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Exercise Behaviors and Valuation of Executive Stock Options*

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1 Introduction

Executive (or employee) stock options (ESOs) have become increasingly popular and currently constitute a certain fraction of total compensation expense of many firms. ESOs are call options that give the option holder the right to buy their firm's stock for a fixed strike price during a specified period of time. Clearly, the exercise of ESOs triggers a dilution of the claims of the firm's existing shareholders, since the firm issues new stocks to the ESO holders. The Financial Accounting Standards Board (FASB) issued in 2004 a revised version of Statement of Financial Accounting Standard No. 123, Share-Based Payments (SFAS 123R), which requires firms to estimate and report the *fair value* of ESOs at the grant date. ESO valuation is now becoming an important issue in many countries, which inspires us to create a reasonable valuation method for ESOs.

ESOs have features different from ordinary market-traded options. While ordinary options usually mature within one year, ESOs have maturity over many years, typically, it is set equal to ten years. Also, ESOs are usually granted at-the-money, namely, its strike price is set equal to the current stock price. During the beginning part of the option's life (called a vesting period), ESO holders cannot exercise their options and must forfeit the options on leaving the firm. Typically, the vesting date is two or three years after the grant date. After the vesting date, ESO holders can exercise the options at any time before maturity date, i.e., ESOs are of American-style. The most significant difference between ESOs and traded options would be that ESO holders cannot sell or otherwise transfer them. An ESO holder leaving the firm is then forced to choose between forfeiting or exercising the options soon after his exit. The lack of transferability implies that ESO holders cannot hedge their positions, and so that their personal valuations depend on their risk preferences and endowments. Thus the non-transferability of ESOs may be realized in mathematical models by maximizing a utility function of ESO holders. Through an empirical analysis using data on ESO exercises from 40 firms, however, Carpenter [5] showed that a simple American option pricing model performs well as an elaborate utility-maximizing model. Hence, this paper considers ESOs with three principal features, namely, *early exercise*, *delayed vesting* and *employment termination*. Additional features such as resetting/reloading provisions and multiple exercising rights are not considered here.

With respect to the modeling of employment termination, there is a significant difference between two elementary models developed by Raupach [17] and Cvitanić et al. [7]: Raupach

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first analyzed the ESO value assuming no exit before maturity and then obtain the unconditional value using a stopping time argument, while Cvitanić et al. adopted the intensity-based framework in Carr and Linetsky [6]. The different modeling yields that Raupach's result has a form of a two-dimensional integral and the result of Cvitanić et al. is represented by a number of explicit but lengthy formulas ([7, pp. 702–710]). The purpose of this paper is to develop a continuous-time barrier option model with an *endogenously* specified flat barrier, following basically the modeling of Raupach. We provide not only a one-dimensional integral formula for the ESO value much more compact than those in Raupach and Cvitanić et al., but also a simple formula for the mean ESO exercise time.

2 Black-Scholes-Merton formulation

Suppose an economy with finite time period $[0,T]$, a complete probability space $(\Omega,\mathcal{F},\mathbb{P})$ and a filtration $(\mathcal{F}_t)_{t\in[0,T]}$. A standard Brownian motion process $W\equiv(W_t)_{t\in[0,T]}$ is defined on $(\Omega,\mathcal{F})$ and takes values in $\mathbb{R}$. The filtration is the natural filtration generated by W and $\mathcal{F}_T=\mathcal{F}$. Let $(S_t)_{t\in[0,T]}$ be the price process of the underlying stock. For S_0 given, assume that $(S_t)_{t\in[0,T]}$ is a geometric Brownian motion process

$$\mathrm{d}S_t=(r-\delta)S_t\mathrm{d}t+\sigma S_t\mathrm{d}W_t,\quad t\in[0,T], \tag{2.1}$$

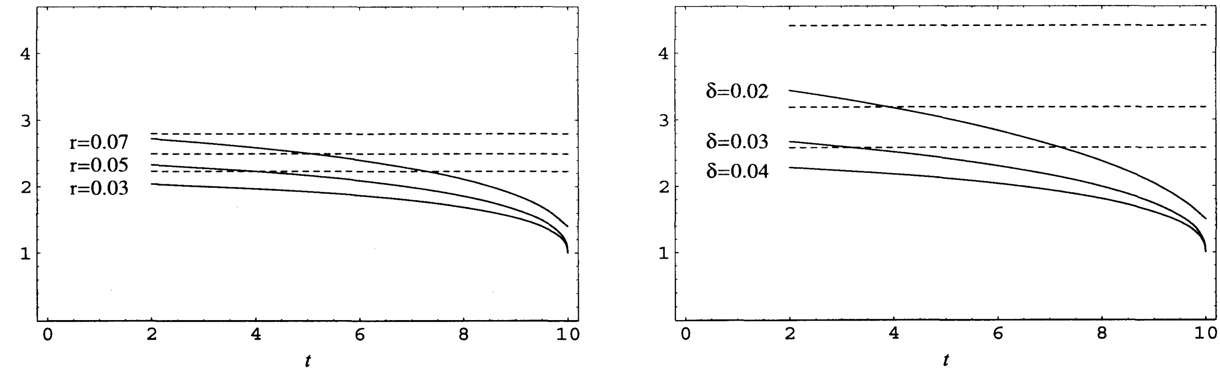
where $r>0$ is the risk-free rate of interest, $\delta\geq 0$ the continuous dividend rate, and $\sigma>0$ the volatility coefficient of stock returns. Assume that all of the coefficients (r,δ,σ) are constant. The stock price process $(S_t)_{t\in[0,T]}$ is represented under the equivalent martingale (i.e., risk-neutral) measure $\mathbb{P}$, which indicates that the stock has mean rate of return r, and the process W is a $\mathbb{P}$-Brownian motion.

