数理解析研究所講究録
第 1675 巻 2010 年 199-211

On Estimating the Growth Optimal Asset Pricing Model under Regime Switching*

中央大学大学院国際会計研究科・石島 博 (Hiroshi Ishijima)
Graduate School of International Accounting
Chuo University
大阪大学大学院理学研究科・坂上 慎也 (Shinya Sakagami)
Graduate School of Science
Osaka University

Abstract

We study an adaptive estimation of growth optimal asset pricing model with the setting of hidden Markov models. We consider a market where risky assets and a risk-free asset are traded in discrete time. We assume that there exists unobservable economic regime which is modeled by time-homogenous Markov process as a state equation. As an observation equation, risky asset price processes are assumed to be lognormal conditioned on unobservable economic regime. We also assume a risk-free asset of which rate of return is switching according to regime. With these assets, we derive a growth optimal portfolio and characterize its first order condition for optimality. We then obtain an asset pricing model which gives the expected excess return for each asset. Conditioned on regime, it is shown to be the covariance of return between the growth optimal portfolio and its constituent risky assets. Hence to estimate the expected excess return of risky assets adaptively, we are to estimate its covariance matrix subject to regime. To implement this, we employ an adaptive filtering technique developed by Elliott (1995). We also implement some numerical examples to estimate the model.

1 Introduction

In this paper, we employ a regime switching model which takes into account hidden states of the market. This model allows us to derive an asset pricing model based on a growth optimal portfolio, and thereby estimate the risk premium of any assets based on the model.

The following gives some background on the theory of growth optimal portfolios and regime switching models.

Growth Optimal Portfolio

First of all, a growth optimal portfolio is a portfolio which maximizes the expected growth rate in portfolio value from the current time to the end of a finite horizon. The expected growth rate can be interpreted as the expected log-return, or the geometric mean, in the portfolio value. Hence these are referred to as a growth optimal portfolio, log-optimal portfolio, and a geometric

*We would like to thank Professor Toshikazu Kimura of Hokkaido University and the participants in the Kyoto University RIMS Workshop on Financial Modeling and Analysis 2009.

mean portfolio, respectively. This portfolio was first proposed by Kelly (1956), who analyzed expected growth rates in relation to information theory, and further progress in theoretical research was made by Cover et. al. (Cover-Thomas, 1990).

On the one hand, this approach has been repeatedly applied to finance, and its validity has been claimed in studies by Hakansson (1971), Thorp (1971) and others. On the other hand, the growth optimal portfolio approach considers only maximization of the expected log utility, hence the approach has been criticized because it does not maximize the expected utility of other risk averse investors. However, Luenberger (1993) showed that the approach is economically sound.

This is not just a sound theoretical framework for performing dynamic portfolio selection as indicated above. The research of Long (1990), Platen and Heath (2006), and others has shown that growth optimal portfolios are a framework for fair asset pricing. That is, they have shown that the Karush-Kuhn-Tucker (KKT) conditions themselves (which are necessary and sufficient for a portfolio to be growth optimal) can be used in asset pricing. More specifically, these KKT conditions express the fact that the relative prices of individual assets, found by dividing by the growth optimal portfolio value as a numeraire, become a martingale under the original probability measure. Furthermore, when this is combined with the theorem (Theorem 1, Long, 1990) which states that the non-existence of opportunities for arbitrage in a market is equivalent to the existence a growth optimal portfolio as a numeraire one, then a growth optimal portfolio can be used for pricing any assets.

An asset pricing formula based on a growth optimal portfolio has a distinguishing feature that it is derived from just the list of assets which are the target of investment universe. The risk premium or the expected excess rate of return of an individual asset is explained to be proportional to that of the growth optimal portfolio. Therefore, this approach can be used in almost the exact same way as the familiar classical capital asset pricing model (CAPM). On the other hand, CAPM always has problems, like how to interpret the market portfolio which is only a conceptual one in theory, and to determine the proper substitution assets when conducting empirical analyses. However, a growth optimal portfolio can be specifically determined from historical data on the list of assets which are the target investment universe. Even if a growth optimal portfolio is not found directly, it is possible to use a benchmark for the list of assets which are the target investment universe, as a proxy for the growth optimal portfolio. This is because many empirical studies have strongly supported that the rate of return of the benchmark satisfies the necessary and sufficient conditions to be a growth optimal portfolio. For example, it has been reported that, if the stocks comprising TOPIX are taken to be the list of assets, the rate of return of TOPIX which is a benchmark of those assets can be used as a proxy for the rate of return of the growth optimal portfolio (Roll 1973, Long 1990, Platen and Heath 2006).

Regime Switching Model

The purpose of this paper is to derive an asset pricing model based on a growth optimal portfolio, which takes regime into account. The term "regime" refers to "hidden" states of the market, such as booms and recessions or bull and bear markets. And the "regime switching model" takes into account the fact that parameters describing existing time series models for asset prices will switch according to the regime. The regime switching model was first proposed in the pioneering research of Hamilton (Hamilton, 1989). The basic concept is to first describe an "observation equation" for the asset price. That is, the equation is given by a logarithmic diffusion model, in which the logarithmic rate of return conforms to a normal distribution, or by an existing time series model of asset prices like an autoregressive processes. At this time, the parameters of the observation equation are placed into correspondence with the regime, and given as many different values as there are regimes. When a certain regime is realized at a discrete time point, the corresponding parameters are regarded as being realized. As a result, the parameters switch in accordance with the regime at discrete time points. On the other hand, it is assumed that regimes conform to a first-order Markov process, and this is called the "state equation". In other words, a regime switching model is described by two equations – an "observation equation" and a "state equation" – and their parameters are inferred from the data. Thus, this model can be regarded as one of the so-called "state-space models", a category which includes things like the Kalman filter (Elliot et al, 1995). Since the research of Hamilton, this model has been actively studied in econometrics, and has shown its effectiveness through theoretical extensions and applications to validation studies.

2 The Model

In this Section, we describe why a growth optimal portfolio serves as a fair pricing framework, and then derive an asset pricing model subject to regime switching.

