}(\sigma)$, define

$$u_\Phi := \int_{B(0,1)} u(x)\Phi(x)dx.$$

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Theorem 1 *For every $d \geq 1$ and $\sigma \in (0,\infty)$, there exists a positive constant $c_1 = c_1(d,\sigma)$ independent of $r \geq 1$, such that for every $\Phi \in \mathcal{N}(\sigma)$ and $u \in L^1(B(0,1), \Phi(x)dx)$,*

$$\begin{aligned}&\int_{B(0,1)} (u(x) - u_\Phi)^2 \Phi(x) dx \\ \leq \;& c_1 r^2 \int_{B(0,1)\times B(0,1)} \frac{(u(x)-u(y))^2}{|x-y|^d} \mathbf{1}_{\{|x-y|\leq 1/r\}} (\Phi(x) \wedge \Phi(y))\, dxdy.\end{aligned}$$

We will prove the above theorem through several lemmas. For the remainder of this section, we fix $\sigma \in (0,\infty)$ and $\Phi \in \mathcal{N}(\sigma)$.

We first prove the following simple lemma. Let

$$u_{B(x,s)} := \frac{1}{\mu_d(B(x,s))} \int_{B(x,s)} u(y)dy.$$

Lemma 2 *For every $B(z,s) \subset B(0,1)$ and every $u \in L^1(B(z,s), dx)$,*

$$\int_{B(z,s)} (u(x) - u_{B(z,s)})^2 dx \leq \frac{1}{\mu_d(B(z,s))} \int_{B(z,s)} \int_{B(z,s)} (u(x)-u(y))^2 dxdy.$$

Proof. By Cauchy-Schwartz inequality,

$$\begin{aligned}\int_{B(z,s)} (u(x) - u_{B(z,s)})^2(x) dx &= \int_{B(z,s)} \left(\frac{1}{\mu_d(B(z,s))} \int_{B(z,s)} (u(x)-u(y))dy \right)^2 dx \\ &\leq \frac{1}{\mu_d(B(z,s))} \int_{B(z,s)} \int_{B(z,s)} (u(x)-u(y))^2 dxdy.\end{aligned}$$

□

Recall Whitney-type coverings (see [SC, Section 5.3.3] for details): We first let

$$\overline{\mathcal{W}} := \left\{ B : \text{ the center of the ball } B \text{ is in } B(0,1) \text{ and } r(B) = \frac{1}{10^3}\rho(B) \right\}$$

where $r(B)$ is the radius of the ball B and $\rho(B)$ denotes the Euclidean distance between the ball B and $B(0,1)^c$. In the sequel, for $\lambda > 0$ and a ball $B = B(x,r)$ centered at x with radius r, we denote λB the concentric ball $B(x, \lambda r)$ with radius λr.

Start $\mathcal{W}$ by picking a ball $B^0 \in \overline{\mathcal{W}}$ with the largest possible radius. Pick the next ball B^1 to be a ball in $\overline{\mathcal{W}}$ which does not intersect B^0 and has maximal radius. Assuming that k balls $B^0, \cdots, B^{k-1}$ have already been picked, pick the next ball B^k to be a ball in $\overline{\mathcal{W}}$ which does not intersect $\cup_{j=0}^{k-1} B^j$ and has maximal radius. Though this procedure, we get a sequence of disjoint balls $\mathcal{W} := \{B^0, \cdots, B^{k-1}, B^k, \cdots\}$ from $\overline{\mathcal{W}}$. Moreover, the Whitney-type decomposition of the unit ball $B(0,1)$ has the following properties (see, for example, page 135 of [SC]).

(1)

$$B(0,1) = \bigcup_{B\in\mathcal{W}} 2B.$$

(2) There exists a positive constant K such that

$$\sup_{y\in B(0,1)} \#\{B \in \mathcal{W} : y \in 10^2 B\} \le K \tag{2}$$

where $\#S$ is the number of elements in the set S.

There exists a ball $B(0) \in \mathcal{W}$ such that $0 \in 2B(0)$. We pick an fix such a ball $B(0)$ and call it the central ball of $\mathcal{W}$. For any $B \in \mathcal{W}$, let γ_B be the straight line segment between the center of B and the origin. Let

$$\overline{\mathcal{W}}(B) := \{A \in \mathcal{W} : 2A \cap \gamma_B \neq \emptyset\}.$$

Now we define the chain $\mathcal{W}(B) := (B_0, B_1, \cdots, B_{l(B)-1})$ with $B_0 = B(0)$ and $B_{l(B)-1} = B$ as follows; Starting from the origin, let y_0 be the first point along γ_B which does not belong to $2B_0$. Define B_1 to be (any) one of balls in $\overline{\mathcal{W}}(B)$ such that $y_0 \in 2B_1$. Inductively, having $B_0, B_1, \cdots, B_k$ constructed, let y_k be the first point along γ_B which does not belong to $\cup_{j=0}^k 2B_j$. Define B_{k+1} to be (any) one of balls in $\overline{\mathcal{W}}(B)$ such that $y_k \in 2B_{k+1}$. When the last chosen is not B, we simply add B as the last ball in $\mathcal{W}(B)$.

Using Lemma 2, the next lemma can be proved easily.

