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Multiple Stopping Problem for Jump Diffusion and Free Boundary Problem[1]

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1 Introduction and Framework

This paper provides the American swing (multiple stopping) option's free boundary problem in a jump diffusion model, and proves that the pair of the American swing put's value and the optimal boundary is a unique solution of the free boundary problem. Recently, Carmona and Touzi (2008) study the American swing put option in the Black and Scholes (BS) model by the so-called martingale approach in the optimal stopping theory, mainly perpetual case. Independently, Ano (2008) provides the optimal multiple stopping rule and the price for the American, Russian and Asian swing put and call options in the Cox-Ross-Rubinstein (CRR) model by using the discrete version of the infinitesimal operator of the reward function. Kifer (2009) studies the Game American swing option in the BS model by Dynkin Game Framework. On the other hand, the free boundary problem for the American (single stopping) put option has been studied by Jacka (1991) for the BS model and by Pham (1997) for the jump diffusion model. However, the free boundary problems of American swing put option in both the BS and the jump diffusion models have not been studied yet. One of the contribution of this paper is this. So that our work is essentially based on the early excellent results in Carmona and Touzi (2008), Ano (2008), Jacka (1991), Pham (1997) and so on.

For the BS model, Carmona and Touzi (2008) show that the optimal multiple stopping rule for the American swing put option is characterized by the multiple optimal stopping boundaries and it is given by $\{\tau_{\ell}^{*}, \cdots, \tau_{1}^{*}\}$ where for a given $\delta > 0$ and each $k = 1, 2, \cdots, \ell$

$$\tau_{k}^{*} = \inf\{\tau_{k+1}^{*} + \delta \leq t \leq T : X_{t}(x) \geq b^{[k]*}(t, x)\}, \tag{1}$$

where we set $\tau_{\ell+1}^{*} \equiv -\delta$. τ_{k}^{*} represents the optimal stopping time when more k stopping chances is allowed. And $t \mapsto b^{[\ell]*}(t, x)$ is increasing and continuous, and $x \mapsto b^{[\ell]*}(t, x)$ is increasing and continuous function. For the CRR model, Ano (2008) shows that each optimal multiple stopping rule for American swing put, Russian swing and Asian swing option is also characterized by the multiple stopping boundaries and is given by the same discrete form of as the one of the American swing put in the BS model. For the jump diffusion model, it is highly expected that the optimal multiple stopping rule of the American put swing has the same form and properties as the ones of both the BS model and CRR the model.

The framework is the usual setting of the jump diffusion market, and it follows completely from Pham (1997), below. Let $(\Omega, \mathcal{F}, P)$ be the probability space with filtration $\mathbb{F} = \{\mathcal{F}_t, 0 \leq t \leq T\}$ be satisfying the usual conditions. On this probability space, the standard Brownian motion B and the homogeneous Poisson random measure $\nu(dt, dy)$ are defined. We assume that

[1]This paper is an abbreviated version of Ano [1].

$\mathbb{F}$ is a σ-field generated from the Brownian motion and the Poisson measure. Intensity measure q of ν is supposed to be $q(dt, dy) = \lambda m(dy)dt$, where $\lambda > 0$ is the intensity of jump of Poisson process $N_t = \nu([0, T] \times \mathbb{R})$ and $m(dy)$ is the random measure on $\mathbb{R}$ being independent of Y_n and N_t. Let the random measure $\tilde{\nu}$ be $\tilde{\nu}(dt, dy) := \nu(dt, dy) - q(dt, dy)$. Bank account process, S is given by the differential equation; for $r > 0$, $dS_t = S_t r dt$. Risky asset price process, X_t is described by

$$\frac{dX_t}{X_{t-}} = \mu dt + \sigma dB_t + \int_{\mathbb{R}} \gamma(y)\tilde{\nu}(dt, dy), \tag{2}$$

where $(\gamma(Y_n)_{n\in\mathbb{N}})$ is the square integrable random jump sizes of X. In order to be an nonnegative price of the risky asset, we suppose that $1 + \gamma > 0$. It is known for jump diffusion model that there are many equivalent martingale measure, that is, the model is incomplete market. Recall the following equivalent martingales

$$\frac{d\tilde{P}}{dP} = \mathcal{E}\left(-\int_0^T \theta_t dB_t + \int_0^T \int_{\mathbb{R}} (p_t(y) - 1)\tilde{\nu}(dt, dy)\right), \tag{3}$$