Consider an ESO written on the stock price process $(S_t)_{t\in[0,T]}$, which has maturity date T and strike price K. Let $t=0$ and $t=T_1$ $(\in(0,T))$ respectively denote the grant and vesting dates of the ESO. Since the ESO can be exercised at any time during $[T_1,T]$, it is adequate to formulate the vested ESO as an ordinary American call option (or its extension) written on a dividend-paying stock in this time interval. For the sake of a clear argument, assume for a while that no exit from the firm occurs before maturity. Let $C(S_t,t)\equiv C(S_t,t;T,K)$ denote the value of the American vanilla call option at time $t\in[T_1,T]$ with maturity date T and strike price K. In the absence of arbitrage opportunities, the value $C(S_t,t)$ is a solution of an *optimal stopping problem*

$$C(S_t,t)=\operatorname*{ess\,sup}_{\tau_e\in[t,T]}\mathbb{E}\left[\mathrm{e}^{-r(\tau_e-t)}(S_{\tau_e}-K)^+\,|\mathcal{F}_t\right], \tag{2.2}$$

for $t\in[T_1,T]$, where $(x)^+=\max(x,0)$ for $x\in\mathbb{R}$, τ_e is a stopping time of the filtration $(\mathcal{F}_t)_{t\geq 0}$, and the conditional expectation is calculated under the measure $\mathbb{P}$. The random variable $\tau_e^*\in[t,T]$ is called an *optimal stopping time* if it gives the supremum value of the right-hand side of (2.2). Let $\mathcal{D}_0=\{(S_t,t)\in\mathbb{R}_+\times[0,T]\}$ denote the whole domain and let

$$\mathcal{D}=\{(S_t,t)\in\mathbb{R}_+\times[T_1,T]\}\subset\mathcal{D}_0$$



(a) $r = 0.03, 0.05, 0.07$, $\delta = 0.05$ (b) $r = 0.03$, $\delta = 0.02, 0.03, 0.04$

Figure 1: Normalized early exercise boundary $(\bar{S}_t/K)_{t\in[T_1,T]}$ ($T_1 = 2$, $T = 10$, $\sigma = 0.3$).

denote the domain after the vesting date. Solving the optimal stopping problem (2.2) is equivalent to finding the points $(S_t, t) \in \mathcal{D}$ for which early exercise of the ESO is optimal. Let $\mathcal{C}$ and $\mathcal{E}$ denote the *continuation region* and *exercise region*, respectively. Clearly, the continuation region $\mathcal{C}$ is the complement of $\mathcal{E}$ in $\mathcal{D}$, and the boundary that separates $\mathcal{E}$ from $\mathcal{C}$ is referred to as an *early exercise boundary*, which is defined by

$$\bar{S}_t = \inf\left\{S \in \mathbb{R}_+ \,|\, C(S,t) = (S-K)^+\right\}, \quad t \in [T_1, T]. \tag{2.3}$$

Since $C(S_t, t)$ is nondecreasing in S_t, the early exercise boundary $(\bar{S}_t)_{t\in[T_1,T]}$ is an *upper* critical stock price, above which it is advantageous to exercise the ESO before maturity. Also, it has been known in [12] that

$$\bar{S}_T = \max\left(1, \frac{r}{\delta}\right) K. \tag{2.4}$$

Let $C_\infty(S)$ be the value of the perpetual American call option at the vesting date T_1 with strike price K and initial stock price $S_{T_1} \equiv S$, i.e., $C_\infty(S) = \lim_{T\to\infty} C(S, T_1; T, K)$. Then, from McKean [16], we have

$$C_\infty(S) = \begin{cases} \dfrac{\bar{S}}{\theta}\left(\dfrac{S}{\bar{S}}\right)^\theta, & S < \bar{S} \\ S - K, & S \geq \bar{S}, \end{cases} \tag{2.5}$$

where $\bar{S}$ $(> K)$ is the value of the perpetual early exercise boundary, given by

$$\bar{S} = \frac{\theta}{\theta - 1} K, \tag{2.6}$$

and $\theta > 1$ is a positive root of the quadratic equation

$$\tfrac{1}{2}\sigma^2\theta^2 + (r - \delta - \tfrac{1}{2}\sigma^2)\theta - r = 0, \tag{2.7}$$

namely,

$$\theta = \frac{1}{\sigma^2}\left\{-(r - \delta - \tfrac{1}{2}\sigma^2) + \sqrt{(r - \delta - \tfrac{1}{2}\sigma^2)^2 + 2\sigma^2 r}\right\}. \tag{2.8}$$

3 Valuation with a barrier option

3.1 Heuristics for the barrier level

For the ESO valuation problem, Hull and White [10] have recently developed a binomial-tree model in the European framework, assuming the early exercise policy such that an executive exercises her/his vested ESOs when the stock price breaches a pre-specified target value, say H ($> K$). If the target value H is attained prior to maturity, then the executive receives $H - K$, otherwise (s)he receives $(S_T - K)^+$ at maturity. Let $M = H/K$ be the early exercise multiple of the strike price. In general, it is quite difficult to estimate M, because it depends on the market situations. In fact, there is no theoretical research for finding an *optimal* value of M in the literature on ESOs. Empirically, Carpenter [5, Table 1] reported that $M \approx 2.75$ on average from a sample of ESO exercises by top executives at 40 firms, whereas Huddart and Lang [9, Table 4] found that $M \approx 2.22$ on average from another sample of exercises by all employees at 8 firms.