Lemma 3 *There exists a positive constant $c = c(d)$ such that for every $B \in \mathcal{W}$, $B_i, B_{i+1} \in \mathcal{W}(B)$ and for every $u \in L^1(B(0,1), \Phi dx)$,*

$$|u_{4B_i} - u_{4B_{i+1}}| \le \sum_{j=0}^{1} \frac{c}{\mu_d(B_{i+j})} \left(\int_{4B_{i+j}} \int_{4B_{i+j}} (u(x) - u(y))^2 dx dy \right)^{1/2}.$$

Proof. Note that

$$\begin{aligned}
&(\mu_d(4B_i \cap 4B_{i+1}))^{1/2} |u_{4B_i} - u_{4B_{i+1}}| \\
= &\left(\int_{4B_i \cap 4B_{i+1}} |u_{4B_i} - u_{4B_{i+1}}|^2 \mu_d(dx) \right)^{1/2} \\
\le &\left(\int_{4B_i} |u(x) - u_{4B_i}|^2 \mu_d(dx) \right)^{1/2} + \left(\int_{4B_{i+1}} |u(x) - u_{4B_{i+1}}|^2 \mu_d(dx) \right)^{1/2}.
\end{aligned}$$

Now the lemma follows from our Lemma 2 and the fact that

$$\mu_d(4B_i \cap 4B_{i+1}) \ge c \max\{\mu_d(B_i), \mu_d(B_{i+1})\}$$

(see Lemma 5.3.7 in [SC]). □

Lemma 4 *There exists a positive constant $c = c(d,\sigma)$ such that for every $B \in \mathcal{W}$, $B_i, B_{i+1} \in \mathcal{W}(B)$ and for every $u \in L^1(B(0,1), \Phi dx)$,*

$$\sqrt{\Phi_B}\,|u_{4B_i} - u_{4B_{i+1}}| \leq \sum_{j=0}^{1} \frac{c}{\mu_d(B_{i+j})}\left(\int_{4B_{i+j}}\int_{4B_{i+j}} (u(x)-u(y))^2(\Phi(x)\wedge\Phi(y))dxdy\right)^{1/2}.$$

Proof. Since the values of Φ are universally comparable to each other on $4B$ for every $B \in \mathcal{W}$, we have from Lemma 3

$$\begin{aligned} &|u_{4B_i} - u_{4B_{i+1}}| \\ &\leq \sum_{j=0}^{1} \frac{c}{(\mu_d(B_{i+j}))^{1/2}(\int_{B_{i+j}}\Phi(y)dy)^{1/2}}\left(\int_{4B_{i+j}}\int_{4B_{i+j}} (u(x)-u(y))^2(\Phi(x)\wedge\Phi(y))dxdy\right)^{1/2}. \end{aligned} \tag{3}$$

Note that

$$\rho(A) = 10^3 r(A) \geq \frac{10^3}{4} r(B) = \frac{1}{4}\rho(B) \qquad \text{for every } A \in \mathcal{W}(B). \tag{4}$$

(See Lemma 5.3.6 in [SC].) Using (1), (4) and the fact that Ψ is non-increasing, there exists a positive constant c independent of B such that

$$\max_{y\in B}\Phi(y) \leq c \min_{y\in A}\Phi(y) \qquad \text{for every } A \in \mathcal{W}(B).$$

Thus we have

$$\Phi_B = \frac{1}{\mu_d(B)}\int_B \Phi(y)dy \leq c\frac{1}{\mu_d(B_i)}\int_{B_i}\Phi(y)dy \qquad \text{for every } B_i \in \mathcal{W}(B). \tag{5}$$

The lemma follows from (3) and (5). □

The proof of the next lemma is similar to that of Theorem 5.3.4 on page 141-143 of [SC]. For reader's convenience, we nevertheless spell out the details of the proof here.

Lemma 5 *There exists a positive constant $c = c(d,\sigma)$ such that for every $u \in L^1(B(0,1), \Phi dx)$,*

$$\int_{B(0,1)} (u(x) - u_\Phi)^2\Phi(x)dx \leq c\sum_{A\in\mathcal{W}} \frac{1}{\mu_d(A)}\int_{4A\times 4A} (u(x)-u(y))^2(\Phi(x)\wedge\Phi(y))dxdy.$$

Proof. Note that

$$
\begin{aligned}
&\int_{B(0,1)} (u(x)-u_\Phi)^2\Phi(x)dx \\
\le\ & 2\int_{B(0,1)} (u(x)-u_{4B(0)})^2\Phi(x)dx + 2\left(\int_{B(0,1)}\Phi(x)dx\right)(u_\Phi - u_{4B(0)})^2 \\
\le\ & 2\int_{B(0,1)} (u(x)-u_{4B(0)})^2\Phi(x)dx + 2\int_{B(0,1)} (u(x)-u_{4B(0)})^2\Phi(x)dx \\
\le\ & 4\sum_{B\in\mathcal{W}}\int_{4B} (u(x)-u_{4B(0)})^2\Phi(x)dx \\
\le\ & 8\sum_{B\in\mathcal{W}}\int_{4B} (u(x)-u_{4B})^2\Phi(x)dx + 8\sum_{B\in\mathcal{W}}(u_{4B}-u_{4B(0)})^2\int_{4B}\Phi(x)dx \\
\le\ & c\sum_{B\in\mathcal{W}}\frac{1}{\mu_d(B)}\int_{4B\times 4B}(u(x)-u(y))^2(\Phi(x)\wedge\Phi(y))dxdy \\
& +c\sum_{B\in\mathcal{W}}\int \mathbf{1}_B(z)\left(|u_{4B}-u_{4B(0)}|(\Phi_B)^{1/2}\right)^2 dz,
\end{aligned}
$$

where in the last inequality, we used the fact that the values of Φ are universally comparable to each other on $4B$ for every $B\in\mathcal{W}$. To establish the lemma, it suffices to deal with the second summation above.

