Approximating the Generalized Capacitated Tree-routing Problem

Ehab Morsy and Hiroshi Nagamochi

Department of Applied Mathematics and Physics,
Graduate School of Informatics
Kyoto University
Yoshida Honmachi, Sakyo, Kyoto 606-8501, Japan

Abstract

We introduce the generalized capacitated tree-routing problem which is described as follows. Given a connected graph $G = (V, E)$ with a sink $s \in V$ and a set $M \subseteq V - \{s\}$ of terminals with a nonnegative demand $q(v)$, $v \in M$, we wish to find a collection of trees rooted at s to send all the demands to s, where the total demand collected by each tree is bounded from above by a demand capacity $\kappa > 0$. Let $\lambda > 0$ denote a bulk capacity of an edge, and each edge $e \in E$ has an installation cost $w(e) \geq 0$ per bulk capacity; each edge e is allowed to have capacity $j\lambda$ for any integer j, which installation incurs cost $jw(e)$. To establish a desired tree routing T_i, each edge e contained in T_i requires $\alpha + \beta q'$ amount of capacity for the total demand q' that passes through edge e along T_i, where $\alpha \geq 0$ and $\beta \geq 0$ are prescribed constants. Term α means a fixed amount used to separate the inside of the routing T_i from the outside while term $\beta q'$ means the net capacity proportional to q'. The objective of GCTR is to find a collection of trees that minimizes the total installation cost of edges. GCTR is a new generalization which unifies several known routing problems in networks with edge/demand capacities.

keyword Approximation algorithm, Graph algorithm, Routing problem, Network optimization.

1 Introduction

In this paper, we introduce the *generalized capacitated tree-routing problem* (GCTR), which is described as follows. Given a connected graph $G = (V, E)$ with a demand capacity $\kappa > 0$, a bulk edge capacity $\lambda > 0$, a sink $s \in V$, and a set $M \subseteq V - \{s\}$ of terminals with a nonnegative demand $q(v)$, $v \in M$, we wish to find a collection $\mathcal{T} = \{T_1, T_2, \ldots, T_\ell\}$ of trees rooted at s to send all the demands to s, where the total demand in the set Z_i of terminals assigned to tree T_i does not exceed the demand capacity κ. Each edge $e \in E$ has an installation cost $w(e) \geq 0$ per bulk capacity; each edge e is allowed to have capacity $j\lambda$ for any integer j, which requires installation cost $jw(e)$. To establish a tree routing T_i through an edge e, we assume that e needs to have capacity at least

$$\alpha + \beta q(Z_i \cap D_{T_i}(v_i^e))$$

for prescribed coefficients $\alpha, \beta \geq 0$, where v_i^e is the tail of e in T_i; α means a fixed amount used to separate the inside and outside of the routing T_i while term $\beta q(Z_i \cap D_{T_i}(v_i^e))$

means the net capacity proportional to the amount $q(Z_i \cap D_{T_i}(v_i^c))$ of demands that passes through edge e along T_i. Hence, given a set $\mathcal{T} = \{T_1, T_2, \ldots, T_\ell\}$ of trees, each edge e needs to have capacity $h_{\mathcal{T}}(e)\lambda$ for the least integer $h_{\mathcal{T}}(e)$ such that

$$\sum_{T_i \in \mathcal{T}:T_i \text{ contains } e} \left(\alpha + \beta q(Z_i \cap D_{T_i}(v_i^c))\right) \leq h_{\mathcal{T}}(e)\lambda,$$

and the total installation cost of edges incurred by $\mathcal{T}$ is given as $\sum_{e \in E} h_{\mathcal{T}}(e)w(e)$, where $h_{\mathcal{T}}(e) = 0$ if no $T_i \in \mathcal{T}$ contains e. The objective of GCTR is to find a set $\mathcal{T}$ of trees that minimizes the total installation cost of edges. We formally state GCTR as follows.

Generalized Capacitated Tree-Routing Problem (GCTR):
Input: A connected graph $G = (V, E)$, an edge weight function $w : E \to R^+$, a demand capacity $\kappa > 0$, an edge capacity $\lambda > 0$, prescribed constants $\alpha, \beta \geq 0$, a sink $s \in V$, a set $M \subseteq V - \{s\}$ of terminals, and a demand function $q : M \to R^+$.
Feasible solution: A partition $\mathcal{M} = \{Z_1, Z_2, \ldots, Z_\ell\}$ of M and a set $\mathcal{T} = \{T_1, T_2, \ldots, T_\ell\}$ of trees of G such that $Z_i \cup \{s\} \subseteq V(T_i)$ and $q(Z_i) \leq \kappa$ hold for each i. The number of copies of an edge $e \in E$ installed in the solution is given by $h_{\mathcal{T}}(e) = \lceil \sum_{T_i:e \in E(T_i)} (\alpha + \beta q(Z_i \cap D_{T_i}(v_i^c)))/\lambda \rceil$, where v_i^c is the tail of e in T_i.
Goal: Minimize the total installation cost of $\mathcal{T}$, that is,

$$\sum_{e \in E} h_{\mathcal{T}}(e)w(e).$$

We have a variant of GCTR if it is allowed to purchase edge capacity in any required quantity. In this model, for each edge e of the underlying network, we assign capacity of $\lambda_e = \alpha|T'| + \beta \sum_{T_i \in T'} q(Z_i \cap D_{T_i}(v_i^c))$ on e, where T' is the set of trees containing e. That is, the total cost of the constructed trees equals $\sum_{e \in E} \lambda_e w(e)$. We call this variant of GCTR. *the fractional generalized capacitated tree-routing problem* (FGCTR).

We easily see that GCTR and FGCTR contain two classical NP-hard problems, the Steiner tree problem and the *bin packing problem* [2]. We see that GCTR with an edge weighted graph G, $\alpha = \lambda = 1$, and $\beta = 0$ is equivalent to the Steiner tree problem in G when $\kappa \geq \sum_{v \in M} q(v)$. whereas it is equivalent to the bin packing problem with bin size κ when G is a complete graph, $w(e) = 1$ for all edges e incident to s and $w(e) = 0$ otherwise. We see that FGCTR also has a similar relationship with the Steiner tree problem and the bin packing problem.

The characteristic of GCTR and FGCTR is their routing capacity which is a linear combination of the number of trees and the total amount of demands that pass through an edge. Such a general form of capacity constraint can be found in some applications.

We here observe that our new problem formulation, GCTR, includes several important routing problems as its special cases such as the *Capacitated Network Design Problem* (CND), the *Capacitated Multicast Tree Routing Problem* (CMTR), and the *Capacitated Tree-Routing Problem* (CTR). See [7] for the definitions of these problems. Table 1 shows a summary of the recent approximation algorithms for CND, CMTR, CTR, and GCTR.

Table 1: Approximation algorithms for CND, CMTR, CTR, and GCTR problems, where $\theta = \lceil \lambda/(\alpha + \beta\kappa) \rceil / \lfloor \lambda/(\alpha + \beta\kappa) \rfloor$.