In all fairness, arbitrary parameters should be excluded from the valuation model if at all possible. To determine the early exercise multiple M in a rational manner, we first give a simple approximation for the early exercise boundary $(\bar{S}_t)_{t\in[T_1,T]}$. There have been several approximations developed for the early exercise boundary, most of which are based on its short-time asymptotics near maturity and hence they are suitable for short-term options; see Barone-Adesi and Whaley [2], Bjerksund and Stensland [4], Goodman and Ostrov [8] and references therein. However, for the purpose of obtaining a practical approximation for the multiple M, we need a much simpler approximation for the boundary $(\bar{S}_t)_{t\in[T_1,T]}$ than those previously established approximations.

Based on the observations in Figure 1, we introduce two naive assumptions on the shape of $(\bar{S}_t)_{t\in[T_1,T]}$ such that (i) the boundary is a square-root function of the remaining time to maturity $\tau = T - t$, starting from $\bar{S}_T$ given in (2.4) at $\tau = 0$, and (ii) the boundary already reaches at the perpetual value $\bar{S}$ given in (2.6) at $\tau = T < \infty$, i.e., $\bar{S}_0 = \bar{S}$. The assumption (i) reflects the boundary behavior close to maturity, while the assumption (ii) is due to the long period up to maturity. From these two assumptions, we propose a *square-root approximation* for the early exercise boundary as

$$\bar{S}_t = \bar{S}_T + (\bar{S} - \bar{S}_T)\sqrt{\frac{T-t}{T}}, \quad t \in [T_1, T]. \tag{3.1}$$

No doubt, complete accuracy cannot be expected from this rough approximation. In particular, for $\bar{S} \gg \bar{S}_T$, it overestimates the true value for small $t \in [T_1, T]$. The principal purpose in proposing the square-root approximation in (3.1) is, however, to generate a concise approximation for H, or equivalently for M. It is a matter of course that there is no definitive method for determining the flat boundary H that approximates the curved boundary $(\bar{S}_t)_{t\in[T_1,T]}$. Among a few alternatives, we adopt here a natural approximation such that H is given by the average level of the curve, which is extensively defined on the whole contract period $[0, T]$, i.e.,

$$H = \frac{1}{T}\int_0^T \bar{S}_t \,\mathrm{d}t = \frac{1}{3}\bar{S}_T + \frac{2}{3}\bar{S}. \tag{3.2}$$

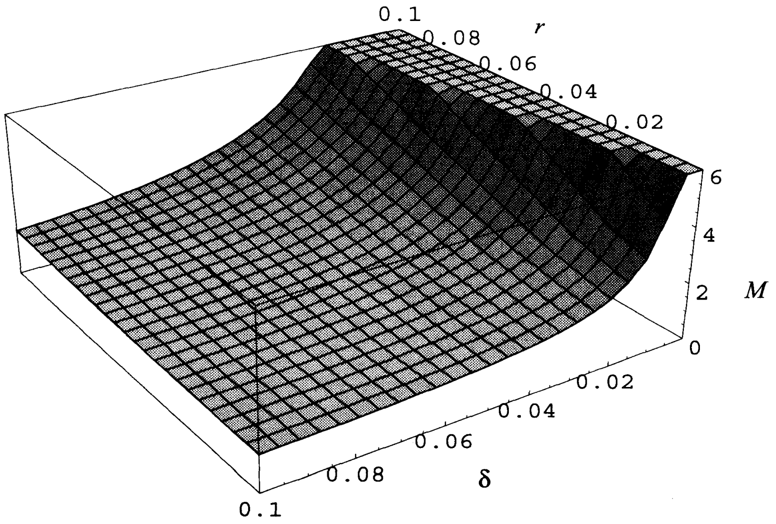


Figure 2: 3D-surface of the early exercise multiple M on the r-δ plane ($\sigma = 0.3$, $(r, \delta) \in (0, 0.1] \times (0, 0.1]$)

Clearly, H is equal to the level dividing the difference between $\bar{S}_T$ and $\bar{S}$ internally in the ratio 2:1, and hence $\bar{S} > H > \bar{S}_T \geq K$. The simple early exercise policy with H would be useful in practical business, and above all things we will see in Section 5 that a European call option with an upper barrier at H in (3.2) exactly follows benchmark results generated by an American binomial-tree model for ESOs.

Remark 1 Figure 2 illustrates a 3-dimensional surface of the early exercise multiple

$$M = \frac{H}{K} = \frac{1}{3}\max\left(1, \frac{r}{\delta}\right) + \frac{2}{3}\frac{\theta}{\theta - 1} \tag{3.3}$$

on the r-δ plane for $\sigma = 0.3$ and $(r, \delta) \in (0, 0.1] \times (0, 0.1]$. We have actually depicted $\min(M, 6)$ in the figure, in which we see that the slope of the surface tends to be rapidly steep as $\delta \to 0$ because $\lim_{\delta \to 0} M = \infty$. This effect of δ on M is consistent with an empirical observation in Bettis et al. [3] such that ESOs are exercised earlier in firms with higher dividend yields. From Figure 2, we see that M is sensitive to r if δ is small, and that M is relatively insensitive to the parameters r and δ if $r < \delta$, being flat on that region; cf. Figure 1. The insensitivity to (r, δ) is a desirable property of the early exercise policy in a versatile economy.