By Lemma 4, we get

$$
\begin{aligned}
&|u_{4B}-u_{4B(0)}|(\Phi_B)^{1/2}\mathbf{1}_B(z) \\
\le\ & \sum_{i=0}^{l(B)-2}|u_{4B_i}-u_{4B_{i+1}}|(\Phi_B)^{1/2}\mathbf{1}_B(z) \\
\le\ & c\sum_{i=0}^{l(B)-1}\frac{1}{\mu_d(B_i)}\left(\int_{4B_i}\int_{4B_i}(u(x)-u(y))^2(\Phi(x)\wedge\Phi(y))dxdy\right)^{1/2}\mathbf{1}_B(z) \\
=\ & c\sum_{i=0}^{l(B)-1}\frac{1}{\mu_d(B_i)}\left(\int_{4B_i}\int_{4B_i}(u(x)-u(y))^2(\Phi(x)\wedge\Phi(y))dxdy\right)^{1/2}\mathbf{1}_{10^4B_i}(z)\mathbf{1}_B(z) \\
\le\ & c\sum_{A\in\mathcal{W}}\frac{1}{\mu_d(A)}\left(\int_{4A}\int_{4A}(u(x)-u(y))^2(\Phi(x)\wedge\Phi(y))dxdy\right)^{1/2}\mathbf{1}_{10^4A}(z)\mathbf{1}_B(z).
\end{aligned}
$$

In the first equality above, we have used the fact that $B\subset 10^4B_i$ (Lemma 5.3.8 in [SC]). Since the balls in $\mathcal{W}$ are disjoint, summing both sides over $B\in\mathcal{W}$ and taking the square, we get

$$
\begin{aligned}
&\sum_{B\in\mathcal{W}}\mathbf{1}_B(z)\left(|u_{4B}-u_{4B(0)}|(\Phi_B)^{1/2}\right)^2 \\
\le\ & c\left(\sum_{A\in\mathcal{W}}\frac{1}{\mu_d(A)}\left(\int_{4A}\int_{4A}(u(x)-u(y))^2(\Phi(x)\wedge\Phi(y))dxdy\right)^{1/2}\mathbf{1}_{10^4A}(z)\right)^2.
\end{aligned}
$$

Integrating over $z \in B(0,1)$, and using Lemma 5.3.12 in [SC] and the fact the balls in $\mathcal{W}$ are disjoint, we have

$$\begin{aligned}
&\sum_{B\in\mathcal{W}} \int \mathbf{1}_B(z) \left(|u_{4B} - u_{4B(0)}|\Phi_B^{1/2}\right)^2 dz \\
\leq\ & c\int \left(\sum_{A\in\mathcal{W}} \frac{1}{\mu_d(A)} \left(\int_{4A}\int_{4A} (u(x)-u(y))^2(\Phi(x)\wedge\Phi(y))dxdy\right)^{1/2} \mathbf{1}_{10^4A}(z)\right)^2 dz \\
\leq\ & c\int \left(\sum_{A\in\mathcal{W}} \frac{1}{\mu_d(A)} \left(\int_{4A}\int_{4A} (u(x)-u(y))^2(\Phi(x)\wedge\Phi(y))dxdy\right)^{1/2} \mathbf{1}_{A}(z)\right)^2 dz \\
\leq\ & c\int \sum_{A\in\mathcal{W}} \frac{1}{(\mu_d(A))^2} \left(\int_{4A}\int_{4A} (u(x)-u(y))^2(\Phi(x)\wedge\Phi(y))dxdy\right) \mathbf{1}_{A}(z) dz \\
\leq\ & c\sum_{A\in\mathcal{W}} \frac{1}{\mu_d(A)} \left(\int_{4A}\int_{4A} (u(x)-u(y))^2(\Phi(x)\wedge\Phi(y))dxdy\right).
\end{aligned}$$

This completes the proof for the lemma. □

Lemma 6 *There exists a positive constant $c = c(d,\sigma)$ such that for every $u \in L^1(B(0,1),\Phi dx)$,*

$$\begin{aligned}
&\int_{B(0,1)} (u(x) - u_\Phi)^2\Phi(x)dx \\
\leq\ & c\int_{B(0,1)\times B(0,1)} \frac{(u(x)-u(y))^2}{|x-y|^{d+\alpha}} \mathbf{1}_{\{|x-y|\leq\frac{1}{10^2}\}}(\Phi(x)\wedge\Phi(y))dxdy.
\end{aligned}$$

Proof. Since $|x-y| \leq 8r(A) \leq \frac{1}{10^2}$ if $x, y \in 4A$, we have for every $A \in \mathcal{W}$

$$\begin{aligned}
&\frac{1}{\mu_d(A)} \int_{4A\times 4A} (u(x)-u(y))^2(\Phi(x)\wedge\Phi(y))dxdy \\
\leq\ & \frac{c}{(r(A))^d} \int_{4A\times 4A} \frac{(u(x)-u(y))^2|x-y|^d}{|x-y|^d} \mathbf{1}_{\{|x-y|\leq\frac{1}{10^2}\}}(\Phi(x)\wedge\Phi(y))dxdy \\
\leq\ & c\int_{4A\times 4A} \frac{(u(x)-u(y))^2}{|x-y|^d} \mathbf{1}_{\{|x-y|\leq\frac{1}{10^2}\}}(\Phi(x)\wedge\Phi(y))dxdy.
\end{aligned}$$

It then follows from Lemma 5 and (2) that

$$\begin{aligned}
&\int_{B(0,1)} (u(x) - u_\Phi)^2\Phi(x)dx \\
\leq\ & c\sum_{A\in\mathcal{W}} \int_{4A\times 4A} \frac{(u(x)-u(y))^2}{|x-y|^d} \mathbf{1}_{\{|x-y|\leq\frac{1}{10^2}\}}(\Phi(x)\wedge\Phi(y))dxdy \\
\leq\ & c\int_{B(0,1)\times B(0,1)} \frac{(u(x)-u(y))^2}{|x-y|^d} \mathbf{1}_{\{|x-y|\leq\frac{1}{10^2}\}}(\Phi(x)\wedge\Phi(y))dxdy.
\end{aligned}$$

$\square$

Due to Lemma 6, we have Theorem 1 for $1 \le r \le 10^2$. So, from now we may assume $r > 10^2$.