Problem		unit demands	general demands
CND	$\alpha = 0, \beta = 1,$ $\kappa = \lambda \in R^+$	$1 + \rho_{\mathrm{ST}}$ [3]	$2 + \rho_{\mathrm{ST}}$ [3]
CMTR	$\alpha = 1, \beta = 0,$ $\lambda = 1, \kappa \in R^+$	$8/5 + (5/4)\rho_{\mathrm{ST}}$ [1], $3/2 + (4/3)\rho_{\mathrm{ST}}$ [8]	$2 + \rho_{\mathrm{ST}}$ [4]
CTR	$\alpha = 1, \beta = 0$ $\lambda \in Z^+, \kappa \in R^+$	$2 + \rho_{\mathrm{ST}}$ [10]	$2 + \rho_{\mathrm{ST}}$ [10]
GCTR	$\alpha, \beta, \kappa, \lambda \in R^+$ with (i) $\lambda \geq \alpha + \beta\kappa$ (ii) $\lambda < \alpha + \beta\kappa$	$2\theta + \rho_{\mathrm{ST}}$ (this paper) 13.037 (this paper)	$2\theta + \rho_{\mathrm{ST}}$ (this paper) 13.037 (this paper)

As observed above, GCTR is a considerably general model for routing problems. In this paper, we first prove that GCTR admits a $(2\lceil \lambda/(\alpha + \beta\kappa) \rceil / \lfloor \lambda/(\alpha + \beta\kappa) \rfloor + \rho_{\mathrm{ST}})$-approximation algorithm if $\lambda \geq \alpha + \beta\kappa$ holds. The high-level description of the proposed algorithm resembles our algorithm for CTR, but we need to derive a new lower bound to the problem. Namely, given an instance $I = (G, w, \kappa, \lambda, \alpha, \beta, s, M, q)$ of GCTR, the main idea of our algorithm is to compute an integer capacity λ' depending on κ, λ, α, and β and then find a feasible tree-routings solution to the instance $I' = (G, w, \kappa, \lambda', s, M, q)$ of CTR. Here such a capacity λ' is chosen so that this set of tree-routings is a feasible solution to the original GCTR instance I.

We observe that it is not straightforward to modify the above algorithm so that it also delivers a constant-factor approximate solution in the case of $\lambda < \alpha + \beta\kappa$. This motivates proposing a different approach for approximating GCTR instances with $\lambda < \alpha + \beta\kappa$. For this, we introduce a new lower bound on GCTR by introducing a generalization of CND, and use a balanced Steiner tree as a base tree from which we construct a collection of trees to send demands to sink. We show that our new algorithm delivers a 13.037-approximate solution to GCTR with $\lambda < \alpha + \beta\kappa$. Based on the same approach, we also prove that FGCTR is 8.529-approximable.

The rest of this paper is organized as follows. Section 2 introduces some notations and several lower bounds on the optimal value of GCTR. Sections 3 and 4 introduce approximation algorithms for GCTR with $\lambda \geq \alpha + \beta\kappa$ and $\lambda < \alpha + \beta\kappa$, respectively. We present an algorithm to FGCTR in Section 5. Section 6 makes concluding remarks.

2 Preliminaries

This section introduces some notations and definitions. An edge-weighted graph is a pair (G, w) of a graph G and a nonnegative weight function $w : E(G) \to R^+$. The length of a shortest path between two vertices u and v in (G, w) is denoted by $d_{(G,w)}(u, v)$. Given a demand function $q : V(G) \to R^+$ and a subgraph H of G, we use $q(H)$ and $q(V(H))$ interchangeably to denote the sum $\sum_{v \in V(H)} q(v)$ of demands of all vertices in $V(H)$.

Let (G, w) be an edge-weighted graph with a terminal set $M \subseteq V(G)$ and a designated vertex s in G. A Steiner minimum tree on $(G, w, M \cup \{s\})$ is a tree of minimum weight of G that spans $M \cup \{s\}$. A shortest path tree on $(G, w, M \cup \{s\})$ rooted at s is a tree that spans $M \cup \{s\}$ such that the distance between s and any vertex $v \in M$ in the tree equals to the shortest distance between s and v in G. Given a Steiner minimum tree and a shortest path tree on $(G, w, M \cup \{s\})$, a "balanced" Steiner tree T is a tree of G that spans $M \cup \{s\}$ and approximates both the shortest path tree and the Steiner minimum tree. That is, for some constants $c_1, c_2 \geq 1$, the distance between s and any vertex $v \in M$ in T is at most c_1 times the shortest distance between s and v in G, and the weight of T is at most c_2 times the weight of a Steiner minimum tree of G.

Let T be a tree. A *subtree* of T is a connected subgraph of T. A set of subtrees in T is called a *tree cover* of T if each vertex in T is contained in at least one of the subtrees. For a subset $X \subseteq V(T)$ of vertices, let $T\langle X \rangle$ denote the minimal subtree of T that contains X (note that $T\langle X \rangle$ is uniquely determined). Now let T be a rooted tree. We denote by $L(T)$ the set of leaves in T. For a vertex v in T, let $Ch(v)$ and $D(v)$ denote the sets of children and descendants of v, respectively, where $D(v)$ includes v. A *subtree* T_v *rooted* at a vertex v is the subtree induced by $D(v)$, i.e., $T_v = T\langle D(v) \rangle$. For a rooted tree T_r, the *depth* of a vertex u in T_r is the length (the number of edges) of the path from r to u.

The rest of this section introduces some lower bounds to GCTR. The first lower bound is based on the Steiner tree problem.

Lemma 2.1. *Given a* GCTR *instance* $I = (G, w, \kappa, \lambda, \alpha, \beta, s, M, q)$, *the minimum cost of a Steiner tree to* $(G, w, M \cup \{s\})$ *is a lower bound on the optimal value to* GCTR *instance* I. $\qquad\square$

The second lower bound is derived from an observation on the distance from vertices to sink s.

Lemma 2.2. *Let* $I = (G, w, \kappa, \lambda, \alpha, \beta, s, M, q)$ *be an instance of* GCTR. *Then*

$$(\alpha + \beta\kappa)/(\kappa\lambda) \sum_{v \in M} q(v) d_{(G,w)}(s, v)$$

is a lower bound on the optimal value to GCTR *instance* I. $\qquad\square$

3 Approximation algorithm for $\lambda \geq \alpha + \beta\kappa$

In this section we present an approximation algorithm to GCTR instances with $\lambda \geq \alpha + \beta\kappa$.