3.2 An up-and-out call option

We shall derive an explicit valuation formula by using a continuous-time model of an up-and-out European call option with an upper barrier at H ($> K$) and a rebate $R = H - K$ (> 0). Let $\Phi(x)$ for $x \in \mathbb{R}$ denote the standard normal cumulative distribution function (cdf), ϕ be its probability density function (pdf), i.e.,

$$\Phi(x) = \int_{-\infty}^{x} \phi(t)\mathrm{d}t \quad \text{with} \quad \phi(t) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{1}{2}t^2},$$

and for $x, y, \tau > 0$ let

$$d_{\pm}(x, y, \tau) = \frac{\log(x/y) + (r - \delta \pm \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}.$$

Then, rearranging the results of Rich [18], we have a much simpler expression for the up-and-out call value as follows:

Lemma 1 *Let $c_H(S_t, t; T, K)$ denote the value of the up-and-out European call option at time $t \in [T_1, T]$, with strike K, barrier at H $(> K)$, rebate R and maturity T. Also, let $c(S_t, t; T, K)$ denote the value of the associated European vanilla call option. Then, we have*

$$\begin{aligned}
&c_H(S_t, t; T, K) \\
&\quad = c(S_t, t; T, K) - c(S_t, t; T, H) - \left(\frac{H}{S_t}\right)^{2\alpha} \left\{ c\left(\frac{H^2}{S_t}, t; T, K\right) - c\left(\frac{H^2}{S_t}, t; T, H\right) \right\} \\
&\quad - (H - K)\mathrm{e}^{-r(T-t)} \left\{ \Phi(d_-(S_t, H, T - t)) - \left(\frac{H}{S_t}\right)^{2\alpha} \Phi(d_-(H, S_t, T - t)) \right\} \\
&\quad + R \left\{ \left(\frac{H}{S_t}\right)^{\alpha+\beta} \Phi(-h_+(H, S_t, T - t)) + \left(\frac{H}{S_t}\right)^{\alpha-\beta} \Phi(-h_-(H, S_t, T - t)) \right\},
\end{aligned}$$

where the parameters α and β are given by

$$\alpha = \frac{1}{\sigma^2}\left(r - \delta - \tfrac{1}{2}\sigma^2\right), \qquad \beta = \sqrt{\alpha^2 + \frac{2r}{\sigma^2}}, \tag{3.4}$$

and $h_{\pm}$ is defined for $x, y, \tau > 0$ as

$$h_{\pm}(x, y, \tau) = \frac{\log(x/y) \pm \beta\sigma^2\tau}{\sigma\sqrt{\tau}}.$$

3.3 Employment termination

We now incorporate the random exit feature into the model. Executives may leave the firm voluntarily or involuntarily during the contract period $[0, T]$. They lose unvested ESOs if they leave the firm during the vesting period $[0, T_1]$. After vesting, however, they may leave the firm, thereby exercising or forfeiting vested ESOs. Let $V^{\circ}(S; T)$ denote the ESO value at grant date $t = 0$ with maturity date T and initial stock price $S_0 = S$, assuming no exit before maturity. This value is equivalent to that of a contingent claim of either receiving at the vesting date the up-and-out European call option with value $c_H(S_{T_1}, T_1; T, K)$ if $S_{T_1} < H$, or exercising it immediately to receive $S_{T_1} - K$ otherwise. Hence, by the risk-neutral valuation principle, we have

$$V^{\circ}(S; T) = \mathrm{e}^{-rT_1}\mathbb{E}\left[c_H(S_{T_1}, T_1; T, K)\mathbf{1}_{\{S_{T_1} < H\}} + (S_{T_1} - K)\mathbf{1}_{\{S_{T_1} \geq H\}} \,\middle|\, \mathcal{F}_0\right], \tag{3.5}$$

where $\mathbf{1}_A$ is the indicator function of A $(\in \mathcal{F})$.

Following Jennergren and Näslund [11], assume that exit from the firm occurs according to a Poisson process with an exogenous constant rate λ (> 0), and that the exit process is independent of the stock price process $(S_t)_{t\in[0,T]}$; cf. Carr and Linetsky [6] for more general point processes with rate dependent on the stock price. Let $V(S; T)$ denote the value of the associated ESO with exit rate λ. Following Raupach [17], we have

Lemma 2 *For $0 < T_1 < T$, we have*

$$V(S;T) = \mathrm{e}^{-\lambda T} V^{\circ}(S;T) + \int_{T_1}^{T} \lambda \mathrm{e}^{-\lambda t} V^{\circ}(S;t)\mathrm{d}t. \tag{3.6}$$

3.4 Valuation formula

To obtain an explicit expression of the conditional expectation in (3.5), let us introduce $\Phi_2(x, y; \gamma)$ for $(x, y) \in \mathbb{R}^2$, which denotes the bivariate standard normal cdf with the correlation coefficient γ ($|\gamma| < 1$), defined by

$$\Phi_2(x, y; \gamma) = \int_{-\infty}^{x} \int_{-\infty}^{y} \phi_2(u, v; \gamma)\mathrm{d}v\mathrm{d}u$$

with

$$\phi_2(u, v; \gamma) = \frac{1}{2\pi\sqrt{1-\gamma^2}} \exp\left\{-\frac{1}{2(1-\gamma^2)}\left(u^2 - 2\gamma uv + v^2\right)\right\}.$$

Then, we obtain

Theorem 1 *Let $V(S;T)$ be the value of ESO at grant date $t = 0$, with strike K, initial stock price $S_0 = S$, exit rate λ, vesting date T_1 and maturity T ($0 < T_1 < T$). Then, we have*

$$V(S;T) = \mathrm{e}^{-\lambda T} V^{\circ}(S;T) + \int_{T_1}^{T} \lambda \mathrm{e}^{-\lambda t} V^{\circ}(S;t)\mathrm{d}t,$$

where $V^{\circ}(S;T)$ denotes the associated ESO value with no exit before maturity, which is given by

$$\begin{aligned} V^{\circ}(S;T) = {} & S\mathrm{e}^{-\delta T_1}\Phi(d_{11}) - K\mathrm{e}^{-rT_1}\Phi(d_{21}) \\ & + S\mathrm{e}^{-\delta T}\psi_S - K\mathrm{e}^{-rT}\psi_K + (H-K)\psi_R. \end{aligned} \tag{3.7}$$