Lemma 7 *There exists a positive constant $c = c(d,\sigma)$ such that for every $r > 10^2$ for every $u \in L^1(B(0,1), \Phi dx)$,*

$$\begin{aligned}
&\int_{B(0,1)} (u(x) - u_\Phi)^2 \Phi(x) dx \\
\le\; & c \int_{B(0,1)\times B(0,1)} \frac{(u(x)-u(y))^2}{|x-y|^d} \mathbf{1}_{\{|x-y|<1/r\}} (\Phi(x) \wedge \Phi(y)) dxdy \\
&+ c \int_{B(0,1-\frac{10}{r})\times B(0,1-\frac{10}{r})} \frac{(u(x)-u(y))^2}{|x-y|^d} \mathbf{1}_{\{|x-y|<\frac{1}{10^2}\}} (\Phi(x) \wedge \Phi(y)) dxdy.
\end{aligned}$$

Proof. By Lemma 5, we have

$$\begin{aligned}
\int_{B(0,1)} (u(x) - u_\Phi)^2 \Phi(x) dx \;&\le\; c \sum_{A \in \mathcal{W}} \int_{4A\times 4A} \frac{(u(x)-u(y))^2}{|x-y|^d} \left(\frac{|x-y|}{r(A)}\right)^d (\Phi(x) \wedge \Phi(y)) dxdy \\
&\le c \left(\sum_{A\in\mathcal{W}:\, r(A)\le\frac{1}{10r}} + \sum_{A\in\mathcal{W}:\, r(A)>\frac{1}{10r}} \right) \int_{4A\times 4A} \frac{(u(x)-u(y))^2}{|x-y|^d} (\Phi(x) \wedge \Phi(y)) dxdy \\
&=: \; I + II.
\end{aligned}$$

If $A \in \mathcal{W}$ and $r(A) \le \frac{1}{10r}$, then $|x-y| \le 8r(A) < \frac{1}{r}$ for every $x, y \in 4A$. So using (2), we have

$$\begin{aligned}
I \;&\le\; c \sum_{A\in\mathcal{W}:\, r(A)\le\frac{1}{10r}} \int_{4A\times 4A} \frac{(u(x)-u(y))^2}{|x-y|^d} \mathbf{1}_{\{|x-y|\le 1/r\}} (\Phi(x) \wedge \Phi(y)) dxdy \\
&\le\; c \int_{B(0,1)\times B(0,1)} (u(x)-u(y))^2 \frac{r^{-\alpha}}{|x-y|^{d+\alpha}} \mathbf{1}_{\{|x-y|<1/r\}} (\Phi(x) \wedge \Phi(y)) dxdy.
\end{aligned}$$

On the other hands, if $A \in \mathcal{W}$ and $r(A) > \frac{1}{10r}$, then for every pair of points x, y in $4A$, we have $|x-y| \le 8r(A) < \frac{1}{10^2}$ and

$$\mathrm{dist}(x, \partial B(0,1)) \;\ge\; \rho(A) - 4r(A) \;>\; 10^2 r(A) \ge \frac{10}{r}.$$

Therefore, using (2) we have

$$\begin{aligned}
II \;&\le\; c \sum_{A\in\mathcal{W}:\, r(A)>\frac{1}{10r}} \int_{4A\times 4A} \frac{(u(x)-u(y))^2}{|x-y|^d} \mathbf{1}_{\{|x-y|\le\frac{1}{10^2}\}} (\Phi(x) \wedge \Phi(y)) dxdy \\
&\le\; c \int_{B(0,1-\frac{10}{r})\times B(0,1-\frac{10}{r})} \frac{(u(x)-u(y))^2}{|x-y|^d} \mathbf{1}_{\{|x-y|<\frac{1}{10^2}\}} (\Phi(x) \wedge \Phi(y)) dxdy.
\end{aligned}$$

□

For our purpose, we need to construct another covering; For each $r > 10^2$, we let $\mathcal{V} = \mathcal{V}_r := \{B^1, \cdots, B^{k(r)}\}$ be a maximum sequence of disjoint balls with radius $\frac{1}{400r}$ that we can put inside $B(0, 1 - \frac{10}{r})$. Note that

$$B(0,1-\frac{10}{r}) \subset \bigcup_{B\in\mathcal{V}} 2B \subset \bigcup_{B\in\mathcal{V}} 10^2 B \subset B(0,1-\frac{9}{r}).$$

For every $y \in B(0,1)$, since $\bigcup_{B\in\mathcal{V}: y\in 2B} B \subset B(y, \frac{3}{400r})$,

$$\#\{B \in \mathcal{V} : y \in 2B\} \cdot \mu_d(B(0, \tfrac{1}{400r})) \leq \mu_d(B(y, \tfrac{3}{400r})).$$

Therefore we have

$$\sup_{y\in B(0,1)} \#\{B \in \mathcal{V} : y \in 2B\} \leq 3^d. \tag{6}$$

Recall that $\rho(B)$ denotes the Euclidean distance between the ball B and $B(0,1)^c$. For balls A and B in $\mathcal{V}$ with $\mathrm{dist}(A,B) > \frac{1}{40r}$ and $\rho(B) \geq \rho(A)$, we construct the path $\gamma_{A,B}$ starting from A in the following way. Let x_A be the center of A and x_B be the center of B. If $|x_B| \geq 1/(400r)$, then let $y_B := \frac{|x_A|}{|x_B|} x_B$ so that x_B is in the straight line segment from y_B to 0. Let $\gamma^2_{A,B}$ be the straight line segment from y_B to x_B. We also let $\gamma^1_{A,B}$ be the shortest path from x_A to y_B with $\gamma^1_{A,B} \subset \partial B(0, |x_A|)$. In this case, $\gamma_{A,B}$ is the union of $\gamma^1_{A,B}$ and $\gamma^2_{A,B}$ starting from x_A and ending at x_B via y_B. If $|x_B| < 1/(400r)$, let $\gamma_{A,B}$ be simply a straight line segment between 0 and x_A.