2.1 Asset Pricing via Growth Optimal Portfolios

We consider a market where one risk-free asset, and n risky assets are traded at discrete time $t = 0, 1, \ldots, T$. Let $\boldsymbol{S}_t = (S_{1t} \ldots S_{it} \ldots S_{nt})'$ be the prices of risky assets at time t which is assumed take finite, non-negative values. The period t is the time interval between $t-1$ and t. The gross return of the assets over the period t, i.e. 1 + the rate of return, is written as $\boldsymbol{X}_t = (X_{1t} \ldots X_{it} \ldots X_{nt})'$, where $X_{it} = S_{it}/S_{it-1}$. Also, we write the gross return of the risk-free asset during period t as $x_{f,t}$.

At each time $t-1$, the investor rebalances her portfolio for the period t. However, it is assumed that there is no exchange of cash flows with the outside, that is, the portfolio is built in a self-financing fashion. The portfolio is characterized by the ratios of the amounts invested in each asset, relative to the entire portfolio value, i.e. by the portfolio weights. The weights for the risk-free asset and the risky assets are denoted, respectively, as $\{b_{0t}, \boldsymbol{b}_t = (b_{1t} \ldots b_{it} \ldots b_{nt})'\}$. The weights are assumed to be constrained as follows:

$$\left\{ b_{0t} \in \mathrm{R}, \boldsymbol{b}_t \in \mathbf{R}^n \,\middle|\, b_{0t} + \boldsymbol{b}_t'\mathbf{1} = 1 \right\} \ ,$$

Under the assumptions above, the portfolio value V_t can be expressed:

$$V_t = V_{t-1} \left(b_{0t} x_{f,t} + \boldsymbol{b}_t' \boldsymbol{X}_t \right) = V_{t-1} X_t \ ,$$

where the gross return of portfolio value is denoted by $X_t \triangleq b_{0t}x_{f,t} + \boldsymbol{b}_t'\boldsymbol{X}_t$. Hence, given the initial investment V_0, then the portfolio value at the end of the horizon can be written as:

$$V_T = V_0 \prod_{t=1}^{T} X_t = V_0 \cdot e^{\sum_{t=1}^{T} \log X_t} = V_0 \cdot e^{\log(V_T/V_0)} ,$$

This last equation is a trivial identity for V_T, and $\log(V_T/V_0)$ gives the gross return over the entire investment period.

This expression shows that, in order to maximize the portfolio value V_T at the end of the term, it is sufficient to maximize the log return of the portfolio during each period $\log X_t$. Therefore, to obtain the growth optimal portfolio, it is sufficient to take the conditional expectation of the log return of the portfolio at the beginning of each period $t-1$, and then to solve the following problem at each period t:

$$\mathbf{P}_t \left| \begin{array}{ll} \underset{\boldsymbol{b}_t}{\text{maximize}} & E_{t-1}[\log X_t] \triangleq E_{t-1}[\log\left(b_{0t}x_{f,t} + \boldsymbol{b}_t'\boldsymbol{X}_t\right)] \\ \text{subject to} & b_{0t} + \boldsymbol{b}_t'\mathbf{1} = 1 \ . \end{array} \right.$$

Since the objective function for the problem $\mathbf{P}_t$ is concave, the Karush-Kuhn-Tucker (KKT) conditions are necessary and sufficient conditions for the optimality. The KKT conditions for the problem $\mathbf{P}_t$ can be stated as $\{b_{0t}^*,\ \boldsymbol{b}_t^*\ ;\ \nu^*\}$, such that:

$$E_{t-1}\left[\frac{x_{f,t}}{X_t^*}\right] = 1 \ , \tag{2.1}$$

$$E_{t-1}\left[\frac{X_{it}}{X_t^*}\right] = 1 \ (i = 1, \ldots, n) \ , \tag{2.2}$$

$$\nu^* = 1 \ . \tag{2.3}$$

Where $\nu \in R$ is the Lagrange multiplier. Also $X_t^* = b_{0t}^* x_{f,t} + \boldsymbol{b}_t^{*\prime}\boldsymbol{X}_t$. In Eqs. (2.1) and (2.2), X_t^* is the gross return of the growth optimal portfolio, and can be expressed as $X_t^* = V_t^*/V_{t-1}^*$. Similarly, since $X_{it} = S_{it}/S_{it-1}$, Eq. (2.2) can be rewritten:

$$\frac{S_{it-1}}{V_{t-1}^*} = E_{t-1}\left[\frac{S_{it}}{V_t^*}\right] \ (i = 1, \ldots, n) \ . \tag{2.4}$$

This equation shows that the relative price of assets, taking the growth optimal portfolio as the numeraire, becomes a martingale under the original probability measure. This asserts that the growth optimal portfolio can be used for pricing an arbitrary asset. Furthermore, Eq. (2.4) implies:

$$S_{it} = E_t\left[\left(\frac{V_T^*}{V_t^*}\right)^{-1} S_{iT}\right] = E_t\Big[\ \left(X_{t+1}^* \cdot X_{t+2}^* \cdot \ldots \cdot X_T^*\right)^{-1} \cdot S_{iT}\ \Big] \ . \tag{2.5}$$

Eq. (2.5) shows that the current price S_{it} of an arbitrary asset i can be obtained by taking the expectation of the discounted asset price at the terminal-time, S_{iT}. Here the discount factor is the inverse of the gross return of the growth optimal portfolio. Furthermore, when this is combined with the Theorem 1 of Long (1990) which states that the non-existence of opportunities for arbitrage in a market is equivalent to the existence a growth optimal portfolio as numeraire one, and that the gross return of the growth optimal portfolio is unique, then we can use the growth optimal portfolio for pricing any assets.

2.2 Asset Pricing under a Regime Switching Model

We will consider on a probability space $(\Omega, \mathcal{F}, P)$. Just as in the previous section, one risk-free asset and n risky assets are traded in the market. It is assumed that K regimes exist in the market at discrete time $t = 0, 1, \dots, T$, and we express regime as $Y \triangleq \{Y_t \; ; \; t = 0, \dots, T\}$ and write $\mathcal{F}_t \triangleq \sigma\left(Y_t : t \in \mathbb{N}\right)$. The state space of regime Y_t is $\{e_1, \dots, e_k, \dots, e_K\}$. Here $e_k \in \mathbb{R}^K$ $(k = 1, \dots, K)$ is a vector whose k-th element is 1, otherwise 0.