where $\mathcal{E}(\cdot)$ is the stochastic exponential seimartingale, θ and p are the predictable processes such that $\mu - r = \theta_t\sigma + \lambda \int_{\mathbb{R}} \gamma(y)(1 - p_t(y))m(dy)$, $p > 0$, $\mathbb{E}(d\tilde{P}/dP) = 1$. According to Pham (1997), we only consider an equivalent martingale measure such that p is independent of t and $\omega \in \Omega$, that is, $p_t(y) = p(y)$ and $p \in L^2(m)$. Hence, by Girsanov's Theorem, ν is still a homogeneous Poisson random measure under $\tilde{P}$ with local characteristics, $\tilde{\lambda} = \lambda \int_{\mathbb{R}} p(y)m(dy), \tilde{m}(dy) = (p(y)m(dy))/(\int_{\mathbb{R}} p(y)m(dy))$ and $\tilde{B}_t = B_t + \int_0^t \theta_s ds$ is $\tilde{P}$-Brownian motion. $(s, t, x) \mapsto X_x^t(x)$ is RCLL for a.s. $\omega \in \Omega$, $X_0^t(x)$ satisfies (2) of the risky asset process on $[t, T]$. Therefore, we have under $\tilde{P}$ a.s.

$$\begin{aligned} X_s^t(x) &= x \exp\left\{-\tilde{\lambda}\tilde{k}(s-t) + \int_t^s \int_{\mathbb{R}} \ln(1+\gamma(y))\nu(du, dy)\right\} \\ &\qquad \cdot \exp\left\{\sigma(\tilde{B}_s - \tilde{B}_t) + \left(r - \frac{1}{2}\sigma^2\right)(s-t)\right\}. \end{aligned} \tag{4}$$

where $\tilde{k} = \int_{\mathbb{R}} \gamma y \tilde{m}(dy)$ is the expectation under $\tilde{P}$ of the jump size. We define $X_x(x) = X_x^0(x)$. To emphasize the initial condition X_0, we use the notation $X_t^0(X_0)$. X is a homogeneous Markov process under $\tilde{P}$.

2 Multiple Stopping Problem

Assume that we can exercise at most $\ell > 1$ times. Our problem is to solve $V^{[\ell]}(t, x) = V^\ell(T-t, x)$ under $\tilde{P}$,

$$V^{[\ell]}(t, x) := \sup_{0 \le \tau_1 < \tau_2 < \cdots < \tau_\ell \le t} \tilde{\mathbb{E}}\left[\sum_{k=1}^{\ell} e^{-r\tau_k} g(X_{\tau_k}(x))\right], \tag{5}$$

where $\tau_\ell \le t$ a.s., and $\tau_k - \tau_{k-1} \ge \delta$ a.s., for each $k = 2, \cdots, \ell$ and a positive constant δ. δ is the length of the refracting time interval which needs to separete two successive exercise. Note that t

is the time to expiry. $T < \infty$ is a given expiration date. We define the state (t, x, k) that we face the stock price x at time t and we have more k exercise chances hereafter. When we exercise the option at the state (t, x, k), the expected reward is given by the sum of the immediate exercise reward at time t, $g(x)$, and the expected maximum reward when we can exercise at most more $k-1$ times after the time $t-\delta$, that is,

$$\begin{aligned} U^{[k]}(t,x) &:= g(x) + \tilde{\mathbb{E}}\left[\sup_{0 \le \tau_1 < \cdots < \tau_{k-1} \le t-\delta} \tilde{\mathbb{E}}\left[\sum_{j=1}^{k-1} e^{-r\tau_j} g(X_{\tau_j}(x)) \right] \Big| \mathcal{F}_t \right] \\ &= g(x) + e^{-r\delta}\tilde{\mathbb{E}}\left[V^{[k-1]}(t-\delta, X_\delta(x)) \right]. \end{aligned} \tag{6}$$

The second inequality follows from the Markov property of X. The boundary condition is $V^{[k]}((k-1)\delta, x) = U^{[k]}((k-1)\delta, x)$ for each $k = 1, 2, \cdots, \ell$ and $x \in \mathbb{R}^+$. For the American put, $V^{[1]}(0, x) = (K - x)^+$, where K is a given positive strike price. From the general theory of the optimal stopping, the optimal stopping time of the American swing put is for each $k = 1, 2, \cdots, \ell$

$$\tau_k^* = \inf\{0 \le s \le T : V^{[k]}(T - s, X_s(x) = U^{[k]}(T - s, X_s(x))\}. \tag{7}$$

We suppose (C): $\tilde{r} := r - \lambda \int_{\gamma(y) \ge 0} \gamma(y) p(y) m(dy) \ge 0$, which means that market price of jump risk, $\tilde{r}$, is nonnegative.

