数理解析研究所講究録
第 1675 巻 2010 年 26-36

Lifetime Ruin, Consumption and Annuity

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1 Introduction

There is growing concern about financial ruin after retirement due to increased longevity and institutional inadequacy in Social Security. We study the self-annuitization and the dynamic optimal portfolio selection to minimize the probability of lifetime ruin. In order to avoid the risk of living after spending out his wealth, there are three financial instruments, a risky asset like corporate stock, risk free asset like bank account, and annuity which guarantee fixed income until death. As a retiree is getting older, the annuity price is becoming cheaper to purchase it. The problem is to find the optimal portfolio of three financial assets and the timing to buy annuity.

The theoretical study has been started by Browne [3]. The extended portfolio problem to minimize the probability of lifetime ruin has formulated and solved by Young [7]. The optimal investment strategy for risky asset, however, needs to borrow large amount of money when his wealth becomes small. This strategy is unrealistic.

Bayraktar et al [1] solved the problem by borrowing constraint or by introducing borrowing rate. The optimal solution is holding only the risky asset afterwards his wealth equals to the risky investment; the policy is divided two domains for wealth : 1) both of risky asset and risk free asset, 2) only risky asset.

When we take an annuity into portfolio on these setting, it is generally difficult to solve it. Because the price of annuity depends on his remaining year of life. This problem has been studied in Milevsky et al [5] by partial differential equation. In this paper, we assume firstly that consumption plan is based on the optimal investment policy of Bayraktar [1], and secondly that the annuity price is exponentially decreasing. We easily obtain the optimal portfolio strategy and the average timing to purchase annuity.

The paper is organized as follows. In section 2, we describe the optimal portfolio problem for a risky asset, risk free asset and annuity. And we explain the consumption plan which is depending the solution of [1]. The objective is set to minimize the probability of life time ruin. In section 3, the optimal investment strategy for risky and risk free assets is obtained by HJB equations according to the consumption plan. In section 4, the annuity price function is defined and we solve wealth processes for optimal investment strategies. We consider the time to purchase annuity as the passage time to cross the annuity price function. We show a proposition on Laplace transform for the time to purchase annuity, which gives the average time to purchase annuity. In section 5 we calculate a numerical example of average Japanese retiree case and conclude the paper.

2 Financial model

Three assets are available for a retiree; the first is a risk free asset whose interest rate is r, and we assume borrowing is not allowed. The second is a risky asset whose price S_t satisfies the stochastic differential equation,

$$dS_t = \mu S_t dt + \sigma S_t dB_t, S_0 = S.$$

The process B_t is a Brownian motion and it is assumed that $\mu > r, \sigma > 0$.

The third security is an annuity which a retiree can purchase by the price A_t after t year of retirement. When he decides the purchase time τ_a, he pays once A_{τ_a} and receives a constant amount c until his death. We assume that c is his minimal consumption level. Let λ be his hazard rate and T be the life expectancy at retirement time. We set the price of annuity at time t,

$$A_t = c(r+\lambda)^{\frac{t-T}{T}}.$$

We assume that the retiree changes his consumption level at a wealth level w_l, which is named realization level of ruin. When his wealth reaches to the realization level w_l, he will decrease his consumption level as the following function;

$$c(w) = \begin{cases} c + c_1 w & ; w > w_l \\ c_2 w & ; w \leq w_l. \end{cases}$$

$c(w)$ is assumed to be continuous at w_l, then

$$c + c_1 w_l = c_2 w_l.$$

Consumption rate is changed to $c_2 = c_1 + \frac{c}{w_l}$ after the time to hit w_l . The consumption function is depicted in Figure 1.