For $A, B \in \mathcal{V}$ with $\rho(B) \geq \rho(A)$, let

$$\overline{\mathcal{V}}(A,B) := \{C \in \mathcal{V} : 2C \cap \gamma_{A,B} \neq \emptyset\}$$

and define the chain $\mathcal{V}(A,B) := (C_0, C_1, \cdots, C_{l(A,B)-1})$ with $C_0 = A$ and $C_{l(A,B)-1} = B$ similar to the chain in the Whitney-type coverings; Starting from the center of A, let y_0 be the first point along $\gamma_{A,B}$ which does not belong to $2C_0$. Define C_1 to be one of balls in $\overline{\mathcal{V}}(A,B)$ such that $y_0 \in 2C_1$. Inductively, having $C_0, C_1, \cdots, C_k$ constructed, let y_k be the first point along $\gamma_{A,B}$ which does not belong to $\cup_{j=0}^k 2C_j$. Define C_{k+1} to be one of balls in $\overline{\mathcal{V}}(A,B)$ such that $y_k \in 2C_{k+1}$. When the last chosen is not B, we add B as the last ball in $\mathcal{V}(A,B)$.

In the sequel, for every path γ in $\mathbb{R}^d$ we denote by $|\gamma|$ the length of γ.

Lemma 8 *There exists a positive constant $c = c(d)$ such that for every $r > 10^2$ and every $A, B \in \mathcal{V}$ with $\rho(B) \geq \rho(A)$, $|\gamma_{A,B}| > \frac{1}{4r}$ and* $\mathrm{dist}(A,B) \leq \frac{1}{50}$,

$$|x-y| \geq \frac{c}{r} \#\overline{\mathcal{V}}(A,B) \geq \frac{c}{r} \#\mathcal{V}(A,B) \geq |\gamma_{A,B}|, \qquad \textit{for every } (x,y) \in 2A \times 2B. \tag{7}$$

In particular,

$$\#\mathcal{V}(A,B) \leq \#\overline{\mathcal{V}}(A,B) \leq cr. \tag{8}$$

Proof. It is easy to see that the length of $\gamma_{A,B}$ is less than or equal to $4|x-y|$ for every $(x,y) \in A \times B$. Thus by using the fact that balls C's in $\overline{\mathcal{V}}(A,B)$ are disjoint and that $\cup_{C \in \overline{\mathcal{V}}(A,B)} C$ is within the $\frac{1}{100r}$-neighborhood of γ_{AB}, we have

$$\#\overline{\mathcal{V}}(A,B) \cdot (\frac{1}{400r})^d = c \sum_{C \in \overline{\mathcal{V}}(A,B)} \mu_d(C) \leq c|x-y|\, r^{1-d}$$

and so $\#\overline{\mathcal{V}}(A,B) \leq cr\,|x-y|$.

On the other hand, since $2C$'s in $\mathcal{V}(A,B)$ covers $\gamma_{A,B}$, it is easy to see that

$$E := \{x \in B(0,1) : \operatorname{dist}(x, \gamma_{A,B}) < \frac{1}{400r}\} \subset \bigcup_{C \in \mathcal{V}(A,B)} 3C$$

and that

$$\mu_d(E) \geq c\,|\gamma_{A,B}|(\frac{1}{r})^{d-1}.$$

Thus

$$c|\gamma_{A,B}|r^{1-d} \leq \mu_d(E) \leq \sum_{C \in \overline{\mathcal{V}}(A,B)} \mu_d(3C) = \#\mathcal{V}(A,B) \cdot (\frac{3}{400r})^d$$

and so $|\gamma_{A,B}| \leq \frac{c}{r}\#\mathcal{V}(A,B)$. The lemma is proved. □

The proof of the next lemma is similar to the one of Lemma 3. So we skip its proof.

Lemma 9 *Let $A, B \in \mathcal{V}$ with $\rho(B) \geq \rho(A)$. There exists a positive constant $c = c(d)$ such that for every $C_i, C_{i+1} \in \mathcal{V}(A,B)$ and for every $u \in L^1(B(0,1), \Phi dx)$,*

$$|u_{2C_i} - u_{2C_{i+1}}|^2 \leq \sum_{j=0}^{1} \frac{c}{(\mu_d(2C_{i+j}))^2} \int_{2C_{i+j}} \int_{2C_{i+j}} (u(x)-u(y))^2 dxdy.$$

Lemma 10 *There exists positive constant $c = c(d,\sigma)$ such that for every $r > 10^2$ and every $A, B \in \mathcal{V}$ with $\rho(B) \geq \rho(A)$ and $|\gamma_{A,B}| \geq \frac{1}{4r}$,*

$$\begin{aligned}
&\int_{2A}\int_{2B} \frac{(u(x)-u(y))^2}{|x-y|^d} \mathbf{1}_{\{|x-y|<\frac{1}{100}\}}(\Phi(x) \wedge \Phi(y))dxdy \\
\leq\ & c(\#\mathcal{V}(A,B))^{1-d} \sum_{C \in \mathcal{V}(A,B)} \int_{2C}\int_{2C} \frac{(u(x)-u(y))^2}{|x-y|^d}(\Phi(x) \wedge \Phi(y))\,dxdy.
\end{aligned}$$