Given an instance $I = (G, w, \kappa, \lambda, \alpha, \beta, s, M, q)$ of GCTR, the main idea of our algorithm is to find a feasible solution $(\mathcal{M} = \{Z_1, \ldots, Z_\ell\}, \mathcal{T} = \{T_1, \ldots, T_\ell\})$ to a CTR instance $I' = (G, w, \kappa, \lambda', s, M, q)$, where $\lambda' = \lfloor \lambda/(\alpha + \beta\kappa) \rfloor$. That is, for each edge e in G, the number of trees of $\mathcal{T}$ containing e is at most $h_{\mathcal{T}}(e)\lambda'$, where $h_{\mathcal{T}}(e)$ denotes the number of copies of e installed in the solution $(\mathcal{M}, \mathcal{T})$ of I'. Note that $q(Z_i) \leq \kappa$

for all $i = 1, 2, \ldots, \ell$. Therefore, for each edge e in G with tail v^e, we have

$$\sum_{T_i \in \mathcal{T} : e \in E(T_i)} (\alpha + \beta q(D_{T_i}(v^e) \cap M)) \leq (\alpha + \beta\kappa)|\{T_i \in \mathcal{T} \mid e \in E(T_i)\}|$$

$$\leq h_{\mathcal{T}}(e)(\alpha + \beta\kappa)\lfloor \lambda/(\alpha + \beta\kappa)\rfloor \leq h_{\mathcal{T}}(e)\lambda.$$

This implies that $(\mathcal{M}, \mathcal{T})$ is a feasible solution to GCTR instance I.

For seeking a simple presentation, we first discuss GCTR instances with $\lfloor \lambda/(\alpha + \beta\kappa)\rfloor = 1$ in the next section.

3.1 Approximation algorithm for $\lfloor \lambda/(\alpha + \beta\kappa)\rfloor = 1$

This section provides an approximate solution to GCTR when $\lfloor \lambda/(\alpha + \beta\kappa)\rfloor = 1$. The algorithm is based on the following a "balanced" partition of a set of terminals.

For a tree T rooted at a vertex r, an ordered partition $\mathcal{Z} = \{Z_1, Z_2, \ldots, Z_p\}$ of a subset of the terminal set M is called κ-*balanced* if the following holds:

(i) $q(Z_i) \leq \kappa$ for $i = 1, 2, \ldots, p$;

(ii) $q(Z_i) > \kappa/2$ for $i = 1, 2, \ldots, p - 1$, and if $p \geq 2$ then $q(Z_{p-1} \cup Z_p) > \kappa$; and

(iii) Each $T\langle Z_j \rangle$ $(j = 1, 2, \ldots, p - 1)$ has no common edge with $T\langle \cup_{j < i \leq p} Z_i \cup \{r\}\rangle$.

Lemma 3.1. *There always exists a κ-balanced partition if* $\max_{v \in M} q(v) \leq \kappa$.

The basic idea of the algorithm is analogous to that for CTR given in the previous chapter. We first compute an approximate Steiner tree T in $(G, w, M \cup \{s\})$, regard T as a tree rooted at s, and then find a κ-balanced partition $\mathcal{M} = \{Z_1, Z_2, \ldots, Z_p\}$ of M in T. For each $Z_i \in \mathcal{M}$, we choose a vertex $t_{Z_i} \in Z_i$ and connect the tree $T\langle Z_i \rangle$ to s by adding a shortest path between s and t_{Z_i} in (G, w). We describe the algorithm in the following form which will be used for the case of $\lfloor \lambda/(\alpha + \beta\kappa)\rfloor \geq 2$.

Algorithm APPROXGCTR
Input: A GCTR instance $I = (G, w, \kappa, \lambda, \alpha, \beta, s, M, q)$.
Output: A solution $(\mathcal{M}, \mathcal{T})$ to I.

Step 1. Compute a ρ_{ST}-approximate solution T to the Steiner tree problem in (G, w) that spans $M \cup \{s\}$ and then regard T as a tree rooted at s.
Define a vertex weight function $d : M \to R^+$ by setting

$$d(v) := d_{(G,w)}(s, v), \quad v \in M.$$

Step 2. Find a partition $\mathcal{M}$ of M.
For each subset $Z \in \mathcal{M}$, assign a vertex $t_Z \in V(T)$ as its hub vertex.
Let S be the set of all hub vertices.

Step 3. For each hub vertex $t \in S$, we choose a shortest path $SP(s, t)$ between s and t in (G, w). For each subset $Z \in \mathcal{M}$, let T_Z be the tree obtained from $T\langle Z \cup \{t_Z\}\rangle$ by adding the edge set in $SP(s, t_Z)$. Let $\mathcal{T} := \{T_Z \mid Z \in \mathcal{M}\}$. $\qquad\square$

For a GCTR instance with $\lfloor\lambda/(\alpha+\beta\kappa)\rfloor = 1$, we realize Step 2 as follows. We compute a κ-balanced partition $\mathcal{M} = \{Z_1, Z_2, \ldots, Z_p\}$ of M. For $j = 1, 2, \ldots, p-1$, we choose a terminal $t_{Z_j} \in Z_j$ with the minimum distance $d(t_{Z_j})$ as its hub vertex, and let $t_{Z_p} := s$ for $j = p$.

Theorem 3.1. *Given a* GCTR *instance with* $\lfloor\lambda/(\alpha+\beta\kappa)\rfloor = 1$, *algorithm* APPROX-GCTR *with the above Step 2 delivers a* $(2\lambda/(\alpha+\beta\kappa) + \rho_{ST})$-*approximate solution.*

Proof. By Property (iii) of κ-balanced partition, each edge in T is used at most once in the union of subtrees in $T' = \{T\langle Z_j\rangle \mid j = 1, 2, \ldots, p-1\}\cup\{T\langle Z_p\cup\{s\}\rangle\}$. Furthermore, the flow on each edge in T is at most $\alpha+\beta\kappa \le \lambda$. On the other hand, the flow on each edge in $SP(s, t_{Z_i})$, $i = 1, 2, \ldots, p-1$, is at most $\alpha+\beta\kappa \le \lambda$. Note that $T' = \{T\langle Z_i\cup\{t_{Z_i}\}\rangle \mid Z_i \in \mathcal{M}\}$ by the choice of hub vertices. Therefore, $(\mathcal{M}, T)$ is feasible and the total weight of the edges to be installed for T is bounded by the weight of T plus the sum of the shortest paths used; i.e., it holds

$$\sum_{e\in E} h_T(e)w(e) \le w(T) + \sum_{1\le i\le p-1} d(t_{Z_i}). \tag{1}$$

For a minimum Steiner tree T^* that spans $M \cup \{s\}$, we have $w(T^*) \le opt(I)$ by Lemma 2.1. Hence $w(T) \le \rho_{ST}\cdot w(T^*) \le \rho_{ST}\cdot opt(I)$ holds. To prove the theorem, it suffices to show that

$$\sum_{1\le i\le p-1} d(t_{Z_i}) \le 2\lambda/(\alpha+\beta\kappa)opt(I). \tag{2}$$

The choice of hub vertices and Property (ii) of κ-balanced partition imply that, for each $i = 1, 2, \ldots, p-1$, we have

$$\sum_{v\in Z_i} q(v)d(v) \ge d(t_{Z_i})\sum_{v\in Z_i} q(v) > d(t_{Z_i})\kappa/2. \tag{3}$$