We assume that regime Y_t is a first-order Markov process, and we denote the time-homogeneous transition probability from regime e_l at time t, to regime e_k at time $t+1$ as:

$$\begin{aligned} \mathbf{P} &= \left(p_{kl}\right)_{1 \leq k,l \leq K} , && (2.6) \\ \text{where } p_{kl} &= \Pr\left(Y_{t+1} = e_k | Y_t = e_l\right) . && (2.7) \end{aligned}$$

Here, p_{kl} satisfies:

$$p_{kl} \geq 0 \; (k, l = 1, \dots, K), \quad \sum_{k=1}^{K} p_{kl} = 1 . \tag{2.8}$$

Now, regime Y_t can be expressed as a state equation as follows:

$$Y_{t+1} = \mathbf{P} Y_t + M_{t+1} , \tag{2.9}$$

where M_{t+1} is a $\mathcal{F}_t$-martingale increment.

On the other hand, given the regime Y_t at time t, the return of risky assets is assumed to be described as follows as an observation equation:

$$\boldsymbol{R}_t\left(Y_t\right) = \boldsymbol{\mu}(Y_t) + \boldsymbol{\Sigma}^{1/2}(Y_t)\boldsymbol{\varepsilon}_t , \tag{2.10}$$

Here, we assume that $\boldsymbol{\mu}(Y_t)$ is the ex-divindend drift parameter. $\boldsymbol{\Sigma}^{1/2}(Y_t) = \left(\sigma_{ij}(Y_t)\right)_{1 \leq i,j \leq n} = \left(\boldsymbol{\sigma}_1'(Y_t) \dots \boldsymbol{\sigma}_i'(Y_t) \dots \boldsymbol{\sigma}_n'(Y_t)\right)'$ is the diffusion parameter. $\boldsymbol{\varepsilon}_t \underset{\text{I.I.D.}}{\sim} N(\mathbf{0}, \mathbf{I})$ is a driving term which follows mutually independent, identical standard normal distribution. Also, we write $\mathcal{R}_t := \sigma\left(\boldsymbol{R}_t : t \in \mathbb{N}\right)$, $\mathcal{G}_t := \sigma\left(Y_t, \boldsymbol{R}_t : t \in \mathbb{N}\right)$.

The drift and diffusion parameters, respectively, are assumed to take values corresponding to the regime from the following state spaces.

$$\{\boldsymbol{\mu}(1), \dots, \boldsymbol{\mu}(k), \dots, \boldsymbol{\mu}(K)\}, \; \{\boldsymbol{\Sigma}(1), \dots, \boldsymbol{\Sigma}(k), \dots, \boldsymbol{\Sigma}(K)\} . \tag{2.11}$$

At this time, we assume that the drift and diffusion parameters in the period t given the regime Y_t at time t:

$$\begin{aligned} \boldsymbol{\mu}(Y_t) &= \sum_{k=1}^{K} \langle Y_t, \; e_k \rangle \, \boldsymbol{\mu}(k) , && (2.12) \\ \boldsymbol{\Sigma}(Y_t) &= \sum_{k=1}^{K} \langle Y_t, \; e_k \rangle \, \boldsymbol{\Sigma}(k) . && (2.13) \end{aligned}$$

Here the operator $\langle \; , \; \rangle$ indicates the inner product. Eq. (2.10) shows that the diffusion and drift parameters which characterize the risk and return structure in the log return of assets will switch according to the regime which dominates that period.

We assume that the return of the risk-free asset is given as follows when the regime Y_t is given at time t.

$$r_t(Y_t) = r_f(Y_t) = \sum_{k=1}^{K} \langle Y_t,\ e_k \rangle\, r_f(k)\ . \tag{2.14}$$

Here the risk-free rate is regarded as taking values corresponding to the regime from the following state space.

$$\{r_f(1), \ldots, r_f(k), \ldots, r_f(K)\}\ . \tag{2.15}$$

Eq. (2.14) signifies the fact that the risk-free rate of interest switches according to the regime which dominates that period.

Given the above assumptions, we construct a portfolio from one risk-free asset and n risky assets. The ratios of amounts invested in the risk-free asset and risky assets after rebalancing at time $t-1$ (i.e. the portfolio weights) are written as $\{b_{0t},\ \boldsymbol{b}_t = (b_{1t} \ldots b_{it} \ldots b_{nt})'\}$. At this time, the rate of return of the entire portfolio in period t, given the regime at time t, can be expressed as follows:

$$\tilde{R}_t(Y_t) = b_{0t} r_t(Y_t) + \boldsymbol{b}_t' \boldsymbol{R}_t(Y_t)\ . \tag{2.16}$$

Using log-linear approximation (Campbell-Viceira, 2002), the log return of the portfolio is:

$$\tilde{R}_t^{\log}(Y_t) = \tilde{R}_t | Y_t - \frac{1}{2}\boldsymbol{b}_t' \boldsymbol{\Sigma}(Y_t) \boldsymbol{b}_t = \mu_P(b_{0t}, \boldsymbol{b}_t;\ Y_t) + \boldsymbol{b}_t' \boldsymbol{\Sigma}^{1/2}(Y_t) \boldsymbol{\varepsilon}_t\ . \tag{2.17}$$

Here we abbreviated as:

$$\mu_P(b_{0t}, \boldsymbol{b}_t;\ Y_t) \triangleq b_{0t} r_f(Y_t) + \boldsymbol{b}_t' \boldsymbol{\mu}(Y_t) - \frac{1}{2}\boldsymbol{b}_t' \boldsymbol{\Sigma}(Y_t) \boldsymbol{b}_t\ . \tag{2.18}$$