Proof. Let $l := \#\mathcal{V}(A,B) \geq 2$. For every $y \in A$ and $x \in B$,

$$\begin{aligned}
&(u(x)-u(y))^2(\Phi(x) \wedge \Phi(y)) \\
\leq\ & (l+2)(\Phi(x) \wedge \Phi(y)) \left(|u(x) - u_{2A}|^2 + |u(x) - u_{2B}|^2 + \sum_{i=0}^{l-1} |u_{2C_i} - u_{2C_{i+1}}|^2 \right) \\
\leq\ & 2l \Big((\Phi(x) \wedge \Phi(y))|u(y) - u_{2A}|^2 + (\Phi(x) \wedge \Phi(y))|u(x) - u_{2B}|^2 \\
&+ \sum_{i=0}^{l-1} (\Phi(x) \wedge \Phi(y))|u_{2C_i} - u_{2C_{i+1}}|^2 \Big).
\end{aligned}$$

Note that from the construction of the chain $\mathcal{V}(A,B)$, it is easy to see that there exists a constant c independent of r such that for every $A, B \in \mathcal{V}$ and $C \in \mathcal{V}(A,B)$,

$$\int_{2A}\int_{2B}(\Phi(x)\wedge\Phi(y))dxdy \le c\int_{2C_i}\int_{2C_{i+1}}(\Phi(x)\wedge\Phi(y))\,dxdy.$$

Obviously

$$\int_{2A}\int_{2B}|u(x)-u_{2B}|^2(\Phi(x)\wedge\Phi(y))dxdy \le \mu_d(2B)\int_{2B}|u(x)-u_{2B}|^2\Phi(x)dx$$

and

$$\int_{2A}\int_{2B}|u(y)-u_{2A}|^2(\Phi(x)\wedge\Phi(y))dxdy \le \mu_d(2A)\int_{2A}|u(y)-u_{2A}|^2\Phi(y)dy.$$

Thus we have, for every $y \in A$ and $x \in B$,

$$\begin{aligned}
&\int_{2A}\int_{2B}(u(x)-u(y))^2(\Phi(x)\wedge\Phi(y))dxdy \\
\le\ & 2l\left(\int_{2A}\int_{2B}(\Phi(x)\wedge\Phi(y))|u(y)-u_{2A}|^2dxdy + \int_{2A}\int_{2B}(\Phi(x)\wedge\Phi(y))|u(x)-u_{2B}|^2dxdy\right.\\
&\left.+\sum_{i=0}^{l-1}\int_{2A}\int_{2B}(\Phi(x)\wedge\Phi(y))|u_{2C_i}-u_{2C_{i+1}}|^2dxdy\right)\\
\le\ & cl\left(\mu_d(2A)|\int_{2A}|u(y)-u_{2A}|^2\Phi(y)dy + \mu_d(2B)\int_{2B}|u(x)-u_{2B}|^2\Phi(x)dx\right.\\
&\left.+\sum_{i=0}^{l-1}|u_{2C_i}-u_{2C_{i+1}}|^2\int_{2C_i}\int_{2C_{i+1}}(\Phi(x)\wedge\Phi(y))\,dxdy\right).
\end{aligned}$$

We apply Lemma 2 to the first two integrals in the above and apply Lemma 9 to the integrals in the summation above. Then using the fact that the values of Φ are universally comparable on each A, B, C_i, we get that

$$\int_{2A}\int_{2B}(u(x)-u(y))^2(\Phi(x)\wedge\Phi(y))dxdy \le cl\sum_{C\in\mathcal{V}(A,B)}\int_{2C}\int_{2C}(u(x)-u(y))^2(\Phi(x)\wedge\Phi(y))\,dxdy. \tag{9}$$

Note that, using (7), we have that for $x \in B$ and $y \in A$ with $|x-y| < \frac{1}{100}$

$$\frac{1}{100} \ge |x-y| \ge c\frac{l}{r} \ge cl\,|z-w|, \qquad \forall z,w \in C \in \mathcal{V}(A,B). \tag{10}$$

Therefore, from (9)-(10), we conclude that

$$\begin{aligned}
&\int_{2A}\int_{2B}\frac{(u(x)-u(y))^2}{|x-y|^d}(\Phi(x)\wedge\Phi(y))\mathbf{1}_{\{|x-y|<\frac{1}{100}\}}dxdy\\
=\ & \left(c\frac{r}{l}\right)^d\int_{2A}\int_{2B}(u(x)-u(y))^2(\Phi(x)\wedge\Phi(y))\mathbf{1}_{\{|x-y|<\frac{1}{100}\}}dxdy\\
\le\ & c\,l^{1-d}\sum_{C\in\mathcal{V}(A,B)}\int_{2C}\int_{2C}\frac{(u(z)-u(w))^2}{|z-w|^d}(\Phi(z)\wedge\Phi(w))\,dzdw.
\end{aligned}$$

□

Recall that $[a]$ denote the largest integer which is no larger than a and define for $C \in \mathcal{V}$

$$C(\mathcal{V}) := \{(A, B) : A, B \in \mathcal{V} \text{ with } \rho(B) \geq \rho(A) \text{ and } C \in \mathcal{V}(A, B)\}.$$

The following is a key lemma to count the number of chains containing each $C \in \mathcal{V}$.