By summing inequality (3) overall $i = 1, 2, \ldots, p-1$, we have

$$(\alpha+\beta\kappa)/(2\lambda)\sum_{1\le i\le p-1} d(t_{Z_i}) < (\alpha+\beta\kappa)/(\kappa\lambda)\sum_{1\le i\le p-1}\sum_{v\in Z_i} q(v)d(v)$$
$$\le (\alpha+\beta\kappa)/(\kappa\lambda)\sum_{t\in M} q(t)d(t).$$

By Lemma 2.2, this proves (2). □

3.2 Approximation algorithm for $\lfloor\lambda/(\alpha+\beta\kappa)\rfloor \ge 2$

This section shows that APPROXGCTR with an additional step, Step 4, can deliver a $(\lfloor 2\lambda/(\alpha+\beta\kappa)\rfloor/\lfloor\lambda/(\alpha+\beta\kappa)\rfloor + \rho_{ST})$-approximate solution for a GCTR instance with $\lfloor\lambda/(\alpha+\beta\kappa)\rfloor \ge 2$. For this, we use the following result on tree covers in a tree to realize Step 2.

For a partition $\mathcal{M}$ of a terminal set M in a rooted tree T and hub vertices t_Z, $Z \in \mathcal{M}$, we denote the set of subsets $Z \in \mathcal{M}$ such that $T\langle Z\cup\{t_Z\}\rangle$ contains a specified edge $e = (x, y) \in E(T)$ with $y \in Ch_T(x)$ by three disjoint sets:

$$\mathcal{M}(e) = \{Z \in \mathcal{M} \mid e \in E(T\langle Z \rangle)\},$$
$$\mathcal{M}_{dwn}(e) = \{Z \in \mathcal{M} \mid Z \subseteq V(T) - V(T_y), t_Z \in V(T_y)\}, \text{ and}$$
$$\mathcal{M}_{up}(e) = \{Z \in \mathcal{M} \mid Z \subseteq V(T_y), t_Z \in V(T) - V(T_y)\}.$$

Lemma 3.2. *Let T be a tree rooted at s with a terminal set $M \subseteq V(T) - \{s\}$, a demand function $q : M \to R^+$, a real κ with $\kappa \geq \max\{q(v) \mid v \in M\}$, a real $\lambda > 0$, and real constants $\alpha, \beta \geq 0$. Given a vertex weight function $d : M \to R^+$, there exist a partition $\mathcal{M} = \cup_{1 \leq j \leq f} \mathcal{C}_j$ of M, and a set $S = \{t_j \in \{\mathrm{argmin}_{t \in Z \in \mathcal{C}_j} d(t)\} \mid j \leq f - 1\} \cup \{t_f = s\}$ of hub vertices such that:*

(i) *$q(Z) \leq \kappa$ for all $Z \in \mathcal{M}$, and $T\langle Z \rangle$ and $T\langle Z' \rangle$ have no common edge for all distinct $Z, Z' \in \mathcal{M}$;*

(ii) *$|\mathcal{C}_j| \leq \lfloor \lambda/(\alpha + \beta\kappa) \rfloor$ for all $j = 1, 2, \ldots, f$, and $\sum_{Z \in \mathcal{C}_j} q(Z) > \lfloor \lambda/(\alpha + \beta\kappa) \rfloor (\kappa/2)$ for all $j = 1, 2, \ldots, f - 1$; and*

(iii) *For $t_Z = t_j$ with $Z \in \mathcal{C}_j$, $j = 1, 2, \ldots, f$, each edge $e \in E(T)$ satisfies*
   (a) *$|\mathcal{M}(e)| \leq 1$,*
   (b) *$|\mathcal{M}_{dwn}(e)| \leq \lfloor \lambda/(\alpha + \beta\kappa) \rfloor - 1$, and*
   (c) *$|\mathcal{M}_{up}(e)| \leq \lfloor \lambda/(\alpha + \beta\kappa) \rfloor - 1$.* $\qquad\square$

We first perform Step 1 of ApproxGCTR. In Step 2, we apply Lemma 3.2 to the Steiner tree T and the function d obtained in Step 1 to get a partition $\mathcal{M} = \cup_{1 \leq j \leq f} \mathcal{C}_j$ of M and a set $S = \{t_1, t_2, \ldots, t_f\}$ of hub vertices that satisfy the conditions of Lemma 3.2, and we set $t_Z = t_j$ for each $Z \in \mathcal{C}_j$, $j = 1, 2, \ldots, f$. Then we perform Step 3 for the set $T' = \{T\langle Z \cup \{t_Z\}\rangle \mid Z \in \mathcal{M}\}$ of induced subtrees of T. Note that each collection $\mathcal{C}_j$, $j = 1, 2, \ldots, f$, contains at most $\lfloor \lambda/(\alpha + \beta\kappa) \rfloor$ subsets from $\mathcal{M}$, all of which can use t_j as a common hub vertex by installing one copy of each edge in $SP(s, t_j)$. We here analyze the installing cost of the resulting tree-routing. Analogously with the previous section, we have

$$\sum_{1 \leq j \leq f-1} d(t_j) \leq \lceil 2\lambda/(\alpha + \beta\kappa) \rceil / \lfloor \lambda/(\alpha + \beta\kappa) \rfloor \, opt(I),$$

since it holds by Lemma 3.2(i)-(ii) that

$$
\begin{aligned}
(\alpha + \beta\kappa)\lfloor \lambda/(\alpha + \beta\kappa) \rfloor/(2\lambda) \sum_{1 \leq j \leq f-1} d(t_j) \;&<\; (\alpha + \beta\kappa)/(\kappa\lambda) \sum_{1 \leq j \leq f-1} \sum_{t \in Z \in \mathcal{C}_j} q(t)d(t) \\
&\leq\; (\alpha + \beta\kappa)/(\kappa\lambda) \sum_{t \in M} q(t)d(t).
\end{aligned}
$$

It should be noted that the flow on an edge $e \in E(T)$ may be more than λ and (1) may not hold for the current tree-routing.

Finally we perform Step 4 in order to modify the assignment of hub vertices so that (1) holds, which implies the $(\lceil 2\lambda/(\alpha + \beta\kappa) \rceil / \lfloor \lambda/(\alpha + \beta\kappa) \rfloor + \rho_{\mathrm{ST}})$-approximability of GCTR with $\lfloor \lambda/(\alpha + \beta\kappa) \rfloor \geq 2$. Consider an edge $e = (x, y)$ in the Steiner tree T, where by definition the number of trees in T' containing e equals $|\mathcal{M}_{dwn}(e)| + |\mathcal{M}_{up}(e)| +$

$|\mathcal{M}(e)|$. Assume that the total number of trees in $\mathcal{T}'$ containing e exceeds $\lfloor \lambda/(\alpha + \beta\kappa)\rfloor$; i.e.,

$$|\mathcal{M}_{dwn}(e)| + |\mathcal{M}_{up}(e)| + |\mathcal{M}(e)| > \lfloor \lambda/(\alpha + \beta\kappa)\rfloor,$$

which implies

$$|\{T' \in \mathcal{T}' \mid e \in E(T')\}| > \lfloor \lambda/(\alpha + \beta\kappa)\rfloor.$$

Step 4 repeats a swapping process for any edge of T shared by more than $\lfloor \lambda/(\alpha + \beta\kappa)\rfloor$ trees of the current $\mathcal{T}'$. See [7] for the details of such a swapping process. Step 4 never changes the set S of hub vertices computed in Lemma 3.2.