Furthermore, letting:

$$\boldsymbol{\mu}_P(b_{0t}, \boldsymbol{b}_t) = \Big(\mu_P(b_{0t}, \boldsymbol{b}_t;\ e_1) \ldots \mu_P(b_{0t}, \boldsymbol{b}_t;\ e_k) \ldots \mu_P(b_{0t}, \boldsymbol{b}_t;\ e_K)\Big)\ ,$$

then we have:

$$\mu_P(b_{0t}, \boldsymbol{b}_t;\ Y_t) = \langle \boldsymbol{\mu}_P(b_{0t}, \boldsymbol{b}_t),\ Y_t \rangle\ . \tag{2.19}$$

Using the tower property, the expected gross return for the entire investment period, given the trajectory of regime is:

$$E\left[\log\left(\frac{V_T}{V_0}\right) \middle| \mathcal{F}_0^R, \mathcal{F}_T^Y\right] = \sum_{t=1}^{T} E\left[\tilde{R}_t^{\log}(Y_t) \middle| \mathcal{F}_0^R, \mathcal{F}_T^Y\right] = \sum_{t=1}^{T} \mu_P\left(b_{0t}, \boldsymbol{b}_t; Y_t\right)\ .$$

Therefore, to maximize the expected growth rate given regime, the investor only have to employ the growth optimal portfolio which is given as the solution to the following problem in each period t.

$$\mathbf{P}(Y_t) \left| \begin{array}{ll} \underset{\{b_{0t}, \boldsymbol{b}_t\}}{\text{maximize}} & \mu_P(b_{0t}, \boldsymbol{b}_t; Y_t) \\ & = b_{0t} r_f(Y_t) + \boldsymbol{b}_t' \boldsymbol{\mu}(Y_t) - \frac{1}{2}\boldsymbol{b}_t' \boldsymbol{\Sigma}(Y_t) \boldsymbol{b}_t \\ \text{subject to} & b_{0t} + \boldsymbol{b}_t' \mathbf{1} = 1\ . \end{array} \right. \tag{2.20}$$

KKT conditions for the problem $\mathbf{P}(Y_t)$ are given as follows:

$$\begin{aligned} \boldsymbol{\Sigma}(Y_t)\boldsymbol{b}_t^* &= \boldsymbol{\mu}(Y_t) - r_f(Y_t)\mathbf{1} \ , && (2.21) \\ b_{0t}^* + \boldsymbol{b}_t^{*\prime}\mathbf{1} &= 1 \ . && (2.22) \end{aligned}$$

Where $\nu \in R$ is the Lagrange multiplier. Given regime, the covariance of return between the risky assets and the growth optimal portfolio is:

$$\mathrm{Cov}\left(\boldsymbol{R}_t,\ \tilde{R}_t^* \middle| \mathcal{F}_{t-1}^R, \mathcal{F}_T^Y\right) = \boldsymbol{\Sigma}(Y_t)\mathbf{b}_t^* \ . \tag{2.23}$$

Equating Eq. (2.21) and Eq. (2.23), we have;

$$\begin{aligned} \boldsymbol{\mu}(Y_t) - r_f(Y_t)\mathbf{1} &= \mathrm{Cov}_{t-1}\left(\boldsymbol{R}_t,\ \tilde{R}_t^* \middle| Y_t\right) \ , \\ \text{Or, } \mu_i(Y_t) - r_f(Y_t) &= \mathrm{Cov}_{t-1}\left(R_{it}, \tilde{R}_t^* \middle| Y_t\right) \ . && (2.24) \end{aligned}$$

Eq. (2.24) also holds for growth optimal portfolio $i = G$ itself, hence:

$$\mu_G(Y_t) - r_f(Y_t) = V_{t-1}\left[\tilde{R}_t^* \middle| Y_t\right] \ . \tag{2.25}$$

From Eqs. (2.24) and (3.1), we obtain:

$$\mu_i(Y_t) - r_f(Y_t) = \beta_i(Y_t)\left(\mu_G(Y_t) - r_f(Y_t)\right) \ . \tag{2.26}$$

Where we define:

$$\beta_i(Y_t) \triangleq \frac{\mathrm{Cov}_{t-1}\left(R_{it},\ \tilde{R}_t^* \middle| Y_t\right)}{V_{t-1}\left[\tilde{R}_t^* \middle| Y_t\right]} \ , \tag{2.27}$$

When $Y_t = e_k$, Eq. (2.26) can be written:

$$\mu_i(k) + \frac{1}{2}\lambda_i(k) - r_f(k) = \beta_i(k)\left(\mu_G(k) + \frac{1}{2}\lambda_G(k) - r_f(k)\right) \ (k = 1, \ldots, K) \ , \tag{2.28}$$

This means that, under conditions where regime Y_t is observed, linear relationships for just the number K of regimes exist between the expected excess rate of return of each asset, and that of the growth optimal portfolio. Furthermore, Eq. (2.26) can be rewritten.

$$E_{t-1}\left[R_{it}|Y_t\right] - (r_f|Y_t) = \beta_i(Y_t)\left(E_{t-1}\left[R_{Gt}|Y_t\right] - (r_f|Y_t)\right) \ . \tag{2.29}$$

Due to Eqs. (2.28) and (2.29), *the asset pricing formula under regime switching (hereafter "regime switching G-CAPM")* of Eq. (2.26) can be interpreted as follows:

"The risk premiums for individual assets and the growth optimal portfolio switch according to the regime. Given regime, the risk premium of individual assets varies proportionally with the risk premium of the growth optimal portfolio, and the degree of that dependence can be grasped by the switching beta $\beta_i(Y_t)$." The growth optimal asset pricing model with the setting of regime switching models, as indicated by Eq. (2.26), is turns out be be effective in the following ways for finding the risk premium of any asset:

- The model have the distinguishing feature which is able to indicate the relationship between the risk premiums of assets given in the list of target investments, and the risk premium of the growth optimal portfolio comprised of those assets. On the other hand, the CAPM is a formula which indicates the relationship between the risk premium of a market portfolio (which is a theoretical concept) and individual assets.
- The growth optimal portfolio can be constructed using the SPOP Scheme (Ishijima-Shirakawa, 2000), and it is even possible to use a benchmark index (S&P 500, DJIA etc.) or a uniform portfolio as a proxy (refer to the validation and theoretical studies in Roll 1973, Long 1990 and Platen and Heath 2006 etc.)
- Estimating the regime is extremely difficult. In particular, the estimation of regime shared between assets becomes ever more difficult as the number of assets increases. In this regard, a good approach for the derived formula is to estimate the regime shared by the two assets, namely: the individual asset and the growth optimal portfolio. This can be done easily using the estimation method given in the following Section.