Lemma 11 *There exists a positive constant $c = c(d)$ such that for every $r > 10^2$, $30 \leq l \leq [16r]$ and $C \in \mathcal{V}$,*

$$\#\left\{(A, B) \in C(\mathcal{V}) : \frac{100 + l}{400r} < |\gamma_{A,B}| \leq \frac{101 + l}{400r}\right\} \leq cl^d. \tag{11}$$

Proof. Without loss of generality, we assume $d \geq 2$. (The case of $d = 1$ is easier.) Fix $r > 10^2$, $30 \leq l \leq [16r]$ and $C \in \mathcal{V}$. We will order $(A, B) \in C(\mathcal{V})$ so that $\rho(B) \geq \rho(A)$. Let x_C be the center of the ball C. If $|x_C| \leq 4/(400r)$, then $|x_B| \leq 6/(400r)$, so the number of possible choice for B is less than $c2^d$. Since $(100 + l)/(400r) \leq |\gamma_{A,B}| \leq (101 + l)/(400r)$, the number of possible choice for A is cl^{d-1}, so (11) holds in this case. We thus assume $|x_C| > 4/(400r)$. Define $H_{x_C} := B(0, |x_C| + 2/(400r)) \setminus B(0, |x_C| - 2/(400r))$. Since $2C \cap \gamma_{A,B} \neq \emptyset$, $H_{x_C} \cap \gamma_{A,B} \neq \emptyset$. Let y'_B be the first point along $\gamma_{A,B}$ (starting from x_B) which belongs to $H_{x_C} \cap \gamma_{A,B}$. Also, let $z_{A,B}$ be the first point along $\gamma_{A,B}$ (starting from x_B) which belongs to $2C$, and let γ_B be the sub-path of $\gamma_{A,B}$ starting from $z_{A,B}$ ending at x_B.

Let $m/(400r) \leq |\gamma_B| < (m + 1)/(400r)$ where $0 \leq m \leq l + 100$ and consider the following two cases:

$$\text{Case (i)} \ \ |y'_B - z_{A,B}| \leq \frac{5}{400r}; \qquad \text{Case (ii)} \ \ |y'_B - z_{A,B}| > \frac{5}{400r}.$$

For Case (i), the number of possible choices for y'_B and B is less than $c2^d$ when C is given and m is fixed. Once y'_B is fixed, the number of possible choice for A is $c(l - m + 106)^{d-1}$, since the arclength between $z_{A,B}$ and x_A along the curve $\gamma_{A,B}$ is at most $\frac{101+l-m}{400r}$ and $|y'_B - z_{A,B}| \leq 5/(400r)$. Summing over m, the number of possible choices for A and B is less than

$$c' \sum_{m=0}^{l+100} (l - m + 106)^{d-1} \leq c'' l^d.$$

For Case (ii), let $i \leq m$ be such that $i/(400r) \leq |z_{A,B} - y_B| < (i+1)/(400r)$ where $y_B := \frac{|x_A|}{|x_B|} x_B$. In this case, $|y_B - y'_B| \leq 4/(400r)$ and $i \geq 1$. Since $y_B \in \partial B(0, |x_A|) \subset H_{x_C}$, given C, the number of possible choices for y_B and B is less than ci^{d-2} when m and i are fixed. Observe that given C and B, y'_B and x_B are determined. Since $x_A \in \partial B(0, |x_A|) \subset H_{x_C}$, given C and B, the number of possible choice for x_A is less than $c((l - m + i + 101)/i)^{d-2}$

when m and i are fixed. Summing over m and i, the number of possible choices for A and B is less than

$$c'\sum_{m=1}^{l+100}\sum_{i=1}^{m} i^{d-2}\left(\frac{l-m-i+101}{i}\right)^{d-2} = c'\sum_{m=1}^{l+100}\sum_{i=1}^{m}(l-m+i+101)^{d-2} \leq c'' l^d.$$

We thus obtain (11). $\square$

Lemma 12 *There exists positive constant $c = c(d,\sigma)$ such that for every $r \geq 10^2$*

$$\begin{aligned}&\sum_{\substack{A,B\in\mathcal{V}\\ dist(A,B)>\frac{1}{3r}}} \int_{2A}\int_{2B} \frac{(u(x)-u(y))^2}{|x-y|^d}\mathbf{1}_{\{|x-y|\leq\frac{1}{100}\}}(\Phi(x)\wedge\Phi(y))dxdy\\ \leq\ & c\int_{B(0,1)\times B(0,1)} (u(x)-u(y))^2\frac{r^2}{|x-y|^d}\mathbf{1}_{\{|x-y|\leq\frac{1}{r}\}}(\Phi(x)\wedge\Phi(y))\,dxdy.\end{aligned}$$

Proof. For $(x,y)\in 2A\times 2B$ with $|x-y|\leq\frac{1}{100}$, it is elementary to check that $|\gamma_{A,B}|<\frac{1}{25}$. Thus, by Lemma 10, we have

$$\begin{aligned}&\sum_{\substack{A,B\in\mathcal{V}\\ \mathrm{dist}(A,B)>\frac{1}{3r}}} \int_{2A}\int_{2B} \frac{(u(x)-u(y))^2}{|x-y|^d}\mathbf{1}_{\{|x-y|\leq\frac{1}{100}\}}(\Phi(x)\wedge\Phi(y))dxdy\\
\leq\ & c\sum_{\substack{A,B\in\mathcal{V}:\rho(B)\geq\rho(A)\\ \frac{130}{400r}<|\gamma_{A,B}|<\frac{1}{25}}} (\#\mathcal{V}(A,B))^{1-d}\sum_{C\in\mathcal{V}(A,B)}\int_{2C}\int_{2C}\frac{(u(x)-u(y))^2}{|x-y|^d}(\Phi(x)\wedge\Phi(y))\,dxdy\\
\leq\ & c\sum_{C\in\mathcal{V}}\left(\sum_{l=30}^{[16r]}\sum_{\substack{(A,B)\in C(\mathcal{V})\\ \frac{100+l}{400r}<|\gamma_{A,B}|\leq\frac{100+l+1}{400r}}}(\#\mathcal{V}(A,B))^{1-d}\right)\\
&\times\int_{2C}\int_{2C}\frac{(u(x)-u(y))^2}{|x-y|^d}\mathbf{1}_{\{|x-y|\leq\frac{1}{r}\}}(\Phi(x)\wedge\Phi(y))\,dxdy.\end{aligned}$$