Therefore, the set $\mathcal{T} = \{T_Z \mid Z \in \mathcal{M}\}$ of tree-routings T_Z obtained from each tree $T\langle Z \cup \{t_Z\}\rangle$ of $\mathcal{T}'$ by adding the edge set of $SP(s, t_Z)$ satisfies (1) and is a $([2\lambda/(\alpha + \beta\kappa)]/\lfloor \lambda/(\alpha + \beta\kappa)\rfloor + \rho_{\mathrm{ST}})$-approximate solution to the given GCTR instance I. Hence we have the following theorem.

Theorem 3.2. GCTR *with* $\lfloor \lambda/(\alpha + \beta\kappa)\rfloor \geq 2$ *is* $([2\lambda/(\alpha + \beta\kappa)]/\lfloor \lambda/(\alpha + \beta\kappa)\rfloor + \rho_{\mathrm{ST}})$-*approximable.* $\qquad\square$

4 Approximation algorithm for $\lambda < \alpha + \beta\kappa$

As we mentioned before, it is not straightforward to modify the algorithm in the previous section so that it also delivers a constant-factor approximate solution in the case of $\lambda < \alpha + \beta\kappa$. In this section, we introduce a new lower bound on GCTR by introducing a generalization of CND in Section 4.1, and use a balanced Steiner tree as a base tree from which we construct a collection of trees to send demands to sink. We prove an approximation algorithm of 13.037 for the problem in this case.

The following lemma introduces another lower bound to GCTR based on the Steiner tree problem which is equivalent to that given in Lemma 2.1 for a GCTR instance with $\alpha \leq \lambda$.

Lemma 4.1. *Let* $I = (G, w, \kappa, \lambda, \alpha, \beta, s, M, q)$ *be an instance of* GCTR *and* T^* *be a minimum cost Steiner tree to* $(G, w, M \cup \{s\})$. *Then* $\lceil \alpha/\lambda\rceil w(T^*)$ *is a lower bound on the optimal value to* I.

Proof. Consider an optimal solution $(\mathcal{M}^* = \{Z_1, \ldots, Z_\ell\}, T^* = \{T_1, \ldots, T_\ell\})$ to I with optimal value $opt(I)$. For each edge $e \in E(T_i)$, $i = 1, 2, \ldots, \ell$, we assume that $e = (u_i^e, v_i^e)$, where $v_i^e \in Ch_{T_i}(u_i^e)$. Let $E(\mathcal{T}^*) = \cup_{T_i \in \mathcal{T}^*} E(T_i)$ ($\subseteq E(G)$), i.e., the set of all edges used in the optimal solution. Then

$$opt(I) = \sum_{e \in E(\mathcal{T}^*)} \left\lceil \sum_{T_i: e \in E(T_i)} (\alpha + \beta q(Z_i \cap D_{T_i}(v_i^e)))/\lambda \right\rceil w(e)$$

$$\geq \lceil \alpha/\lambda\rceil \sum_{e \in E(\mathcal{T}^*)} w(e) \geq \lceil \alpha/\lambda\rceil \sum_{e \in E(T^*)} w(e),$$

since the edge set $E(\mathcal{T}^*)$ contains a tree that spans $M \cup \{s\}$ in G. $\qquad\square$

4.1 Generalized capacitated network design problem

In this section, we propose a generalized version of CND, *the generalized capacitated network design problem* (GCND), which defines a new lower bound to the optimal value of GCTR. We show that such a lower bound can be used to construct a constant factor approximation algorithm to GCTR instances with $\lambda < \alpha + \beta\kappa$. We are given a graph $G = (V, E)$ with a bulk edge capacity $\lambda > 0$, a sink $s \in V$, and a set $M \subseteq V - \{s\}$ of terminals with a nonnegative demand $q(v)$, $v \in M$. The problem asks to choose a path P_v from each terminal $v \in M$ to the sink along which the demand $q(v)$ of v is sent to s. Each edge $e \in E$ has an installation cost $w(e) \geq 0$ per bulk capacity; each edge e is allowed to have capacity $j\lambda$ for any integer j, which requires installation cost $jw(e)$. Hence, given a set $\mathcal{P} = \{P_v \mid v \in M\}$ of paths of G, each edge e in $E(\mathcal{P}) = \cup_{v \in M} E(P_v)$ needs to have capacity $k_{\mathcal{P}}(e)\lambda$ for the least integer $k_{\mathcal{P}}(e)$ such that

$$\alpha + \beta \sum_{v \in M: P_v \text{ contains } e} q(v) \leq k_{\mathcal{P}}(e)\lambda,$$

where $k_{\mathcal{P}}(e) = 0$ if no path contains e. The total installation cost of edges incurred by $\mathcal{P}$ is given as $\sum_{e \in E(\mathcal{P})} k_{\mathcal{P}}(e)w(e)$. The objective of GCND is to minimizes the total installation cost of edges. The problem is formally stated as follows.

Generalized Capacitated Network Design Problem (GCND):
Input: A connected graph $G = (V, E)$, an edge weight function $w : E \to R^+$, an edge capacity $\lambda > 0$, and prescribed constants $\alpha, \beta \geq 0$, a sink $s \in V$, a set $M \subseteq V - \{s\}$ of terminals, and a demand function $q : M \to R^+$.
Feasible solution: A set $\mathcal{P} = \{P_v \mid v \in M\}$ of paths of G such that $\{s, v\} \subseteq V(P_v)$ holds for each $v \in M$. The number of copies of an edge e in $E(\mathcal{P}) = \cup_{v \in M} E(P_v)$ installed in the solution is given by $k_{\mathcal{P}}(e) = \lceil (\alpha + \beta \sum_{v:e \in E(P_v)} q(v))/\lambda \rceil$.
Goal: Minimize the total installed cost, that is,

$$\sum_{e \in E(\mathcal{P})} k_{\mathcal{P}}(e)w(e).$$

The following lemma follows directly from the definitions of GCND and GCTR.

Theorem 4.1. *Let* $I' = (G, w, \lambda, \alpha, \beta, s, M, q)$ *and* $I = (G, w, \kappa, \lambda, \alpha, \beta, s, M, q)$ *be two instances of* GCND *and* GCTR, *respectively. Then the optimal value of* I' *is a lower bound to the optimal value of* I.