Let W_t be the wealth value of portfolio, w_0 be the initial wealth and α_t be the investment value into the risky asset at time t. The wealth process under borrowing constraint satisfies

$$dW_t = \{r(W_t - \alpha_t)^+ + \mu\alpha_t - c(W_t)\}dt + \sigma\alpha_t dB_t, \quad t < \tau_a, \tag{1}$$

before the time to purchase annuity τ_a, and after purchasing annuity it is

$$W_t = c, \quad t \geq \tau_a.$$

The objective function is to minimize the probability of lifetime ruin and it is defined as

$$h(w) = \inf_{\{0 \leq \alpha_t \leq w\}} Pr[\tau_0 < \tau_d | W_0 = w],$$

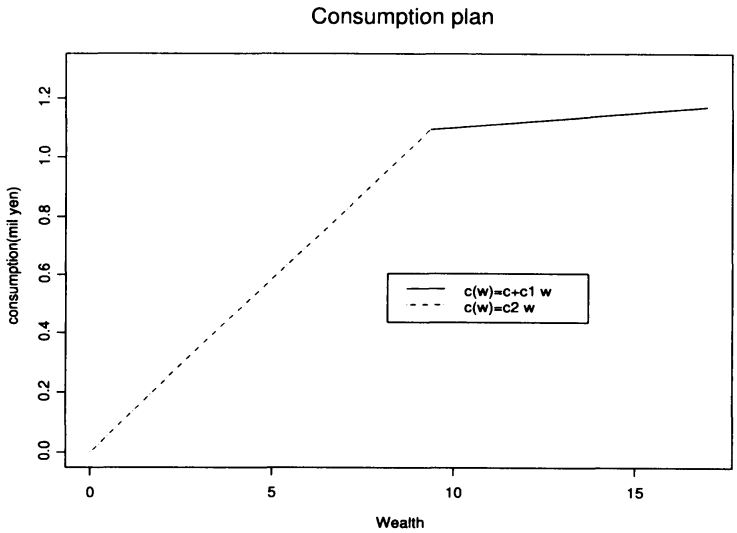

Figure 1: consumption

where τ_0 is the time of ruin and τ_d is the time of death. Let N denote a Poisson process with constant hazard rate λ such that N is independent of the Brownian motion B. The probability of lifetime ruin is expressed

$$Pr[\tau_0 < \tau_d | W_0 = w] = E[\exp(-\lambda \tau_0) | W_0 = w],$$

where the ruin time is defined as $\tau_0 = \inf\{t : W_t < c\}$. Applying Itô's formulas, we have

$$dh(W_t) = \mathcal{L}h(W_t) I_{\{W_t > \alpha_t\}} + h'(W_t)\sigma dB_t - h(W_t)(dN_t - \lambda dt)$$

where $\mathcal{L}h(w)$ is defined by

$$\mathcal{L}h(w) = [(r - c_1)w + (\mu - r)\alpha - c]h'(w) + \frac{1}{2}\sigma^2\alpha^2 h''(w) - \lambda h(w).$$

By defining $r_1 = r - c_1 > 0$, the Hamiltonian-Jacobi-Bellman equation for optimality is obtained as

$$\lambda h(w) = (r_1 w - c)h'(w) + \min_{0 < \alpha \leq w} [(\mu - r)\alpha h'(w) + \frac{1}{2}\sigma^2\alpha^2 h''(w)]. \qquad (2)$$

By $\alpha = (\mu - r)h'(w)/\{\sigma^2 h''(w)\}$, we have an ordinary differential equation as

$$\lambda h(w) = (r_1 w - c)h'(w) - m\frac{h'(w)^2}{h''(w)}, \quad m = \frac{(\mu - r)^2}{2\sigma^2}.$$

Whenever his wealth increases more than c/r_1, he purchases a console bond whose interest equals to $c + c_1 w$ forever. Then a boundary conditions is

$$h(c/r_1) = 0.$$

If his wealth is smaller than c, the probability ruin is 1. Then another boundary condition is

$$h(c_-) = 1.$$

The probability of ruin under borrowing constraints in the case of propositnal consumption of Proposition 2.2 in Bayraktar and Young [1] is given by

$$h(w) = h_0 \left(\frac{c}{r_1} - w\right)^d, \tag{3}$$

with

$$d = \frac{1}{2r_1}\left((r_1 + \lambda + m) + \sqrt{(r_1 + \lambda + m)^2 - 4r_1\lambda}\right) > 1,$$

and

$$m = \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2, \quad h_0 \geq 1.$$