Applying (7), we see that

$$\begin{aligned}&\sum_{\substack{A,B\in\mathcal{V}\\ \mathrm{dist}(A,B)>\frac{1}{3r}}} \int_{2A}\int_{2B} \frac{(u(x)-u(y))^2}{|x-y|^d}\mathbf{1}_{\{|x-y|\leq\frac{1}{100}\}}(\Phi(x)\wedge\Phi(y))dxdy\\
\leq\ & c\sum_{C\in\mathcal{V}}\left(\sum_{l=30}^{[16r]} l^{1-d}\cdot\#\left\{(A,B)\in C(\mathcal{V}):\frac{100+l}{400r}<|\gamma_{A,B}|\leq\frac{101+l}{400r}\right\}\right)\\
&\times\int_{2C}\int_{2C}\frac{(u(x)-u(y))^2}{|x-y|^d}\mathbf{1}_{\{|x-y|\leq\frac{1}{r}\}}(\Phi(x)\wedge\Phi(y))\,dxdy.\end{aligned}$$

By Lemma 11,

$$\sum_{l=30}^{[16r]} l^{1-d}\cdot \#\left\{(A,B)\in C(\mathcal{V}) : \frac{100+l}{400r} < |\gamma_{A,B}| \le \frac{101+l}{400r}\right\} \le c\sum_{l=30}^{[16r]} l^1 \le cr^2.$$

Thus we conclude that

$$\begin{aligned}
&\sum_{\substack{A,B\in\mathcal{V}\\ \mathrm{dist}(A,B)>\frac{1}{3r}}} \int_{2A}\int_{2B} \frac{(u(x)-u(y))^2}{|x-y|^d}\mathbf{1}_{\{|x-y|\le\frac{1}{100}\}}(\Phi(x)\wedge\Phi(y))dxdy\\
\le\ & cr^2\sum_{C\in\mathcal{V}}\int_{2C}\int_{2C}\frac{(u(x)-u(y))^2}{|x-y|^d}\mathbf{1}_{\{|x-y|\le\frac{1}{r}\}}(\Phi(x)\wedge\Phi(y))\,dxdy\\
\le\ & c\int_{B(0,1)\times B(0,1)}(u(x)-u(y))^2\frac{r^2}{|x-y|^d}\mathbf{1}_{\{|x-y|\le\frac{1}{r}\}}(\Phi(x)\wedge\Phi(y))\,dxdy.
\end{aligned}$$

In the last inequality above, we have used (6). □

Proof of Theorem 1: By Lemma 7, it is enough to show the following claim; there exists constant $c=c(d,\sigma)>0$ such that for every $r>10^2$ and $u\in L^1(B(0,1),\Phi dx)$

$$\begin{aligned}
&\int_{B(0,1-\frac{10}{r})\times B(0,1-\frac{10}{r})}\frac{(u(x)-u(y))^2}{|x-y|^d}\mathbf{1}_{\{|x-y|\le\frac{1}{100}\}}(\Phi(x)\wedge\Phi(y))dxdy\\
\le\ & c\int_{B(0,1)\times B(0,1)}(u(x)-u(y))^2\frac{r^2}{|x-y|^d}\mathbf{1}_{\{|x-y|\le\frac{1}{r}\}}(\Phi(x)\wedge\Phi(y))dxdy. \qquad (12)
\end{aligned}$$

Note that

$$\begin{aligned}
&\int_{B(0,1-\frac{10}{r})\times B(0,1-\frac{10}{r})}\frac{(u(x)-u(y))^2}{|x-y|^d}\mathbf{1}_{\{|x-y|\le\frac{1}{100}\}}(\Phi(x)\wedge\Phi(y))dxdy\\
\le\ & \sum_{A,B\in\mathcal{V}}\int_{2A}\int_{2B}\frac{(u(x)-u(y))^2}{|x-y|^d}\mathbf{1}_{\{|x-y|\le\frac{1}{100}\}}(\Phi(x)\wedge\Phi(y))dxdy\\
\le\ & \sum_{\substack{A,B\in\mathcal{V}\\ \mathrm{dist}(A,B)\le\frac{1}{3r}}}\int_{2A}\int_{2B}\frac{(u(x)-u(y))^2}{|x-y|^d}\mathbf{1}_{\{|x-y|\le\frac{1}{r}\}}(\Phi(x)\wedge\Phi(y))dxdy\\
&+\sum_{\substack{A,B\in\mathcal{V}\\ \mathrm{dist}(A,B)>\frac{1}{3r}}}\int_{2A}\int_{2B}\frac{(u(x)-u(y))^2}{|x-y|^d}\mathbf{1}_{\{|x-y|\le\frac{1}{100}\}}(\Phi(x)\wedge\Phi(y))dxdy\\
\le\ & cr^2\int_{B(0,1)\times B(0,1)}\frac{(u(x)-u(y))^2}{|x-y|^d}\mathbf{1}_{\{|x-y|\le\frac{1}{r}\}}(\Phi(x)\wedge\Phi(y))dxdy\\
&+\sum_{\substack{A,B\in\mathcal{V}\\ \mathrm{dist}(A,B)>\frac{1}{3r}}}\int_{2A}\int_{2B}\frac{(u(x)-u(y))^2}{|x-y|^d}\mathbf{1}_{\{|x-y|\le\frac{1}{100}\}}(\Phi(x)\wedge\Phi(y))dxdy.
\end{aligned}$$

In the last inequality above, we have used (6) and the fact $r^2\ge 1$. Thus (12) follows from Lemma 12. □

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