Proof. Let $opt(I)$ and $opt(I')$ denote the optimal values of I and I', respectively. Consider an optimal solution $(\mathcal{M}^* = \{Z_1, \ldots, Z_\ell\}, \mathcal{T}^* = \{T_1, \ldots, T_\ell\})$ to GCTR instance I. For each $i = 1, 2, \ldots, \ell$ and $v \in Z_i$, let P_v be the path from v to s in T_i. We observe that $\mathcal{P} = \{P_v \mid v \in M\}$ is a feasible solution to GCND instance I'. Moreover, for $E(\mathcal{P}) = \cup_{v \in M} E(P_v)$ and $E(\mathcal{T}^*) = \cup_{T_i \in \mathcal{T}^*} E(T_i)$, it hold $E(\mathcal{P}) = E(\mathcal{T}^*)$ and $k_{\mathcal{P}}(e) \leq h_{\mathcal{T}^*}(e)$. Hence, it holds

$$opt(I') \leq \sum_{e \in E(\mathcal{P})} k_{\mathcal{P}}(e)w(e) \leq \sum_{e \in E(\mathcal{T}^*)} h_{\mathcal{T}^*}(e)w(e) = opt(I).$$

$\square$

Before constructing an approximate solution to GCND, we present two lower bounds to the problem. The first lower bound is based on the Steiner tree problem, where the proof is similar to that of Lemma 2.1.

Lemma 4.2. *Given a* GCND *instance* $I' = (G, w, \lambda, \alpha, \beta, s, M, q)$, *the minimum cost of a Steiner tree that spans* $M \cup \{s\}$ *is a lower bound on the optimal value to* I'. $\square$

The second lower bound is based on a linear combination of both the Steiner tree problem and the distances from s to all terminals.

Lemma 4.3. *Let* $I' = (G, w, \lambda, \alpha, \beta, s, M, q)$ *be an instance of* GCND *and* T^* *be a minimum cost Steiner tree that spans* $M \cup \{s\}$. *Then*

$$(\alpha/\lambda) \sum_{e \in E(T^*)} w(e) + (\beta/\lambda) \sum_{v \in M} q(v) d_{(G,w)}(s, v)$$

is a lower bound on the optimal value to I'.

Proof. Consider an optimal solution $\mathcal{P} = \{P_v \mid v \in M\}$ to GCND instance I', and let $E(\mathcal{P}) = \cup_{v \in M} E(P_v)$. Let $opt(I')$ denote the optimal value to I'. Then we have

$$
\begin{aligned}
opt(I') &= \sum_{e \in E(\mathcal{P})} \lceil (\alpha + \beta \sum_{v: e \in E(P_v)} q(v))/\lambda \rceil w(e) \\
&\geq (\alpha/\lambda) \sum_{e \in E(\mathcal{P})} w(e) + (\beta/\lambda) \sum_{e \in E(\mathcal{P})} (w(e) \sum_{v: e \in E(P_v)} q(v)) \\
&= (\alpha/\lambda) \sum_{e \in E(\mathcal{P})} w(e) + (\beta/\lambda) \sum_{v \in M} (q(v) \sum_{e \in E(P_v)} w(e)) \\
&\geq (\alpha/\lambda) \sum_{e \in E(T^*)} w(e) + (\beta/\lambda) \sum_{v \in M} q(v) d_{(G,w)}(s, v),
\end{aligned}
$$

since $E(\mathcal{P})$ contains a tree that spans $M \cup \{s\}$ in G and $\sum_{e \in E(P_v)} w(e) \geq d_{(G,w)}(s, v)$ holds for all $v \in M$. $\square$

Now we construct an approximate solution to a GCND instance $I' = (G, w, \lambda, \alpha, \beta, s, M, q)$ based on a tree balanced an approximate Steiner tree and a shortest path tree in G. Let T^* and T^{ast} denote optimal and ρ_{ST}-approximate solutions to the Steiner tree problem to $(G, w, M \cup \{s\})$, respectively. This implies that $w(T^{ast}) \leq \rho_{ST} \cdot w(T^*)$. Regard T^* and T^{ast} as trees rooted at s. Let T^{spt} be a shortest path tree that spans $M \cup \{s\}$ rooted at s. Let T be a balanced Steiner tree that approximates both T^{ast} and T^{spt}. Note that T can be found in polynomial time [5, 6]. Namely, given T^{ast}, T^{spt}, and a real number $\gamma > 0$, there is a balanced Steiner tree T such that

$$w(T) \leq (1 + 2/\gamma) w(T^{ast}), \text{ and} \tag{4}$$

$$d_{(T,w)}(s, v) \leq (1 + \gamma) d_{(G,w)}(s, v), \quad \text{for all } v \in M. \tag{5}$$

Let v^e denote the tail of edges e in T. Inequalities (4) and (5) imply that

$$
\begin{aligned}
\sum_{e \in E(T)} \lceil (\alpha + \beta q(T_{v^e}))/\lambda \rceil w(e) \;\leq\;& \sum_{e \in E(T)} ((\alpha + \beta q(T_{v^e}))/\lambda + 1)w(e) \\
=\;& (\alpha/\lambda + 1)w(T) + (\beta/\lambda)\sum_{v \in M} q(v)d_{(T,w)}(s,v) \\
\leq\;& (\alpha/\lambda + 1)\rho_{\mathrm{ST}}(1 + 2/\gamma)w(T^*) \\
& + (\beta/\lambda)(1 + \gamma)\sum_{v \in M} q(v)d_{(G,w)}(s,v) \\
\leq\;& \rho_{\mathrm{ST}}(1 + 2/\gamma)w(T^*) + \max\{\rho_{\mathrm{ST}}(1 + 2/\gamma),(1 + \gamma)\} \\
& ((\alpha/\lambda)w(T^*) + (\beta/\lambda)\sum_{v \in M} q(v)d_{(G,w)}(s,v)). \qquad (6)
\end{aligned}
$$

Hence Lemmas 4.2 and 4.3 prove that the right hand side of (6) is bounded from above by

$$
\big(\rho_{\mathrm{ST}}(1 + 2/\gamma) + \max\{\rho_{\mathrm{ST}}(1 + 2/\gamma),(1 + \gamma)\}\big)opt(I'),
$$

where $opt(I')$ denotes the optimal value to I'. This proves the following theorem.

Theorem 4.2. *Let* $I' = (G,w,\lambda,\alpha,\beta,s,M,q)$ *be an instance of* GCND *with optimal value* $opt(I')$. *Then, for any* $\gamma > 0$, *there is a Steiner tree* T *that spans* $M \cup \{s\}$ *rooted at* s *such that*

$$
\sum_{e \in E(T)} \lceil (\alpha + \beta q(T_{v^e}))/\lambda \rceil w(e) \leq \mu \cdot opt(I'),
$$

where v^e *is the tail of* e *in* T *and* $\mu = \rho_{\mathrm{ST}}(1 + 2/\gamma) + \max\{\rho_{\mathrm{ST}}(1 + 2/\gamma),(1 + \gamma)\}$. *Furthermore, such a tree* T *can be computed in polynomial time.* $\qquad \square$

4.2 Approximation algorithms to GCTR

In this section we present two approximation algorithms for a GCTR instance with $\lambda < \alpha + \beta\kappa$. Our proposed algorithms are based on κ-balanced partition and the results described in Section 4.1.