The coefficients ψ_S, ψ_K and ψ_R are defined by

$$\begin{aligned} \psi_S &= \Phi_2(-d_{11}, d_{12}; -\rho) - \Phi_2(-d_{11}, d_{13}; -\rho) \\ &\quad - \left(\frac{H}{S}\right)^{2(\alpha+1)} \left\{\Phi_2(d_{31}, d_{32}; \rho) - \Phi_2(d_{31}, d_{33}; \rho)\right\}, \\ \psi_K &= \Phi_2(-d_{21}, d_{22}; -\rho) - \Phi_2(-d_{21}, d_{23}; -\rho) \\ &\quad - \left(\frac{H}{S}\right)^{2\alpha} \left\{\Phi_2(d_{41}, d_{42}; \rho) - \Phi_2(d_{41}, d_{43}; \rho)\right\}, \\ \psi_R &= \left(\frac{H}{S}\right)^{\alpha+\beta} \Phi_2(h_{11}, -h_{12}; -\rho) + \left(\frac{H}{S}\right)^{\alpha-\beta} \Phi_2(h_{21}, -h_{22}; -\rho), \end{aligned}$$

where

$$\rho = \sqrt{\frac{T_1}{T}},$$

and

$$
\begin{aligned}
d_{11} &= d_+(S,H,T_1), & d_{12} &= d_+(S,K,T), & d_{13} &= d_+(S,H,T),\\
d_{21} &= d_-(S,H,T_1), & d_{22} &= d_-(S,K,T), & d_{23} &= d_-(S,H,T),\\
d_{31} &= d_+(H,S,T_1), & d_{32} &= d_+(H^2/K,S,T), & d_{33} &= d_+(H,S,T),\\
d_{41} &= d_-(H,S,T_1), & d_{42} &= d_-(H^2/K,S,T), & d_{43} &= d_-(H,S,T),\\
h_{11} &= h_+(H,S,T_1), & h_{12} &= h_+(H,S,T), & &\\
h_{21} &= h_-(H,S,T_1), & h_{22} &= h_-(H,S,T). & &
\end{aligned}
$$

Remark 2 To compute the integral in (3.6) numerically, we have to evaluate $V^\circ(S;t)$ for t close to T_1, where the computation often fails if we directly use the expression in (3.7). The reason is that the bivariate normal cdf Φ_2 becomes highly correlated as $t \to T_1$, which causes degeneration. However, we can derive the analytical limit $V^\circ(S;T_1)$ directly from (3.7), which does not include Φ_2: For $\rho = \pm 1$, it is easily verified that

$$
\Phi_2(x,y;1) = \Phi(x \wedge y) \quad \text{and} \quad \Phi_2(x,y;-1) = \begin{cases} \Phi(x) - \Phi(-y), & x+y>0 \\ 0, & x+y \le 0, \end{cases}
$$

for $(x,y) \in \mathbb{R}^2$. Using these relations, we have

$$
\begin{aligned}
\lim_{T\to T_1} \psi_S &= \Phi(-d_{11}) - \Phi(-d_+(S,K,T_1))\\
\lim_{T\to T_1} \psi_K &= \Phi(-d_{21}) - \Phi(-d_-(S,K,T_1))\\
\lim_{T\to T_1} \psi_R &= 0,
\end{aligned}
$$

and hence

$$
\begin{aligned}
\lim_{T\to T_1} V^\circ(S;T) &= S\mathrm{e}^{-\delta T_1}\Phi(d_+(S,K,T_1)) - K\mathrm{e}^{-rT_1}\Phi(d_-(S,K,T_1)) \\
&= c(S,0;T_1,K),
\end{aligned} \tag{3.8}
$$

which is consistent with the expected result.

Consider the perpetual case with $T = \infty$, for which we can obtain a closed-form solution, being *exact* in the Black-Scholes-Merton formulation.

Theorem 2 *Let $V_\infty(S)$ be the exact value of the perpetual ESO with exit rate λ, i.e., $V_\infty(S) = \lim_{T\to\infty} V(S;T)$. Then, we have*

$$
V_\infty(S) = \begin{cases} \displaystyle\int_{T_1}^{\infty} \lambda \mathrm{e}^{-\lambda t} V^\circ(S;t)\mathrm{d}t, & \lambda > 0 \\ S\mathrm{e}^{-\delta T_1}\Phi(d_+(S,\bar{S},T_1)) - K\mathrm{e}^{-rT_1}\Phi(d_-(S,\bar{S},T_1)) & \\ \displaystyle\quad + \frac{\bar{S}}{\theta}\left(\frac{S}{\bar{S}}\right)^{\theta} \Phi(h_-(\bar{S},S,T_1)), & \lambda = 0. \end{cases} \tag{3.9}
$$

4 Mean exercise time

In empirical studies of Carpenter [5] and Huddart and Lang [9], they displayed descriptive statistics for sample variables characterizing the exercise policy of ESOs, which include the average of the ratio of stock price at the time of ESO exercise to the strike price (i.e., M in this paper) and the average time of ESO exercise in years. The latter characteristic quantity still plays a critical role in the SFAS 123R proposal that recommends a modified Black-Scholes model where the stated maturity is replaced with the expected ESO life time. From samples of ten-year ESOs, the average exercise time has been reported as 5.83 years in [5] and 3.4 years in [9].