3 An Adaptive Estimation of Regime Switching G-CAPM

The model to be estimated is indicated as Eq. (2.24). With the notation:

$$\boldsymbol{\Sigma}(Y_t) := E\left[\left(\begin{array}{c} \tilde{R}_t^* - \mu_G(Y_t) \\ \boldsymbol{R}_t - \boldsymbol{\mu}(Y_t) \end{array}\right)\left(\begin{array}{c} \tilde{R}_t^* - \mu_G(Y_t) \\ \boldsymbol{R}_t - \boldsymbol{\mu}(Y_t) \end{array}\right)' \middle| Y_t\right]$$

where $\boldsymbol{\Sigma}(Y_t) = \left(\boldsymbol{\sigma}_0(Y_t)' \boldsymbol{\sigma}_1(Y_t)' \ldots \boldsymbol{\sigma}_i(Y_t)' \ldots \boldsymbol{\sigma}_n(Y_t)'\right)'$, Eq. (2.24) can be rewritten as

$$\boldsymbol{\mu}(Y_t) - r_f(Y_t)\mathbf{1} = \mathrm{Cov}_{t-1}\left(\tilde{R}_t^*, \boldsymbol{R}_t \middle| Y_t\right) = \boldsymbol{\sigma}_0'(Y_t) \ . \tag{3.1}$$

Hence our task is to estimate $\Sigma(Y_t)$, namely $\Sigma(k)$ $(k = 1, \ldots, K)$.

An adaptive estimation of the parameter $\boldsymbol{\Theta}$ is implemented by so-called "EM algorithm". In this algorithm, an initial value $\boldsymbol{\Theta}^{(0)}$ is set appropriately, and then alternating iterations $(j = 1, 2, \ldots)$ are performed comprised of Expectation Steps and Maximization Steps. At iteration j, the parameter $\boldsymbol{\Theta}^{(j-1)}$ is updated to a better one $\boldsymbol{\Theta}^{(j)}$. That is, iterations comprised of E-steps and M-steps monotonically increase the likelihood function.

Iteration is repeated until the parameter estimate values are no longer updated, and when $\boldsymbol{\Theta}^{(J-1)} \approx \boldsymbol{\Theta}^{(J)}$, the estimate value for parameter $\boldsymbol{\Theta}$ is set as $\hat{\boldsymbol{\Theta}} = \boldsymbol{\Theta}^{(J)}$.

Below we shall discuss the EM algorithm used for estimating the model parameters in adaptive manner. Write $\mathscr{R}_t = \sigma(R_1, \ldots, R_t)$ and $\mathscr{Y}_t = \sigma(Y_1, \ldots, Y_t)$. We then define the Q-function as:

$$Q\left(\boldsymbol{\Theta}^{(j+1)}; \boldsymbol{\Theta}^{(j)}, \mathscr{R}\right) := E^{\boldsymbol{\Theta}^{(j)}}\left[\log \frac{dP^{\boldsymbol{\Theta}^{(j+1)}}}{dP^{\boldsymbol{\Theta}^{(j)}}} \middle| \mathscr{R}_T\right] \ . \tag{3.2}$$

Also we let

$$\Lambda_T := \left.\frac{dP^{\boldsymbol{\Theta}^{(j+1)}}}{dP^{\boldsymbol{\Theta}^{(j)}}}\right|_{\mathscr{Y}_T} \ .$$

To estimate $\mathbf{\Sigma}(k)$, consider Λ_T as

$$\Lambda_T = \frac{f\left(\mathscr{R}_T, \mathscr{Y}_T; \mathbf{\Theta}^{(j+1)}\right)}{f\left(\mathscr{R}_T, \mathscr{Y}_T; \mathbf{\Theta}^{(j)}\right)} = \prod_{t=1}^{T} \frac{f\left(\boldsymbol{R}_t, Y_t \middle| \mathscr{R}_{T-1}, \mathscr{Y}_{T-1}; \mathbf{\Theta}^{(j+1)}\right)}{f\left(\boldsymbol{R}_t, Y_t \middle| \mathscr{R}_{T-1}, \mathscr{Y}_{T-1}; \mathbf{\Theta}^{(j)}\right)} .$$

Here $f\left(\mathscr{R}_T, \mathscr{Y}_T; \mathbf{\Theta}\right)$ is the complete likelihood function. Given the model (2.10),

$$\begin{aligned} &f\left(\boldsymbol{R}_t, Y_t \middle| \mathscr{R}_{T-1}, \mathscr{Y}_{T-1}; \mathbf{\Theta}^{(j+1)}\right) \\ &= \frac{1}{(2\pi)^{\frac{n}{2}} \left|\mathbf{\Sigma}^{(j+1)}(Y_t)\right|^{\frac{1}{2}}} \exp\left[-\frac{1}{2}\left(\boldsymbol{R}_t - \mu^{(j+1)}(Y_t)\right)' \left(\mathbf{\Sigma}^{(j+1)}(Y_t)\right)^{-1} \left(\boldsymbol{R}_t - \mu^{(j+1)}(Y_t)\right)\right] . \end{aligned}$$

Then

$$\begin{aligned} &\log \Lambda_T \\ &= -\frac{1}{2}\sum_{t=1}^{T}\sum_{k=1}^{K} \langle Y_t, e_k\rangle \left[\log\left|\mathbf{\Sigma}^{(j+1)}(k)\right| + \left(\boldsymbol{R}_t - \mu^{(j+1)}(k)\right)' \left(\mathbf{\Sigma}^{(j+1)}(k)\right)^{-1} \left(\boldsymbol{R}_t - \mu^{(j+1)}(k)\right)\right] \\ &+A\left(\boldsymbol{\mu}^{(j)}, \mathbf{\Sigma}^{(j)}\right) . \end{aligned}$$