3 Optimal investment strategy

Before the wealth reaches to the realization level w_l, his consumption function is $c(w) = c + c_1 w$. Then his optimal portfolio to risky asset is given for the wealth domain $0 \leq w \leq c/r_1$ from ([1]),

$$\alpha(w) = \begin{cases} g\left(\frac{c}{r_1} - w\right) & w_l \leq w \leq c/r_1; \cdots (i) \\ w & w \leq w_l, \end{cases} \tag{4}$$

with $g = \frac{\mu - r}{\sigma^2(d-1)}$. We define w_l as the wealth by which a retiree realizes the probability of ruin because his risk free asset investment becomes zero. The optimal investment for $w < w_l$ is bounded by borrowing constraints, that is, $\alpha(w) = w$. It implies that $g(c/r_1 - w_l) = w_l$. We obtain the realization wealth level w_l as follows,

$$w_l = \frac{g}{g+1}\frac{c}{r_1}. \tag{5}$$

It depends only on parameters of sharp ratio $(\mu - r)/\sigma$ and d, c, c_1 but not on the initial wealth w_0. After the wealth becomes smaller than w_l, his consumption is changed to $c(w) = c_2 w$, then his wealth process satisfies

$$dW_t = [(r - c_2)W_t + (\mu - r)\alpha]dt + \sigma\alpha dB_t, \quad W_t < w_l, t < \tau_a.$$

The HJB equation is similarly as (2)

$$\lambda h(t) = r_2 w h'(w) + \min_{0 \leq \alpha \leq w}[(\mu - r)\alpha h'(w) + \frac{1}{2}\sigma^2\alpha^2 h''(w)],$$

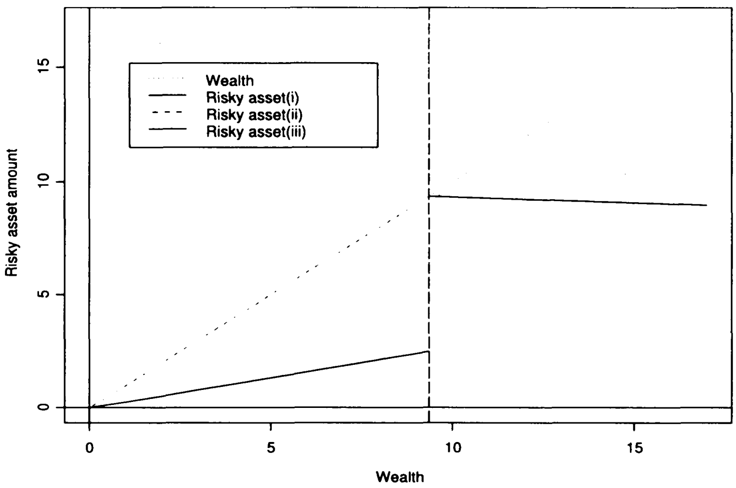

Figure 2: Optimal Investment: Risky & Riskless assets

where we set $r_2 = r - c_2 < 0$. The optimal ruin probability is decreasing function as given in the case of proportional consumption of Theorem 2.9 in [1],

$$h(w) = \begin{cases} h_1 \left(\frac{w}{w_l}\right)^{-a}, & \frac{\mu - r}{\sigma^2(a+1)} < 1; \\ h_1 \left(\frac{w}{w_l}\right)^{-k}, & \frac{\mu - r}{\sigma^2(a+1)} \geq 1, \end{cases}$$

with $h_1 = h(w_l)$ and

$$\begin{aligned} a &= \frac{1}{2r_2}\left(-(r_2 + \lambda + m) - \sqrt{(r_2 + \lambda + m)^2 - 4r_2\lambda}\right) > 0 \\ k &= \frac{1}{\sigma^2}\left((\mu - c_2 - \frac{1}{2\sigma^2}) + \sqrt{(\mu - c_2 - \frac{1}{2\sigma^2})^2 + 2\sigma^2\lambda}\right) > 0. \end{aligned}$$

The optimal investment to risky asset is given respectively for the two cases,

$$\alpha(w) = \begin{cases} w, & \frac{\mu - r}{\sigma^2(a+1)} \geq 1 \cdots (ii), \\ \frac{\mu - r}{\sigma^2(a+1)} w, & \frac{\mu - r}{\sigma^2(a+1)} < 1; \cdots (iii). \end{cases} \tag{6}$$

The optimal investment to risky asset is shown in Figure 2; The risky asset line (i) in right hand side of dashed line of w_l is below the wealth of dotted line of 45 degrees, which means that $w - \alpha_t$ is investment into riskless asset. In the left hand side of dashed line of w_l, there are two cases of risky investment; in the case (ii) of equation (6), the risky investment equals to his wealth, however, in the case (iii) of (6), he cuts the risky asset to the solid line (iii) in Figure 2.