Let X denote the exercise time of a vested ESO with strike K, initial stock price $S_0 = S$, vesting date T_1 and maturity T ($0 < T_1 < T$). Clearly, the exercise time of an unvested ESO is defined as zero. Assume that $X = T$ when null payoff is paid at maturity. In addition, let $L(S;T) = \mathbb{E}[X \mid \mathcal{F}_0]$ denote the mean exercise time. To obtain an explicit expression for $L(S;T)$ in our barrier-option model, we begin with the analysis of a first passage time to the barrier: Let $\tau_H(S)$ denote the first passage time at which the stock price process S_t starting from $S_0 = S$ hits the barrier H ($> S$) for the first time, i.e., $\tau_H(S) = \inf\{t \geq 0 \mid S_t \geq H, S_0 = S\}$. Then, by the theory of diffusion processes, we have

Lemma 3 *Let $\bar{F}_H(t;S) = \mathbb{P}\{\tau_H(S) > t\}$ denote the complementary cdf of the first passage time $\tau_H(S)$, and define the Laplace transform of $\bar{F}_H(t;S)$ by*

$$\bar{F}_H^*(\lambda;S) = \int_0^\infty \mathrm{e}^{-\lambda t} \bar{F}_H(t;S)\mathrm{d}t,$$

for $\lambda \in \mathbb{C}$ ($\mathrm{Re}(\lambda) > 0$). Then, we have

$$\bar{F}_H(t;S) = \Phi\left(\frac{\log(H/S) - \alpha\sigma^2 t}{\sigma\sqrt{t}}\right) - \left(\frac{H}{S}\right)^{2\alpha} \Phi\left(-\frac{\log(H/S) + \alpha\sigma^2 t}{\sigma\sqrt{t}}\right), \tag{4.1}$$

and

$$\bar{F}_H^*(\lambda;S) = \frac{1}{\lambda}\left\{1 - \left(\frac{H}{S}\right)^{\alpha-\beta^*}\right\}, \tag{4.2}$$

where the parameters α and β^* are given by

$$\alpha = \frac{1}{\sigma^2}\left(r - \delta - \tfrac{1}{2}\sigma^2\right), \qquad \beta^* = \sqrt{\alpha^2 + \frac{2\lambda}{\sigma^2}}. \tag{4.3}$$

Consider the case that no exit occurs before maturity, in which a vested ESO can be exercised either at (i) the vesting date T_1 if $S_{T_1} \geq H$, or (ii) $\min\{T_1 + \tau_H(S_{T_1}), T\}$ otherwise. Hence, the exercise time X can be defined in terms of τ_H as

$$\begin{aligned} X &= T_1 \mathbf{1}_{\{S_{T_1} \geq H\}} + \min\{T_1 + \tau_H(S_{T_1}), T\} \mathbf{1}_{\{S_{T_1} < H\}} \\ &= T_1 \mathbf{1}_{\{S_{T_1} \geq H\}} + \Big(\min\{\tau_H(S_{T_1}), T - T_1\} + T_1\Big) \mathbf{1}_{\{S_{T_1} < H\}} \\ &= T_1 + \min\{\tau_H(S_{T_1}), T - T_1\} \mathbf{1}_{\{S_{T_1} < H\}}. \end{aligned} \tag{4.4}$$

Let $\bar{G}(t) = \mathbb{P}\{X > t\}$ $(t \geq 0)$ define the complementary cdf of X. Then, from (4.4), we have

$$\bar{G}(t) = \mathbb{P}\{\min\{\tau_H(S_{T_1}), T - T_1\}\mathbf{1}_{\{S_{T_1} < H\}} > t - T_1\}, \quad t \geq 0,$$

which yields that $\bar{G}(t) = 1$ for $t < T_1$, and $\bar{G}(t) = 0$ for $t \geq T$. By the equivalence relation for $t \in [T_1, T)$

$$\Big\{\min\{\tau_H(S_{T_1}), T - T_1\}\mathbf{1}_{\{S_{T_1} < H\}} > t - T_1\Big\} \equiv \Big\{\tau_H(S_{T_1})\mathbf{1}_{\{S_{T_1} < H\}} > t - T_1\Big\},$$

we obtain for $t \in [T_1, T)$

$$\begin{aligned}
\bar{G}(t) &= \mathbb{P}\{\tau_H(S_{T_1})\mathbf{1}_{\{S_{T_1} < H\}} > t - T_1\} \\
&= \int_0^H \bar{F}_H(t - T_1; S')p(S', T_1; S, 0)\mathrm{d}S' \\
&= \Phi_2\left(-d_{21}, -d_-(S, H, t); \sqrt{\frac{T_1}{t}}\right) - \left(\frac{H}{S}\right)^{2\alpha} \Phi_2\left(d_{41}, -d_-(H, S, t); -\sqrt{\frac{T_1}{t}}\right),
\end{aligned}$$

from Lemma 3. Taking the limit $t \to T_1$ in $\bar{G}(t)$ and using the relations

$$\Phi_2(x, x; 1) = \Phi(x) \quad \text{and} \quad \Phi_2(x, -x; -1) = 0$$

for $x \in \mathbb{R}$, we have

$$\lim_{t \to T_1} \bar{G}(t) = \Phi(-d_{21}) = \mathbb{P}\{S_{T_1} < H\}. \tag{4.5}$$

Also, letting $t \to T$, we have

$$\lim_{t \to T} \bar{G}(t) = \Phi_2\left(-d_{21}, -d_{23}; \rho\right) - \left(\frac{H}{S}\right)^{2\alpha} \Phi_2\left(d_{41}, -d_{43}; -\rho\right). \tag{4.6}$$