Hence,

$$\begin{aligned} &E^{\mathbf{\Theta}^{(j)}}\left[\log \Lambda_T | \mathscr{R}_T\right] \\ &= -\frac{1}{2}\sum_{t=1}^{T}\sum_{k=1}^{K} \xi_{k,t}^{(j)} \left[\log\left|\mathbf{\Sigma}^{(j+1)}(k)\right| + \left(\boldsymbol{R}_t - \mu^{(j+1)}(k)\right)' \left(\mathbf{\Sigma}^{(j+1)}(k)\right)^{-1} \left(\boldsymbol{R}_t - \mu^{(j+1)}(k)\right)\right] \\ &+\hat{A}\left(\boldsymbol{\mu}^{(j)}, \mathbf{\Sigma}^{(j)}\right) . \end{aligned}$$

Here $\xi_{k,t}^{(j)} := E^{\mathbf{\Theta}^{(j)}}\left[\langle Y_t, e_k\rangle | \mathscr{R}_T\right]$. Differentiate this equation with respect to $\mathbf{\Sigma}^{(j+1)}(k)$ to equate to $\mathbf{O}$, we obtain the re-estimate of $\hat{\mathbf{\Sigma}}^{(j+1)}(k, T)$, at time $t = T$:

$$\begin{aligned} \mathbf{\Sigma}^{(j+1)}(k,T) &= \sum_{t=1}^{T} \xi_{k,t}^{(j)} \left(\boldsymbol{R}_t - \boldsymbol{\mu}^{(j+1)}(k)\right)\left(\boldsymbol{R}_t - \boldsymbol{\mu}^{(j+1)}(k)\right)' \Big/ \sum_{t=1}^{T} \xi_{k,t}^{(j)} \\ &= \left(\hat{\mathcal{T}}_{k,T}^{(j)}(\boldsymbol{R}\boldsymbol{R}') - \hat{\mathcal{T}}_{k,T}^{(j)}(\boldsymbol{R})\boldsymbol{\mu}^{(j+1)\prime}(k) - \boldsymbol{\mu}^{(j+1)}(k)\hat{\mathcal{T}}_{k,T}^{(j)}(\boldsymbol{R}') + \boldsymbol{\mu}^{(j+1)}(k)\boldsymbol{\mu}^{(j+1)\prime}(k)\hat{\mathcal{O}}_{k,T}^{(j)}\right) \Big/ \hat{\mathcal{O}}_{k,T}^{(j)} \end{aligned} \tag{3.3}$$

The filtering techniques under $P^{\mathbf{\Theta}^{(j)}}$ are given in Elliott et al (1995). Define a new probability measure $\bar{P}$ by

$$\left.\frac{d\bar{P}}{dP}\right|_{\mathcal{G}_t} = \Lambda_t = \prod_{\tau=1}^{t} \lambda_\tau = \frac{\phi\left(\boldsymbol{R}_t\right) \sum_{k=1}^{K} \langle Y_t, e_k\rangle \left|\Sigma(k)\right|^{\frac{1}{2}}}{\phi\left(\boldsymbol{\varepsilon}_t\right)} .$$

Also to construct P from $\bar{P}$, define P by putting

$$\left.\frac{dP}{d\bar{P}}\right|_{\mathcal{G}_t} = \bar{\Lambda}_t = \prod_{\tau=1}^{t} \bar{\lambda}_\tau ,$$

where $\bar{\lambda}_t = \lambda_t^{-1}$ $(t \geq 1)$.

Let $\{H_t : t \in \mathbb{N}\}$ be any process adapted to $\mathcal{G}_t$ and consider its filtering problem. With the notation of

$$\gamma_t(H_t) := \bar{E}\left[\bar{\Lambda}_t H_t \middle| \mathcal{R}_t\right] ,$$

the filter for H_t is then given as

$$\hat{H}_t = E\left[H_t \middle| \mathcal{R}_t\right] = \frac{\bar{E}\left[\bar{\Lambda}_t H_t \middle| \mathcal{R}_t\right]}{\bar{E}\left[\bar{\Lambda}_t \middle| \mathcal{R}_t\right]} = \frac{\gamma_t(H_t)}{\gamma_t(1)} .$$

We remark that the estimate of $\gamma_t(H_t)$ is expressed in terms of $\gamma_t(H_t Y_t)$, since $\gamma_t(H_t) = \langle \gamma_t(H_t Y_t), \mathbf{1} \rangle$. Also the recursion for $\gamma_t(H_t Y_t)$ can be easily obtained. As summary, by deriving the recursive equation for $\gamma_{t,t}(H_t) = \gamma_t(H_t Y_t)$, we then obtain $\gamma_t(H_t)$ and $\gamma_t(1)$, and we thus arrive at the filter $\hat{H}_t = E[H_t|\mathcal{R}_t] = \gamma_t(H_t)/\gamma_t(1)$. Our concern is the estimators for the followgs:

$$\begin{array}{ll} \textit{State}: & Y_t \\ \textit{Occupation time}: & \mathcal{O}_{k,t} := \sum_{\tau=1}^{t} \langle Y_\tau, e_k \rangle \\ \textit{Number of jumps}: & \mathcal{J}_{kl,t} := \sum_{\tau=1}^{t} \langle Y_{\tau-1}, e_k \rangle \langle Y_\tau, e_l \rangle \\ \textit{Observation process}: & \mathcal{T}_{k,t}(f) := \sum_{\tau=1}^{T} \langle Y_\tau, e_k \rangle f(\boldsymbol{R}_\tau) \end{array}$$