Lemma 1 *There exists $M > 0$ such that, for all $t_1, t_2 \in [0, T], x_1, x_2 \in \mathbb{R}^+$,*

$$|V^{[k]}(t_1, x_1) - V^{[k]}(t_2, x_2)| \le M[|t_1 - t_2|^{1/2} + |x_1 - x_2|]. \tag{8}$$

This lemma will be used in the proof of the smooth fit condition. Let $\mathcal{L}$ be the parabolic integrodifferential operator of $X_t(x)$.

$$\begin{aligned} \mathcal{L}v^{[k]} &= -\frac{\partial v^{[k]}}{\partial t} - rv^{[k]} + rx\frac{\partial v^{[k]}}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 v^{[k]}}{\partial x^2} \\ &+ \lambda \int_{\mathbb{R}} \left[v^{[k]}(t, x(1+\gamma(y)) - v^{[k]}(t,x) - \gamma(y) x \frac{\partial v^{[k]}}{\partial x}(t,x) \right] p(y) m(dy). \end{aligned} \tag{9}$$

Under (C) and $\sigma > 0$, the free boundary problem of American put in the jump-diffusion model is formulated in Pham (1997) as follows;

- $\mathcal{L}v^{[1]} = 0, \ x > b^{[1]}(t)$,
- $\lim_{x \downarrow b^{[1]}(t)} v^{[1]}(t, x) = K - b^{[1]}(t), \ \ t \in (0, T]$
- $\lim_{x \downarrow b^{[1]}(t)} v_x^{[1]}(t, x) = -1, \ \ t \in (0, T]$
- $v^{[1]}(0, x) = (K - x)^+, \ \ x \in \mathbb{R}^+$
- $v^{[1]} > (K - x)^+, x > b^{[1]}(t)$,
- $v^{[1]} = (K - x)^+, x \le b^{[1]}(t)$.

American put option's price and the optimal stopping boundary, $(V^{[1]}, b^{[1]})$, is the unique solution pair $(v^{[1]}, b^{[1]})$ of this free boundary problem. Further, the followings are proved; (1) The maximum expected reward, $x \mapsto V^{[1]}(t, x)$, is nonincreasing and convex, for every $t \in [0, T]$. (2) The optimal single stopping boundary, $t \mapsto b^{[1]}(t)$, is continuous and nonincreasing with $b^{[1]}(0) = K$, for every $x \in \mathbb{R}^+$. (3) For all $t \in (0, T]$, there exists a positive constant $M > 0$ such that $0 \le b^{[1]BS}(t) - b^{[1]}(t) \le M\sqrt{t}$, where $b^{[1]BS}(t)$ is the optimal stopping boundary for the BS market.

Proposition 1 *For each $k = 1, \cdots, \ell$,*

(i) $x \mapsto V^{[k]}(t,x)$ *is nonincreasing and convex on* $\mathbb{R}^+$ *for each* $t \in [0,T]$.

(ii) $t \mapsto V^{[k]}(t,x)$ *is nondecreasing on* $[0,T]$ *for each* $x \in \mathbb{R}^+$.

(iii) $V^{[k]}(t,x) \geq V^{[k-1]}(t,x)$ *for each* $(t,x) \in (0,T] \times \mathbb{R}^+$.