Algorithm ApproxGCTR
Input: An instance $I = (G,w,\kappa,\lambda,\alpha,\beta,s,M,q)$ of GCTR.
Output: A solution $(\mathcal{M},\mathcal{T})$ to I.

Step 1. Compute a tree T that spans $M \cup \{s\}$ rooted at s.
Find a κ-balanced partition $\mathcal{M} = \{Z_1,Z_2,\ldots,Z_p\}$ of M in T.

Step 2. For each $i = 1,2,\ldots,p-1$, assign a vertex t_{Z_i} in $T\langle Z_i\rangle$ as its hub vertex and let T_{Z_i} be the tree obtained from $T\langle Z_i\rangle$ by adding the edge set of a shortest path $SP(s,t_{Z_i})$ between s and t_{Z_i} in G.
Let $t_{Z_p} := s$ and $T_{Z_p} := T\langle Z_p \cup \{s\}\rangle$.

Step 3. For each $i = 1,2,\ldots,p$,
Regard T_{Z_i} as a tree rooted at s.

Install $\lceil((\alpha + \beta q(Z_i \cap D_{T_{Z_i}}(v_i^e)))/\lambda\rceil$ copies of each edge $e \in E(T_{Z_i})$ with tail v_i^e in T_{Z_i}.

Step 4. Let $\mathcal{T} = \{T_{Z_i} \mid i = 1, 2, \ldots, p\}$ and output $(\mathcal{M}, \mathcal{T})$. $\qquad\square$

Note that the demand capacity constraint on each tree in $\mathcal{T}$ is obviously satisfied by the definition of κ-balanced partition. It is also easy to observe that the edge capacity constraint remains satisfied on each edge installed on the graph. Thereby $(\mathcal{M}, \mathcal{T})$ is feasible to I. It remains to discuss the approximation ratio of the algorithm. We consider two versions of algorithm APPROXGCTR by realizing Steps 1 and 2 in two different ways as follows.

(A) We compute a tree T in the first step by any ρ_{ST}-approximation algorithm to the Steiner tree problem, and choose $t_{Z_i} \in Z_i$, $i = 1, 2, \ldots, p - 1$, in Step 2 to be a terminal of the minimum distance $d_{(G,w)}(s, t_{Z_i})$ in Z_i, and

(B) we compute a tree T in the first step by using Theorem 4.2, and, for each $i = 1, 2, \ldots, p - 1$, we choose t_{Z_i} in Step 2 to be a vertex of the minimum depth in T.

Theorem 4.3. *For a* GCTR *instance* I *with* $\lambda < \alpha + \beta\kappa$, *algorithm* APPROXGCTR *with Steps 1 and 2 as defined in* **(A)** *delivers an approximate solution* $(\mathcal{M}, \mathcal{T})$ *with approximation ratio of* $2\xi + \min\{\lceil(\alpha + \beta\kappa)/\lambda\rceil, \lceil\beta\kappa/\lambda\rceil + 1\}\rho_{\mathrm{ST}}$, *where* $\xi = \lambda\lceil(\alpha + \beta\kappa)/\lambda\rceil/(\alpha + \beta\kappa)$. $\qquad\square$

Note that the ratio in Theorem 4.3 may not be constant due to the factor $\lceil\beta\kappa/\lambda\rceil$. We show in the next theorem that algorithm APPROXGCTR with Steps 1 and 2 as defined in **(B)** admits a constant factor approximate solution.

Theorem 4.4. *For a* GCTR *instance* I *with* $\lambda < \alpha + \beta\kappa$, *algorithm* APPROXGCTR *with Steps 1 and 2 as defined in* **(B)** *delivers an approximate solution* $(\mathcal{M}, \mathcal{T})$ *with approximation ratio of* $2\xi + 2\rho_{\mathrm{ST}} + 4\sqrt{2\xi\rho_{\mathrm{ST}}}$, *where* $\xi = \lambda\lceil(\alpha + \beta\kappa)/\lambda\rceil/(\alpha + \beta\kappa)$. $\qquad\square$

Note that the approximation ratio given in Theorem 4.4 is bounded from above by

$$(2\xi + 2\rho_{\mathrm{ST}} + 4\sqrt{2\mu\rho_{\mathrm{ST}}}) < (4 + 2\rho_{\mathrm{ST}} + 8\sqrt{\rho_{\mathrm{ST}}}) < 17.057$$

for the best known ratio $\rho_{\mathrm{ST}} = 1 + \frac{\ln 3}{2}$ to the Steiner tree problem (since $\xi < 2$).

We show that the bound can be improved by choosing the best one from both solutions constructed by using **(A)** and **(B)** in Steps 1 and 2.

Theorem 4.5. *For a* GCTR *instance* I *with* $\lambda < \alpha + \beta\kappa$, *there exists an approximate solution* $(\mathcal{M}, \mathcal{T})$ *with approximation ratio of*

$$\min\{2\xi + \lceil(\alpha + \beta\kappa)/\lambda\rceil\rho_{\mathrm{ST}}, 2\xi + 2\rho_{\mathrm{ST}} + 4\sqrt{2\xi\rho_{\mathrm{ST}}}\} \leq 13.037.$$

Proof. Let $j = \lceil(\alpha + \beta\kappa)/\lambda\rceil$. Note that $\lambda < \alpha + \beta\kappa$ implies that $j = \lceil(\alpha + \beta\kappa)/\lambda\rceil \geq 2$. Since $j - 1 < (\alpha + \beta\kappa)/\lambda \leq j$, ξ is bounded from above by

$$\xi = \lambda\lceil(\alpha + \beta\kappa)/\lambda\rceil/(\alpha + \beta\kappa) < j/(j - 1).$$

First consider the case where $\lceil(\alpha + \beta\kappa)/\lambda\rceil \leq 6$. In this case, for the best known ratio $\rho_{\text{ST}} = 1 + \frac{\ln 3}{2}$ to the Steiner tree problem, the approximation factor $2\xi + \lceil(\alpha + \beta\kappa)/\lambda\rceil\rho_{\text{ST}}$ proved in Theorem 4.3 is bounded from above by

$$2\xi + \lceil(\alpha + \beta\kappa)/\lambda\rceil\rho_{\text{ST}} \leq 11.696,$$

which is obtained when $j = \lceil(\alpha + \beta\kappa)/\lambda\rceil = 6$ (and hence $\xi < j/(j-1) = 6/5$).