Adaptive estimators for these are given in Elliott et al (1995) as follows:

$$\begin{aligned} \gamma_t(Y_t) &= \sum_{k=1}^{K} \langle \gamma_{t-1}(Y_{t-1}), \Gamma_k(\boldsymbol{R}_t) \rangle \boldsymbol{p}_i , \\ \gamma_{t,t}\left(\mathcal{J}_t^{kl}\right) &= \sum_{k=1}^{K} \left\langle \gamma_{t,t}\left(\mathcal{J}_{t-1}^{kl}\right), \Gamma_k(\boldsymbol{R}_t) \right\rangle \boldsymbol{p}_k + \langle \gamma_{t-1}(Y_{t-1}), \Gamma_k(\boldsymbol{R}_t) \rangle p_{lk} \boldsymbol{e}_l , \\ \gamma_{t,t}(\mathcal{O}_{k,t}) &= \sum_{l=1}^{K} \langle \gamma_{t-1,t-1}(\mathcal{O}_{k,t-1}, \Gamma_l(\boldsymbol{R}_t)) \rangle \boldsymbol{p}_l + \langle \gamma_t(Y_t), \Gamma_k(\boldsymbol{R}_t) \rangle \boldsymbol{p}_k , \\ \gamma_{t.t}(\mathcal{T}_{k,t}) &= \sum_{l=1}^{K} \langle \gamma_{t-1,t-1}(\mathcal{T}_{k,t}(f)), \Gamma_l(\boldsymbol{R}_{t+1}) \rangle \boldsymbol{p}_l + \langle \gamma_{t-1}(Y_{t-1}), \Gamma_k(\boldsymbol{R}_t) \rangle f(\boldsymbol{R}_t) \boldsymbol{p}_k . \end{aligned}$$

Using these estimators, we are able to re-estimate $\boldsymbol{\Sigma}^{(j+1)}(k,t)$ which provides risk premiums for risky assets in an adaptive manner.

4 Numerical Example

We generate the sample data according to the two-dimensional and two-state regime switching normal model. The parameter set used are shown in Table 1. Under these settings, we estimate the regime switching G-CAPM by two algorithms: Baum-Welch and Elliott algorithms. We evaluate them in the viewpoint of the error in estimated value when compared to the true ones.

Estimated transition probabilities using Baum-Welch and Elliott algorithms are shown in Table 2. Values in parentheses shows the errors of estimated values from true ones. Also estimated mean vectors and covariance matrices for each of regime, using Baum-Welch and Elliott algorithms, are shown in Table 3 and 4.

Both Baum-Welch and Elliott algorithms monotonically increase the likelihood as the iteration has been repeated. In this numerical experiment, the log-likelihood function is locally maximized in the second iteration. Our finding is that the same initial values for both algorithms leads to exactly the same results.

P	Regime 1	Regime 2
Regime 1	0.4	0.5
Regime 2	0.6	0.5

Regime 1	Asset 1	Asset 2
Mean	5	-5
Covariance	2	1
	1	2

Regime 2	Asset 1	Asset 2
Mean	-10	10
Covariance	1	0.5
	0.5	1

Figure 1: Parameter set used in the numerical experiments. Table on the left shows the transition probability matrix. Two tables on the right show the mean vector and covariance matrix for each of regime.

Figure 2: Estimated transition probabilities with the sample size of 10, 100, 1,000, 10,000 and 100,000. The results by Baum-Welch and Elliott algorithms are shown in the left and right tables, respectively.

T = 100	Regime 1	Regime 2
Regime 1	0.3830	0.5385
	(-4.2553)	(7.6923)
Regime 2	0.6170	0.4615
	(2.8369)	(-7.6923)

T = 100	Regime 1	Regime 2
Regime 1	0.3830	0.5385
	(-4.2553)	(7.6923)
Regime 2	0.6170	0.4615
	(2.8369)	(-7.6923)

T = 1,000	Regime 1	Regime 2
Regime 1	0.4026	0.5226
	(0.6424)	(4.5113)
Regime 2	0.5974	0.4774
	(-0.4283)	(-4.5113)

T = 1,000	Regime 1	Regime 2
Regime 1	0.4026	0.5226
	(0.6424)	(4.5113)
Regime 2	0.5974	0.4774
	(-0.4283)	(-4.5113)

T = 10,000	Regime 1	Regime 2
Regime 1	0.4000	0.5212
	(-0.0108)	(4.2422)
Regime 2	0.6000	0.4788
	(0.0072)	(-4.2422)

T = 10,000	Regime 1	Regime 2
Regime 1	0.4000	0.5212
	(-0.0108)	(4.2422)
Regime 2	0.6000	0.4788
	(0.0072)	(-4.2422)

T = 100,000	Regime 1	Regime 2
Regime 1	0.4003	0.5025
	(0.0855)	(0.4963)
Regime 2	0.5997	0.4975
	(-0.0570)	(-0.4963)

T = 100,000	Regime 1	Regime 2
Regime 1	0.4003	0.5025
	(0.0855)	(0.4963)
Regime 2	0.5997	0.4975
	(-0.0570)	(-0.4963)

5 Conclusion

We derived a growth optimal asset pricing model with the setting of hidden Markov models and slightly extended Elliott algorithm (1995) to estimate the model in an adaptive manner.

References

[1] Breiman, L. (1961), "Optimal Gambling Systems for Favorable Games," *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability*, I, 65-78.

[2] Campbell, John Y. and Luis M. Viceira (2002), *Strategic Asset Allocation: Portfolio Choice for Long-Term Investors*, Oxford University Press.

[3] Cover, T.M. and J.A. Thomas (1990), *Elements of Information Theory*, Wiley-Interscience.

[4] Dempster, A. P., N. M. Laird and D. B. Rubin (1977), "Maximum Likelihood from Incomplete Data via the EM Algorithm," *Journal of the Royal Statistical Society Series B*, 39, 1-38.

[5] Elliott, R. J., L. Aggoun and J. B. Moore (1995), *Hidden Markov Models: Estimation and Control*, Springer-Verlag.

[6] Hakansson, N.H. (1971), "Capital Growth and the Mean-Variance Approach to Portfolio Selection," *Journal of Financial and Quantitative Analysis*, 6(1), 517-557.

[7] Hamilton, J.D. (1989), "A New Approach to the Economic Analysis of Nonstationary Time Series and the Business Cycle," *Econometrica*, 57, 357-384.

[8] Hamilton, J.D. (1990), "Analysis of Time Series Subject to Changes in Regime," *Journal of Econometrics*, 45, 39-70.