Proof. We have

$$V^{[\ell]}(t,x) \quad = \quad \sup_{0 \leq \tau_1 < \tau_2 < \cdots < \tau_\ell \leq t} \tilde{\mathbb{E}} \left[\sum_{k=1}^{\ell} e^{-r\tau_k} (K - X^0_{\tau_k}(x))^+ \right]$$

So that (i) follows from the pathwise solution of $X^t_x(x)$ in (8) provided that the reward $x \mapsto (K-x)^+$ of the American put is nonincreasing and convex. (ii) follows from the fact that if $\tau_k \in \mathcal{T}_{0,t}$, then $\tau_k \in \mathcal{T}_{0,s}$ for any $s \geq t$. (iii) follows from the reward function $g(x)$ is nonnegative and the value function $V^{[k]}(t,x)$. □

Lemma 2 *For each $k = 1, \cdots, \ell$,*

(i) $x \mapsto U^{[k]}(t,x)$ *is nonincreasing and convex on* $\mathbb{R}^+$ *for all* $t \in [0,T]$.

(ii) $U^{[k]}(t,x) \geq U^{[k-1]}(t,x)$ *for all* $(t,x) \in (0,T] \times \mathbb{R}^+$.

Proof. (i) follows from $g(x) = (K-x)^+$, Proposition 1 (i) with the pathwise solution of $X^t_x(x)$. (ii) follows from Proposition 1 (iii) and $U^{[k]}(t,x)$ in (6). □

Proposition 2 *For each $k = 1, \cdots, \ell$,*

(i) $V^{[k]}(t,x) = U^{[k]}(t,x)$, *for* $0 \leq x \leq b^{[k]}(t)$

(ii) $V^{[k]}(t,x) > U^{[k]}(t,x)$, *for* $b^{[k]}(t) < x$.

(iii) $V^{[k]}(t,x) > 0$ *for all* $(t,x) \in (0,T] \times \mathbb{R}^+$.

Proof. It follows from the Proposition 1 and Lemma 1 (i) (ii). The proof of (iii) follows from Proposition 2.1 in Pham (1997) and Proposition 1 (iii). □

From Propositions 1 and 2, we have the stopping region for each $k = 1, 2, \cdots, \ell$,

$$\begin{aligned} D^{[k]} &:= \{(t,x) \in (0,T] \times \mathbb{R}^+ : V^{[k]}(t,x) = U^{[k]}(t,x)\} \\ &= \{(t,x) \in (0,T] \times \mathbb{R}^+ : x \leq b^{[k]}(t)\}. \end{aligned} \tag{10}$$

The continuation region is given by

$$\begin{aligned} C^{[k]} &:= \{(t,x) \in (0,T] \times \mathbb{R}^+ : V^{[k]}(t,x) > U^{[k]}(t,x)\} \\ &= \{(t,x) \in (0,T] \times \mathbb{R}^+ : x > b^{[k]}(t)\}. \end{aligned} \tag{11}$$

Since $x \mapsto V^{[k]}$ is continuous, $D^{[k]}$ is open and $C^{[k]}$ is closed for each $k = 1, 2, \cdots, \ell$.

Proposition 3 *For each $k = 1, 2, \cdots, \ell$, $D^{[k-1]} \subseteq D^{[k]}$.*

Proof. It follows from the Propositions 1 and 2. □

3 Free Boundary Problem

Proposition 4 *For each $k = 1, 2, \cdots, \ell$, the value function of American swing put option satisfies;*

$$\begin{aligned}
&\text{(i)} && V^{[k]}(t,x) \text{ is smooth in } C^{[k]} \text{ and } \mathcal{L}V^{[k]}(t,x) = 0 \text{ in } C^{[k]}, && (12)\\
&\text{(ii)} && \lim_{x \downarrow b^{[k]}(t)} V^{[k]}(t,x) = U^{[k]}(t, b^{[k]}(t)), t \in (0,T], && (13)\\
&\text{(iii)} && V^{[k]}((k-1)\delta, x) = U^{[k]}((k-1)\delta, x), x \in \mathbb{R}^+. && (14)
\end{aligned}$$

Proof. (i) follows from the martingale property of $\{e^{-rs}V^{[k]}(t-s, X_s(x))\}_{\{s\in[0,\tau_k^*(t,x)]\}}$, where $\tau_k^* = \inf\{s \in [0,t] : (t-s, X_s(x)) \notin C^{[k]}\}$, Itô's formula, and the smoothness property for the operator $\mathcal{L}$. (ii) follows from the continuity of $V^{[k]}$ and $b^{[k]} < K$. (iii) is just the boundary condition. □

It's well-known that the smooth fit condition does not necessarily holds. For example, it is not hold for the pure jump model (i.e., $\sigma = 0$). Pham (1997) proves the the condition (C) is one of the sufficient condition for the smooth fit. This condition inherits the swing option.