The explicit expression for the complementary cdf $\bar{G}$ leads to

Theorem 3 *Let $L(S;T)$ denote the mean exercise time of a vested ESO with strike K, initial stock price $S_0 = S$, exit rate λ, vesting date T_1 and maturity T $(0 < T_1 < T)$. Then, we have*

$$\begin{aligned}
L(S;T) = T_1 \mathrm{e}^{-\lambda T_1} + \int_{T_1}^T \mathrm{e}^{-\lambda t}\Bigg\{&\Phi_2\left(-d_{21}, -d_-(S, H, t); \sqrt{\frac{T_1}{t}}\right) \\
&- \left(\frac{H}{S}\right)^{2\alpha} \Phi_2\left(d_{41}, -d_-(H, S, t); -\sqrt{\frac{T_1}{t}}\right)\Bigg\}\mathrm{d}t.
\end{aligned}$$

Corollary 1 *The mean exercise time $L(S;T)$ is*

(i) *a concave increasing function of maturity T; and*

(ii) *a convex decreasing function of exit rate λ.*

Using Lemma 3, we can derive a closed-form solution of the mean exercise time for the perpetual ESO, which gives an *upper bound* of $L(S;T)$ for the associated finite-lived ESO, due to the property (i) in Corollary 1. Similarly to the perpetual value $V_\infty(S)$, it also gives the exact result for the perpetual case in the Black-Scholes-Merton formulation.

Table 1: A comparison of values of executive stock options ($T = 10$, $T_1 = 2$, $S = K = 1$, $r = 0.03$, $\lambda = 0.0$).

δ	σ	binomial	$V^\circ(S;T)$	R.E.(%)	quadratic	R.E.(%)
0.02	0.2	0.2429	0.2415	(-0.58)	0.2474	(1.85)
	0.3	0.3406	0.3385	(-0.62)	0.3466	(1.76)
	0.4	0.4323	0.4298	(-0.58)	0.4395	(1.67)
0.03	0.2	0.2043	0.2034	(-0.44)	0.2084	(2.01)
	0.3	0.3010	0.2996	(-0.47)	0.3069	(1.96)
	0.4	0.3915	0.3899	(-0.41)	0.3991	(1.94)
0.04	0.2	0.1743	0.1732	(-0.63)	0.1776	(1.89)
	0.3	0.2682	0.2670	(-0.45)	0.2737	(2.05)
	0.4	0.3569	0.3554	(-0.42)	0.3641	(2.02)
0.05	0.2	0.1499	0.1484	(-1.00)	0.1524	(1.67)
	0.3	0.2406	0.2389	(-0.71)	0.2451	(1.87)
	0.4	0.3268	0.3251	(-0.52)	0.3333	(1.99)

Theorem 4 *Let $L_\infty(S)$ denote the exact mean exercise time of the perpetual ESO with exit rate λ, i.e., $L_\infty(S) = \lim_{T\to\infty} L(S;T)$. Then, for $\lambda > 0$ we have*

$$L_\infty(S) = T_1 \mathrm{e}^{-\lambda T_1} + \frac{1}{\lambda}\left\{\Phi(-d_-(S,\bar{S},T_1)) - \left(\frac{S}{\bar{S}}\right)^{\alpha-\beta^*} \Phi(h^*_-(\bar{S},S,T_1))\right\}, \tag{4.7}$$

where

$$h^*_-(\bar{S},S,T_1) = \frac{\log(\bar{S}/S) - \beta^*\sigma^2 T_1}{\sigma\sqrt{T_1}}.$$

Also, let $L^\circ_\infty(S) = \lim_{T\to\infty} L^\circ(S;T)$ for $\lambda = 0$. Then, we have

$$L^\circ_\infty(S) = \begin{cases} T_1 - \log\left(\frac{\bar{S}}{S}\right)\left\{T_1\Phi(-d_-(S,\bar{S},T_1)) - \frac{\sqrt{T_1}}{\alpha\sigma}\phi(-d_-(S,\bar{S},T_1))\right\}, & \alpha > 0 \\ +\infty, & \alpha \le 0. \end{cases}$$

5 Computational results

First we examine the accuracy of the formula for the ESO value $V(S;T)$ in Theorem 1. To see the quality of the formula, we numerically compare it with benchmark results generated by a binomial-tree model, which extends the standard American option model by introducing delayed vesting as well as random exit. The extended American binomial-tree model can be easily implemented by modifying the backward induction algorithm for valuing the standard American option; see the appendices of [1, 5, 19] for specific algorithms.

To accelerate the convergence of binomial-tree values, we use the 3-point Richardson scheme of extrapolating the 1000-, 2000- and 3000-period binomial values. As a standard set of parameters, we use $S = K = 1$, $T_1 = 2$ and $T = 10$ unless otherwise stated. As we saw in Figure 1

Table 2: A comparison of values of executive stock options ($T = 10$, $T_1 = 2$, $S = K = 1$, $r = 0.03$, $\lambda = 0.1$).