[9] Hamilton, J.D. (1994), *Time Series Analysis*, Princeton University Press.

[10] Ishijima, H. and H. Shirakawa (2000), "The Optimal Log-Utility Asset Management under Incomplete Information," *Asia-Pacific Financial Markets*, 7(2), 145-154.

[11] Kelly, J. L. (1956), "A New Interpretation of Information Rate," *Bell System Technical Journal*, 35, 917-926.

[12] Long, J.B. (1990), "The Numeraire Portfolio," *Journal of Financial Economics*, 26, 29-69.

[13] Luenberger, D.G. (1993), "A Preference Foundation for Log Mean-variance Criteria in Portfolio Choice Problems," *Journal of Economic Dynamics and Control*, 17, 887-906.

[14] Luenberger, D.G. (1997), *Investment Science*, Oxford University Press.

[15] Platen, E. and D. Heath (2006), *A Benchmark Approach to Quantitative Finance*, Springer.

[16] Roll, R. (1973), "Evidence on the "Growth-Optimum" Model," *Journal of Finance*, 28(3), 551-566.

[17] Thorp, E. O. (1971), "Portfolio Choice and the Kelly criterion," *Proceedings of the 1971 Business and Economics Section of the American Statistical Association*, 215-224.

T = 100	Regime 1	Asset 1	Asset 2
	Mean	4.9266	-5.2289
		(-1.4671	)(4.5790)
	Covariance	1.6809	1.0451
		(-15.9563	)(4.5115)
		1.0451	1.9738
		(4.5115	)(-1.3100)
	Regime 2	Asset 1	Asset 2
	Mean	-9.9895	9.9963
		(-0.1048	)(-0.0365)
	Covariance	0.7189	0.3139
		(-28.1087	)(-37.2216)
		0.3139	0.7526
		(-37.2216	)(-24.7365)
T = 1,000	Regime 1	Asset 1	Asset 2
	Mean	4.9518	-5.1184
		(-0.9638	)(2.3680)
	Covariance	1.9963	1.0833
		(-0.1870	)(8.3323)
		1.0833	1.9927
		(8.3323	)(-0.3660)
	Regime 2	Asset 1	Asset 2
	Mean	-9.9270	10.0359
		(-0.7300	)(0.3586)
	Covariance	0.9145	0.4583
		(-8.5495	)(-8.3389)
		0.4583	1.0141
		(-8.3389	)(1.4081)

T=100	Regime 1	Asset 1	Asset 2
	Mean	4.9266	-5.2289
		(-1.4671	)(4.5790)
	Covariance	1.6809	1.0451
		(-15.9563	)(4.5115)
		1.0451	1.9738
		(4.5115	)(-1.3100)
	Regime 2	Asset 1	Asset 2
	Mean	-9.9895	9.9963
		(-0.1048	)(-0.0365)
	Covariance	0.7189	0.3139
		(-28.1087	)(-37.2216)
		0.3139	0.7526
		(-37.2216	)(-24.7365)
T=1,000	Regime 1	Asset 1	Asset 2
	Mean	4.9518	-5.1184
		(-0.9638	)(2.3680)
	Covariance	1.9963	1.0833
		(-0.1870	)(8.3323)
		1.0833	1.9927
		(8.3323	)(-0.3660)
	Regime 2	Asset 1	Asset 2
	Mean	-9.9270	10.0359
		(-0.7300	)(0.3586)
	Covariance	0.9145	0.4583
		(-8.5495	)(-8.3389)
		0.4583	1.0141
		(-8.3389	)(1.4081)

Figure 3: Estimated mean vectors and covariance matrices with the sample size of 100, 1,000. The results by Baum-Welch and Elliott algorithms are shown in the left and right tables, respectively.

T = 10,000	Regime 1	Asset 1	Asset 2
	Mean	5.0233	-5.0227
		(0.4652	)(0.4547)
	Covariance	2.0253	0.9979
		(1.2671	)(-0.2132)
		0.9979	1.9658
		(-0.2132	)(-1.7103)
	Regime 2	Asset 1	Asset 2
	Mean	-9.9865	10.0038
		(-0.1350	)(0.0379)
	Covariance	1.0231	0.4942
		(2.3094	)(-1.1600)
		0.4942	0.9864
		(-1.1600	)(-1.3618)
T = 100,000	Regime 1	Asset 1	Asset 2
	Mean	5.0163	-4.9887
		(0.3256	)(-0.2254)
	Covariance	2.0094	1.0077
		(0.4682	)(0.7714)
		1.0077	2.0102
		(0.7714	)(0.5122)
	Regime 2	Asset 1	Asset 2
	Mean	-10.0019	10.0014
		(0.0189	)(0.0140)
	Covariance	1.0074	0.5066
		(0.7372	)(1.3237)
		0.5066	1.0078
		(1.3237	)(0.7829)

T=10,000	Regime 1	Asset 1	Asset 2
	Mean	5.0233	-5.0227
		(0.4652	)(0.4547)
	Covariance	2.0253	0.9979
		(1.2671	)(-0.2132)
		0.9979	1.9658
		(-0.2132	)(-1.7103)
	Regime 2	Asset 1	Asset 2
	Mean	-9.9865	10.0038
		(-0.1350	)(0.0379)
	Covariance	1.0231	0.4942
		(2.3094	)(-1.1600)
		0.4942	0.9864
		(-1.1600	)(-1.3618)
T=100,000	Regime 1	Asset 1	Asset 2
	Mean	5.0163	-4.9887
		(0.3256	)(-0.2254)
	Covariance	2.0094	1.0077
		(0.4682	)(0.7714)
		1.0077	2.0102
		(0.7714	)(0.5122)
	Regime 2	Asset 1	Asset 2
	Mean	-10.0019	10.0014
		(0.0189	)(0.0140)
	Covariance	1.0074	0.5066
		(0.7372	)(1.3237)
		0.5066	1.0078
		(1.3237	)(0.7829)

Figure 4: Estimated mean vectors and covariance matrices with the sample size of 10,000, 100,000. The results by Baum-Welch and Elliott algorithms are shown in the left and right tables, respectively.