Proposition 5 *Assume (C). For each $k = 1, 2, \cdots, \ell$, the value function of American swing option is continuously differentiable with respect to x, in $(0,T] \times \mathbb{R}^+$, especially, across the optimal stopping boundary,*

$$\lim_{(s,x)\to(t,b^{[k]}(t))} \frac{\partial V^{[k]}}{\partial x}(s,x) = \lim_{(s,x)\to(t,b^{[k]}(t))} \frac{\partial U^{[k]}}{\partial x}(s,x), \ t \in (0,T]. \tag{15}$$

For $k = 1$,

$$\lim_{(s,x)\to(t,b^{[1]}(t))} \frac{\partial V^{[1]}}{\partial x}(s,x) = -1, \ t \in (0,T]. \tag{16}$$

Outline of Proof. $V_x^{[k]}$ is uniformly bounded in $[0,T] \times \mathbb{R}^+$ and $V_t^{[k]}$ is locally bounded in $(0,T] \times \mathbb{R}^+$. $V^{[k]}$ satisfies $\mathcal{L}V^{[k]} \le 0$, i.e.,

$$\begin{aligned}
\frac{1}{2}\sigma^2 x^2 \frac{\partial^2 V^{[k]}}{\partial x^2} &\le rV^{[k]} + \frac{\partial V^{[k]}}{\partial t} - rx\frac{\partial V^{[k]}}{\partial x}\\
&\quad -\lambda \int_{\mathbb{R}} \left(V^{[k]}(t, x(1+\gamma(y))) - V^{[k]}(t,x) - \gamma(y)x\frac{\partial V^{[k]}}{\partial x}(t,x) \right) p(y)m(dy).
\end{aligned}$$

Since $V^{[k]}$ is Lipschitz in x, uniformly in t, the integrand term of RHS above is bounded above by $M|x|(1 + V_x^{[k]}) \int_{\mathbb{R}} |\gamma(y)|p(y)m(dy)$. From $\gamma \in L^1(\tilde{m})$, it follows that $V_{xx}^{[k]}$ is locally bounded in $(0,T] \times \mathbb{R}^+$. This with the convexity of $x \mapsto V^{[k]}(t,x)$ yields that $V_x^{[k]}(t,x)$ is continuous in $(0,T] \times \mathbb{R}^+$. □

Theorem 1 *Assume the condition (C). For each $k = 1, 2, \cdots, \ell$, the pair of the value function of American swing option and optimal stopping boundary is the unique solution pair of $(v^{[k]}, b^{[k]})$*

with $v : [0,T] \times \mathbb{R}^+ \mapsto \mathbb{R}$, $x \mapsto v(t,x)$ *is nonincreasing and convex, and* $b : (0,T] \mapsto \mathbb{R}$, $0 \le b^{[k]}(t) < K$, *of the free boundary problem;*

- $\mathcal{L}v^{[k]} = 0, \ x > b^{[k]}(t),$ (17)
- $\lim_{x \downarrow b^{[k]}(t)} v^{[k]}(t,x) = U^{[k]}(t, b^{[k]}(t)), \quad t \in (0,T]$ (18)
- $\lim_{(s,x)\to(t,b^{[k]}(t))} v_x^{[k]}(s,x) = \lim_{(s,x)\to(t,b^{[k]}(t))} U_x^{[k]}(s,x), \quad t \in (0,T]$ (19)
- $v^{[k]}((k-1)\delta, x) = U^{[k]}((k-1)\delta, x), \quad x \in \mathbb{R}_+$ (20)
- $v^{[k]}(t,x) > U^{[k]}(t,x), \quad x > b^{[k]}(t),$ (21)
- $v^{[k]}(t,x) = U^{[k]}(t,x), \quad x \le b^{[k]}(t).$ (22)