Next consider the case where $\lceil(\alpha + \beta\kappa)/\lambda\rceil \geq 7$. We have $\xi < j/(j-1) \leq 7/6$ and hence the approximation factor $2\xi + 2\rho_{\text{ST}} + 4\sqrt{2\xi\rho_{\text{ST}}}$ proved in Theorem 4.4 is bounded from above by

$$2\xi + 2\rho_{\text{ST}} + 4\sqrt{2\xi\rho_{\text{ST}}} \leq 13.037$$

since $2\xi + 2\rho_{\text{ST}} + 4\sqrt{2\xi\rho_{\text{ST}}}$ is an increasing function of ξ over $[1, 2)$. This completes the proof of the theorem. $\square$

5 Approximation algorithm to FGCTR

In this section we present an approximation algorithm for a FGCTR instance by modifying the algorithm given in Section 4.2. We first introduce the following lower bound on the optimal value to FGCTR. The proof of the lemma is similar to that of Lemma 2.2.

Lemma 5.1. *Let $I = (G, w, \kappa, \alpha, \beta, s, M, q)$ be an instance of FGCTR. Then*

$$(\alpha + \beta\kappa)/\kappa \sum_{v \in M} q(v)d_{(G,w)}(s, v)$$

is a lower bound on the optimal value to I. $\square$

The fractional generalized capacitated network design problem (FGCND) is a variant of GCND in which it is allowed to purchase edge capacity in any required quantity. Namely, we assign capacity of $\lambda_e = \alpha + \beta\sum_{v:e\in E(P_v)} q(v)$ on each edge e in $E(\mathcal{P}) = \cup_{v\in M}E(P_v)$. That is, the total cost of installed capacities equals $\sum_{e\in E(\mathcal{P})} \lambda_e w(e)$. Corresponding results to that in Sections 4.1 and 4.2 can be obtained similarly.

Theorem 5.1. *Let $I' = (G, w, \alpha, \beta, s, M, q)$ and $I = (G, w, \kappa, \alpha, \beta, s, M, q)$ be two instances of FGCND and FGCTR, respectively. Then the optimal value to I' is a lower bound on the optimal value to I.* $\square$

Theorem 5.2. *Let $I' = (G, w, \alpha, \beta, s, M, q)$ be an instance of FGCND and let $opt(I')$ be the optimal value to I'. Then, for any $\gamma > 0$, there is a Steiner tree T that spans $M \cup \{s\}$ rooted at s such that*

$$\sum_{e \in E(T)} (\alpha + \beta q(T_{v^e}))w(e) \leq \max\{\rho_{\text{ST}}(1 + 2/\gamma), (1 + \gamma)\}opt(I'),$$

where v^e is the tail of e in T. $\square$

Now, we are ready to present a formal algorithm to FGCTR based on the above results.

Algorithm APPROXFGCTR
Input: An instance $I = (G, w, \kappa, \alpha, \beta, s, M, q)$ of FGCTR.
Output: A solution $(\mathcal{M}, \mathcal{T})$ to I.

Step 1. Compute a $(\max\{\rho_{ST}(1+2/\gamma), (1+\gamma)\})$-approximate Steiner tree T that spans $M \cup \{s\}$ rooted at s by Theorem 5.2.
Find a κ-balanced partition $\mathcal{M} = \{Z_1, Z_2, \ldots, Z_p\}$ of M in T.

Step 2. For each $i = 1, 2, \ldots, p - 1$, choose a vertex t_{Z_i} in $T\langle Z_i \rangle$ with the minimum depth in T and let T_{Z_i} be the tree obtained from $T\langle Z_i \rangle$ by adding the edge set of a shortest path $SP(s, t_{Z_i})$ between s and t_{Z_i} in G.
Let $t_{Z_p} := s$ and $T_{Z_p} := T\langle Z_p \cup \{s\} \rangle$.

Step 3. Let $\mathcal{T} = \{T_{Z_i} \mid i = 1, 2, \ldots, p\}$ and output $(\mathcal{M}, \mathcal{T})$. □

Theorem 5.3. *For a* FGCTR *instance* I, *algorithm* APPROXFGCTR *delivers an approximate solution* $(\mathcal{M}, \mathcal{T})$ *with approximation ratio of* 8.529. □

6 Conclusion

In this paper, we have studied the generalized capacitated tree-routing problem (GCTR), a new routing problem formulation under a multi-tree model with a general routing capacity, which unifies several important routing problems such as the capacitated network design problem (CND), the capacitated multicast tree routing problem (CMTR), and the capacitated tree-routing problem (CTR). We have proved $(2\lfloor \lambda/(\alpha + \beta\kappa) \rfloor / \lfloor \lambda/(\alpha + \beta\kappa) \rfloor + \rho_{ST})$-approximation algorithm and 13.037-approximation algorithm for GCTR with $\lambda \geq \alpha + \beta\kappa$ and $\lambda < \alpha + \beta\kappa$, respectively. We also have proved that FGCTR is 8.529-approximable. It would be interesting to design better algorithms to GCTR and FGCTR without relying on "balanced" Steiner tree.

References

[1] Z. Cai, Z.-Z. Chen, G.-H. Lin, L. Wang, An improved approximation algorithm for the capacitated multicast tree routing problem. In Proceedings of the 2nd Annual International Conference on Combinatorial Optimization and Applications (CO-COA), LNCS 5165 (2008) 286-295.

[2] M. R. Garey and D. S. Johnson, Computers and Inntractability: A Guide to the Theory of NP-Completeness, W. H. Freeman (1979).

[3] R. Hassin, R. Ravi, F. S. Salman, Approximation algorithms for a capacitated network design problem, Algorithmica, 38 (2004) 417-431.

[4] R. Jothi, B. Raghavachari, Approximation algorithms for the capacitated minimum spanning tree problem and its variants in network design, In proceedings of ICALP 2004, LNCS 3142 (2004) 805-818.

[5] S. Khuller, B. Raghavachari, N. N. Young, Balancing minimum spanning and shortest path trees, Algorithmica, 14 (1993) 305-322.

[6] G.-H. Lin and G. Xue, Balancing Steiner minimum trees and shortest-path trees in the rectilinear plane, In Proceedings of the 1999 IEEE International Symposium on Circuits and Systems (ISCAS), 6 (1999) 117-120.

[7] E. Morsy, Approximation algorithms to the capacitated tree-routings in networks, PhD Thesis (2009).

[8] E. Morsy, H. Nagamochi, An improved approximation algorithm for capacitated multicast routings in networks, Theoritical Computer Science, 390(1) (2008) 81-91.

[9] E. Morsy and H. Nagamochi, Approximating the generalized capacitated tree-routing problem, In Proceedings of the 14th Annual International Computing and Combinatorics Conference (COCOON), LNCS 5092 (2008) 621-630.

[10] E. Morsy, H. Nagamochi, Approximating capacitated tree-routings in networks, In Proceedings of the 4th Annual Conference on Theory and Applications of Models of Computation (TAMC07), LNCS 4484 (2007) 342-353.