δ	σ	binomial	$V(S;T)$	R.E.(%)	quadratic	R.E.(%)
0.02	0.2	0.1717	0.1709	(-0.47)	0.1741	(1.40)
	0.3	0.2425	0.2413	(-0.49)	0.2458	(1.36)
	0.4	0.3097	0.3083	(-0.45)	0.3139	(1.36)
0.03	0.2	0.1466	0.1462	(-0.27)	0.1492	(1.77)
	0.3	0.2167	0.2161	(-0.28)	0.2204	(1.71)
	0.4	0.2831	0.2824	(-0.25)	0.2880	(1.73)
0.04	0.2	0.1268	0.1264	(-0.32)	0.1291	(1.81)
	0.3	0.1950	0.1947	(-0.15)	0.1988	(1.95)
	0.4	0.2604	0.2599	(-0.19)	0.2652	(1.84)
0.05	0.2	0.1103	0.1100	(-0.27)	0.1123	(1.81)
	0.3	0.1767	0.1762	(-0.28)	0.1799	(1.81)
	0.4	0.2403	0.2400	(-0.12)	0.2449	(1.91)

that the early exercise boundary is relatively less sensitive to risk-free rate r than the other market parameters, and hence we fix $r = 0.03$ as a sample value. Also, in order to accelerate the numerical integration appeared in $V(S;T)$, we approximate $V^{\circ}(S;u)$ for $u \in [T_1, T]$ by a cubic spline curve interpolating $V^{\circ}(S;T_1) = c(S, 0; T_1, K)$ at $u = T_1$ and values $V^{\circ}(S;u)$ at adjacent points $u = T_1 + 1, \ldots, T$. The spline approximation quickly generates a very accurate result, of which first four digits coincide with those of the direct numerical integration result.

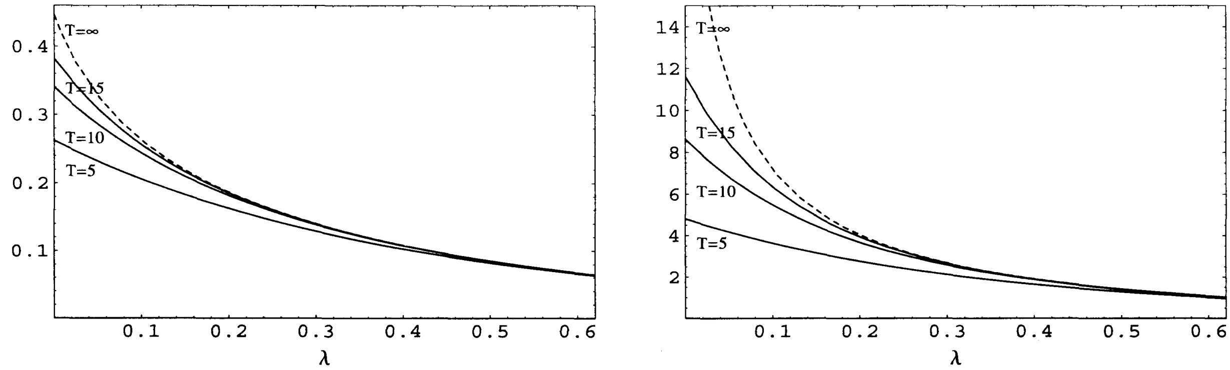
Tables 1 and 2 compare our approximation for $V(S;T)$ with the benchmark result (referred to as "binomial") and the quadratic approximation of Kimura [14] (referred to as "quadratic") for some combinations of the parameters δ, σ and λ, where "R.E.(%)" stands for the relative percentage error with respect to the benchmark. These tables show that

(i) the larger dividend rate δ, the better our approximation; and

(ii) the quality of our approximation is much better than the quadratic approximation.

The first property (i) certainly reflects the gap between $\bar{S}$ and $(\bar{S}_t)_{t\in[T_1,T]}$ for small δ, which was shown in Figure 1(b). The second property (ii) is a little bit unexpected result, since the barrier option models have been rather considered as an *ad hoc* approximation in the previous literature. From the tables, we see that our approximations always underestimate the benchmark results and the absolute values of the relative percentage errors are less than 1% for all cases, whereas the quadratic approximations overestimate the benchmarks and the relative percentage errors are less than about 2%.

To see the effects of the length of maturity to the ESO value, Figure 3(a) illustrates the values $V(S;T)$ for $T = 5, 10, 15$ as functions of exit rate λ, where we added the perpetual value $V_{\infty}(S)$ for $T = \infty$ drawn in a dashed line. We observe from this figure that the perpetual value



(a) ESO value (b) mean exercise time

Figure 3: Values and mean exercise times of executive stock options with different maturities ($T_1 = 2$, $S = K = 1$, $r = 0.05$, $\delta = 0.03$, $\sigma = 0.3$).

is an upper bound of $V(S;T)$ for $T < \infty$, which is almost insensitive to the maturity date, e.g., if $\lambda > 0.3$ for $T = 10$. This robustness is due to the fact that the ESO exercise for large λ is actually driven by exit not by maturity. Hence, for large λ, $V(S;T)$ can be well approximated by $V_\infty(S)$ even for finite-lived cases.

Figure 3(b) also illustrates the effects of maturity to the mean exercise time $L(S;T)$, where the dashed line represents $L_\infty(S)$ for $T = \infty$. Clearly, this figure provides numerical validation of Corollary 1. In the same way as for the ESO value, we apply the cubic spline approximation to the numerical integration appeared in $L(S;T)$. That is, we approximate the complementary cdf $\bar{G}(t)$ for $t \in [T_1, T]$ by a cubic spline curve evaluated at a sequence of discrete points $t = T_1, T_1 + \Delta t, \ldots, T$. We used the time step $\Delta t = 0.1$ in the experiment. From Figure 3(b), we see that the mean exercise time $L(S;T)$ also has the robustness similar to $V(S;T)$ for large λ, and further that the mean exercise time is differently affected by the length of maturity especially for $\lambda \approx 0$. We have $L^\circ(S;5) = 4.8073$ for the short-term case ($T = 5$) and $L^\circ(S;10) = 8.6316$ for the standard-term case ($T = 10$), which implies that the ESO exercise occurs mostly at maturity for $T = 5$, but for $T = 10$ it may occur before maturity by breaching the barrier. We also have $L^\circ_\infty = \infty$ because $\alpha < 0$ for this particular case.

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