4 Time to purchase annuity

4.1 Annuity price function

The price of annuity is decreasing function of time and then we simply assume that it is an exponential function. Then we get $A_t = c(r+\lambda)^{\frac{t-T}{T}}$ as follows;

$$A_t = A_0 e^{-\beta t}.$$

The annuity price of retirement time is assumed to be $A_0 = \frac{c}{r+\lambda}$ and be A_T the price of end timing of life expectancy T, $A_T = c$, then

$$\beta = \frac{-1}{T}\log(r+\lambda). \tag{7}$$

The price of annuity is accordingly

$$A_t = c(r+\lambda)^{\frac{t-T}{T}}.$$

The purchasing policy is assumed that he will buy annuity after τ_l when he realize the probability of life time ruin. Moreover it is also assumed if his wealth at τ_l is smaller than the annuity price A_{τ_l} and then he will not buy annuity to simplify the solution. This assumption is that the order of first hitting time is for $w_0 > A_0$

$$\tau_l < \tau_a. \tag{8}$$

In Figure 3, we depict the annuity price function curve, the wealth level w_l and the initial wealth w_0.

4.2 Controlled wealth process

The optimal investment to risky asset is $\alpha(w) = g(\frac{c}{r_1} - w)$ for the domain $W_t \in (w_l, c/r_1]$ as (4) and the consumption before w_l is $c(w) = c + c_1 w$. Consequently the wealth process of (1) is starting from $W_0 = w_0$ and satisfies

$$dW_t = \{r_1 W_t + (\mu - r)\alpha - c\}dt + \sigma\alpha(W_t)dB_t,$$

before reaching w_l. The solution is

$$W_t = c/r_1 - (c/r_1 - w_0)\exp\{(r_1 - \frac{m(2d-1)}{(d-1)^2})t - \frac{\mu - r}{\sigma(d-1)}B_t\}.$$

Let τ_l be the first passage time from w_0 to w_l and the Laplace transform is given by Borodin [2],

$$E[\exp(-\theta\tau_l)] = \left(\frac{w_l r_1 - c}{w_0 r_1 - c}\right)^{\{\sqrt{\nu_l + 2\theta/\sigma_l^2} + \nu_l\}}$$

with $\sigma_l = g\sigma, \nu_l = \frac{r_1}{\sigma_l^2} - (d - \frac{1}{2})$.

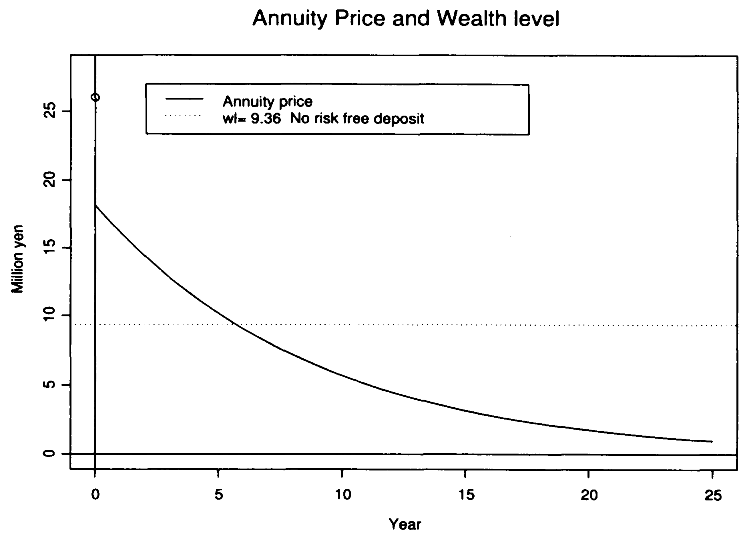

Figure 3: Annuity price

We assume that the retiree dare not purchase annuity before his wealth decreases to w_l because his wealth w_l is far more than c which he receives as annuity. This assumption implies

$$\tau_a > \tau_l. \tag{9}$$

There are two cases of optimal investment strategies for the domain $W_t \in (c, w_l]$ as (6), and the consumption after w_l is $c(w) = c_2 w$.