Outline of Proof. From the preceding propositions, it follows that $(V^{[k]}, b^{[k]})$ is the solution of the free boundary problem for each k. Reversely, let $(v^{[k]}, b^{[k]})$ be the pair in Theorem 1. Thus, $v^{[k]}$ is $\mathcal{C}^1$ in x, piecewise $\mathcal{C}^1$ in t. From Itô's formula, it follows that

$$\begin{aligned}
&e^{-rt} v^{[k]}(t-s, X_s(x)) \\
&= v^{[k]}(t,x) + \int_0^s \mathcal{L}v^{[k]}(t-u, X_u(x))du + \int_0^s e^{-rs} v_x^{[k]}(t-u, X_u(x))\sigma X_u(x) d\tilde{B}_u \qquad (23)\\
&\quad + \int_0^s \int_{\mathbb{R}} e^{-ru}[v^{[k]}(t-u, X_u(x)(1+\gamma(y))) - v^{[k]}(t-u, X_u(x))][\nu(du,dy) - \tilde{\lambda}\tilde{m}(dy)du].
\end{aligned}$$

Since, $x \mapsto v^{[k]}(t,x)$ is nonincreasing and convex, and $v^{[k]} \ge U^{k]}$, it implies that $v_x^{[k]}$ is bounded on $\mathbb{R}^+$. Thus the two stochastic integral are $\tilde{P}$-martingales. Using (23), $\mathcal{L}v^{[k]}$, and the several facts shown already, we can show that $V^{[k]} \le v^{[k]} \le V^{[k]}$. If $x > b^{[k]}(t)$, then $\mathcal{L}v^{[k]} = 0$, and if $x \le b^{[k]}(t)$, then it follows from $x \mapsto v^{[k]}(t,x)$ is nonincreasing, $v^{[k]}(t, b^{[k]}) = U^{[k]}(t, b^{[k]}(t))$, $\{x(1+\gamma(y)) \ge b^{[k]}(t)\} I_{\{x \le b^{[k]}(t)\}} \subset \{\gamma(y) \ge 0\} I_{\{x \le b^{[k]}(t)\}}$ and $x \le b^{[k]}(t) \le kK$ that

$$\mathcal{L}v^{[k]}(t,x) \quad \le \quad -K \left[r - \lambda \int_{\gamma(y)\ge 0} \gamma(y)p(y)m(dy) \right].$$

From (C), it follows that $\mathcal{L}v^{[k]}(t,x) \le 0$ and $\{e^{-rs}v^{[k]}(t-s, X_s(x))\}_{\{0\le s\le t\}}$ is $\tilde{P}$-supermartingale. Then, from $v^{[k]} \ge U^{[k]}$, it follows that for all $\tau \in \mathcal{T}_{0,t}$,

$$v^{[k]}(t,x) \ge \tilde{\mathbb{E}}[e^{-r\tau} v^{[k]}(t-\tau, X_\tau(x))] \ge \tilde{\mathbb{E}}[U^{[k]}(t-\tau, X_\tau(x))],$$

which implies that $v^{[k]} \ge V^{[k]}$. The reverse inequality can be shown. Thus, we have $V^{[k]} \le v^{[k]} \le V^{[k]}$. □

Theorem 2 *For* $k = 2, \cdots, \ell$, $b^{[k]}(t) \ge b^{[k-1]}(t)$ *in* $(0,T]$.

Proof. From Proposition 3 and the closeness of the stopping region of $D^{[k]}$ for each k, it follows. Another simple proof is an induction on k. Use that $b^{[k]}(t)$ is the unique implicit solution of the integral equation

$$V^{[k]}(t, b^{[k]}(t)) = (K - b^{[k]}(t))^+ + e^{-r\delta}\tilde{\mathbb{E}}\left[V^{[k-1]}(t-\delta, X_\delta^0(b^{[k]}(t)))\right], \ \forall t \in (0,T]$$

and $V^{[k]}(t, b^{[k]}(t)) \ge V^{[k-1]}(t, b^{[k-1]}(t))$ in Proposition 1(iii). □

4 Future Study

Several important properties of the optimal stopping boundaries such that "the optimal stopping boundary $b^{[k]}(t)$ for each $k = 2, 3, \cdots, \ell$ is continuous in $(0, T]$" are still not proved. For the numerical solutions of the value functions and the optimal stopping boundaries for each $k = 1, \cdots, \ell$, we need further explicit or implicit properties of the reward function, $U^{[k]}(t, x)$, since in order to solve the free boundary problem with the number of stopping chances, ℓ (FBP(ℓ)), it requires sequential calculation of the unique solution pairs for the FBP(1), FBP(2), $\cdots$, FBP($\ell - 1$), starting from FBP(1).

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