(a) The first case is $\alpha(w) = w$ if $\frac{\mu - r}{\sigma^2(a+1)} \geq 1$.

The wealth process satisfies

$$dW_t = [r_2 + (\mu - r)]W_t dt + \sigma W_t dB_t, W_{\tau_l} = w_l$$

and the solution is

$$W_t = w_l \exp\{(\mu - c_2 - \frac{1}{2}\sigma^2)s + \sigma B_s\}, \quad s = t - \tau_l.$$

Let τ_{la} be the first passage time to cross A_t from w_l as

$$\tau_{la} = \inf\{s : W_s = A_s | s = t - \tau_l\}.$$

Because the annuity price is $A_t = A_0 e^{-\beta t}$, the passage time τ_{la} satisfies

$$\tau_{la} = \inf\{s : \left(\frac{A_0}{w_l}\right) = \exp\{(\mu - c_2 + \beta - \frac{1}{2}\sigma^2)s + \sigma B_s\}\}.$$

Then Laplace transform of τ_{la} is

$$E[\exp(-\theta\tau_{la})] = \left(\frac{c}{(r+\lambda)w_l}\right)^{\{\sqrt{\nu_a+2\theta/\sigma^2}+\nu_a\}},$$

with $\nu_a = \frac{\mu-c_2+\beta}{\sigma^2} - \frac{1}{2}$.

(b) The second case is $\alpha(w) = \frac{\mu-r}{\sigma^2(a+1)}w$ if $\frac{\mu-r}{\sigma^2(a+1)} < 1$.

The wealth process satifies

$$dW_t = (r_2 + \frac{2m}{a+1})W_t dt + \frac{\mu-r}{\sigma(a+1)}W_t dB_t, W_{\tau_l} = w_l.$$

and the solution is

$$W_t = w_l \exp\{(r_2 + \frac{2m}{a+1} - \frac{m}{(a+1)^2})s + \frac{\mu-r}{\sigma(a+1)}B_s\}, \quad s = t - \tau_l.$$

Similarly Laplace transform of τ_{la} is

$$E[\exp(-\theta\tau_{la})] = \left(\frac{c}{(r+\lambda)w_l}\right)^{\{\sqrt{\nu_b+2\theta/\sigma_b^2}+\nu_b\}},$$

with $\nu_b = (r_2 + \frac{2m}{a+1})/\sigma_b^2 - \frac{1}{2}, \quad \sigma_b = \frac{\mu-r}{\sigma(a+1)}$.

The time to purchase annuity τ_a is passage time to cross the curve A_t after τ_l . Then it is sum of two independent stopping times as

$$\tau_a = \tau_l + \tau_{la}.$$

Therefor we state the following proposition;

Proposition 4.1 *The time to purchase annuity τ_a satisfies the following Laplace transform;*

$$\begin{aligned} E[\exp(-\theta\tau_a)] &= \left(\frac{w_l r_1 - c}{w_0 r_1 - c}\right)^{\{\sqrt{\nu_l+2\theta/\sigma_l^2}+\nu_l\}} \\ &\quad \times \left(\frac{c}{(r+\lambda)w_l}\right)^{\{\sqrt{\nu_a+2\theta/\sigma^2}+\nu_a\}}, \end{aligned}$$

where

$$\sigma_l = g\sigma, \nu_l = \frac{r_1}{\sigma_l^2} - (d - \frac{1}{2}),$$

$$\nu_a = \begin{cases} (\mu - c_2)/\sigma^2 - \frac{1}{2}, & \sigma_a = \sigma \qquad \frac{\mu-r}{\sigma^2(a+1)} \geq 1 \\ (r_2 + \frac{2m}{a+1})/\sigma_b^2 - \frac{1}{2}, & \sigma_a = \frac{\mu-r}{\sigma(a+1)} \qquad \frac{\mu-r}{\sigma^2(a+1)} < 1. \end{cases}$$

4.3 Mean time of purchasing annuity

The expected price of annuity A_{τ_a} is calculated by the proposition 4.1,

$$\hat{A}_a := E[A_{\tau_a}] = \frac{c}{r+\lambda} E[\exp(-\beta\tau_a)]$$

The average time to purchase annuity $\hat{\tau}_a$ is obtained by

$$\hat{A}_a = A_0 e^{-\beta\hat{\tau}_a}.$$

Corollary 4.2 *The expected price of annuity is*

$$\begin{aligned} \hat{A}_a &= \frac{c}{(r+\lambda)} \left(\frac{w_l r - c(w_l)}{w_0 r - c(w_0)} \right)^{\{\sqrt{\nu_l + 2\beta/\sigma_l^2} + \nu_l\}} \\ &\quad \times \left(\frac{c}{(r+\lambda) w_l} \right)^{\{\sqrt{\nu_a + 2\beta/\sigma^2} + \nu_a\}} \end{aligned}$$

The time of $\hat{A}_a$ is

$$\hat{\tau}_a := \frac{\log A_0 - \log \hat{A}_a}{\beta}$$

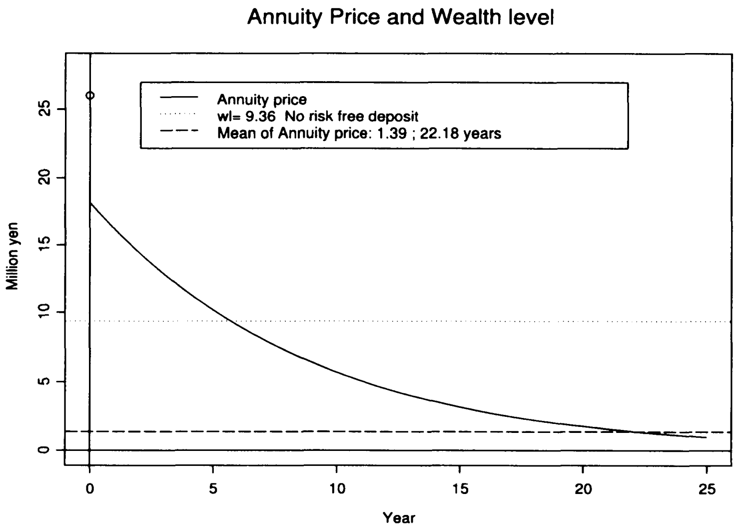

Figure 4: Mean time of purchasing annuity

5 Summary and numerical example of Japanese retirees

The average retirement year is 60, and the life expectancy at 60 is $T = 25$ years in Japan, then we set hazard rate $\lambda = 0.04$. The interest rate of risk free asset is $r = 1.5\%$, and risky asset's parameters are $\mu = 0.06, \sigma = 0.2$. The initial wealth is set to $w_0 = 26$ million yen which is the average retirement income and satisfies the condition (8). By (5), we get $w_l = 9.36$ million yen which is depicted in figure-4 as the dotted line.

The continuous consumption is $c(w) = 1 + 0.01w$ before τ_l the wealth is less than w_l, and $c_2 = c_1 + c/w_l = 0.0783$ after τ_l. It means that even if his wealth W_t exceeds w_l after τ_l, his consumption is $c_2 W_t$.

The console bond price which produces income c perpetually is $c/r_1 = 200$ million yen which makes the probability of life time ruin zero. The annuity price at age 60 is $A_0 = c/(r + \lambda) = 18.18$ million yen and the price at age $85(T = 25)$ is $A_T = 1$ million yen. From (7) we get $\beta = 0.116$.

Finally, from the corollary we calculate the expectation of annuity price to purchase which is $\hat{A} = 1.39$. The retiree could wait to purchase individual annuity up to the time $\hat{\tau}_a = 22.18$ years after retirement time. The retiree could buy annuity at age 82.18 in average which depicted in figure-4 as the broken line.

In our setting, the objective function is simply minimizing the probability of ruin. If his wealth exceeds the annuity price, then it is optimal to buy the annuity even if the price is very expensive. In this paper we assume the annuity purchase policy as (8) to avoid this unrealistic situation.

The optimal investment policy in Fleming [4] for maximizing CRRA utility has the same solution as minimizing the probability of ruin afterward risk realization as Bayraktar [1] mentioned, It implies that if we maximize utility of consumption, it might avoid the annuity purchase policy assumption because the utility balances the sacrifice of present consumption value against the probability of life time ruin.

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