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Representation of Ultrametric Minimum Cost Spanning Tree Games as Cost Allocation Games on Rooted Trees

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Abstract

A minimum cost spanning tree game is called ultrametric if the cost function on the edges of the underlying network is an ultrametric. We show that every ultrametric minimum cost spanning tree game is represented as a cost allocation game on a rooted tree and give an $O(n^2)$ time algorithm to find such a representation, where n is the number of players. Using the known results on the time complexity of solutions of cost allocation games on rooted trees, we then show that there exist $O(n^2)$ time algorithms for computing the Shapley value, the nucleolus and the egalitarian allocation of the ultrametric minimum cost spanning tree games.

1 Introduction

Let $N = \{1, \cdots, n\}$, where $n \geq 1$ is an integer. Suppose that K_{N_0} is the complete graph whose vertex set is $N_0 = N \cup \{0\}$ and a function w which assigns a nonnegative cost $w(e)$ to each edge e of K_{N_0} is given. A minimum cost spanning tree game (MCST game for short) is a cooperative (cost) game (N, c_w) defined as follows: for $S \subseteq N$ let $c_w(S)$ be the cost of a minimum cost spanning tree of the subgraph of K_{N_0} induced by $S \cup \{0\}$. Bird [2] showed that the core of an MCST game is always nonempty by explicitly constructing a core allocation, which is often called a Bird allocation (also see [8]).

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An ultrametric MCST game is an MCST game where the cost function w on the edges of the underlying graph is an ultrametric, i.e., for each distinct $i, j, k \in N_0$ we have

$$w(i,k) \leq \max\{w(i,j), w(j,k)\}. \tag{1}$$

An ultrametric MCST game is not only of interest in its own right but also associated with every (general) MCST games in the following way. Let (N, c_w) be an arbitrary MCST game, which may not be ultrametric. For each $i, j \in N_0$ let $\bar{w}(i,j)$ be the maximum of $w(k,l)$ over all the edges (k,l) in the path from i to j in some minimum cost spanning tree of K_{N_0}. The cost function $\bar{w}$ thus defined is known to be an ultrametric (see [19]), and conversely, each ultrametric function is derived in this way (see [17]). Bird [2] showed that the core of the MCST game (N, c_w) contains that of ultrametric MCST game $(N, c_{\bar{w}})$ associated with the cost function $\bar{w}$. Bird called the latter core the *irreducible core* and the irreducible core of an MCST game (N, c_w) and the associated game $(N, c_{\bar{w}})$ have been studied by many authors (e.g. [2], [1], [14] and [19]).

Cost allocation games on rooted trees are another class of cooperative (cost) games. Let $T = (V, A)$ be a rooted tree whose set of leaves is $N = \{1, \ldots, n\}$ and let l be a function which assigns a nonnegative length $l(a)$ to each edge a of T. For $S \subseteq N$ define $t_l(S)$ as the total length of edges that belongs to some path from a leaf $i \in S$ to the root. We call the resulting game (N, t_l) a cost allocation game on a rooted tree. This class of games is equivalent to the games studied by Megiddo [15] and the standard tree games [9] (see [12]). Any cost allocation game on a rooted tree is submodular and there exist efficient algorithms for computing solutions like the nucleolus and the egalitarian allocation for them ([15], [7], [12]).

In this paper, we show that any ultrametric MCST game can be represented as a cost allocation game on a rooted tree. It follows that for an ultrametric MCST game we can compute the Shapley value, the nucleolus and the egalitarian allocation in $\mathrm{O}(n^2)$ time. It should be noted here, in contrast, that computing solutions of a general MCST game are intractable: computing the nucleolus of the MCST games is NP-hard [5] and testing membership in the core of MCST games is co-NP-complete [4]. The computational complexities of the Shapley value and the egalitarian allocation of the MCST games are still open problems.

The rest of this paper is organized as follows. In Section 2, we give definitions from cooperative game theory and review basic results of ultrametric MCST games and cost allocation games on rooted trees. In Section 3, we show that every ultrametric can be represented by an equidistant rooted tree and give an $\mathrm{O}(n^2)$ time algorithm to find such a representation. In Section 4, we show that every ultrametric minimum cost spanning tree game is reduced to a cost allocation game on a rooted tree. Section 5 gives conclusion of this paper.

2 Preliminaries

In this section, we give definitions from cooperative game theory, and review basic results of ultrametric MCST games and cost allocation games on rooted trees.

We denote by $\mathbb{R}$ the set of real numbers and by $\mathbb{R}_+$ the set of nonnegative real numbers.

2.1 Cooperative games

A *cooperative (cost) game* (N, c) is a pair of a finite set $N = \{1, \cdots, n\}$ and a function $c: 2^N \to \mathbb{R}$ with $c(\emptyset) = 0$. We call $N = \{1, \cdots, n\}$ the set of the *players* and the function c is called the *characteristic function.* In the context of this paper, the value $c(S)$ for $S \subseteq N$ is interpreted as the total cost of some activity when only the members in S cooperate.

A cooperative game (N, c) is *subadditive* if for all $S, T \subseteq N$ with $S \cap T = \emptyset$ we have $c(S) + c(T) \geq c(S \cup T)$. Also, a game (N, c) is *submodular* (or *concave*) if for all $S, T \subseteq N$ we have $c(S) + c(T) \geq c(S \cup T) + c(S \cap T)$. The *core* of the cooperative game (N, c) is defined as follows

$$\mathrm{core}(c) = \{x \mid x \in \mathbb{R}^N, \forall S \subseteq N: x(S) \leq c(S), x(N) = c(N)\}, \tag{2}$$

where $x(S) = \sum_{i \in S} x(i)$ for $S \subseteq N$. Note that the directions of the inequalities in the usual definition of the core are reversed. The core of a submodular game is nonempty [18].

The *Shapley value* $\Phi: N \to \mathbb{R}$ of game (N, c) is defined as

$$\Phi(i) = \sum_{i \notin S \subseteq N} \frac{|S|!(n - |S| - 1)!}{n!}(c(S \cup \{i\}) - c(S)) \quad (i \in N). \tag{3}$$

If game (N, c) is submodular, the Shapley value of (N, c) is in the core.

For a vector $x \in \mathbb{R}^N$ let us denote by $\tilde{x}$ the vector in $\mathbb{R}^N$ obtained by rearranging the components of x in nondecreasing order. For vectors $\tilde{x}$ and $\tilde{y}$ in $\mathbb{R}^n$ we say $\tilde{x}$ is *lexicographically greater than* $\tilde{y}$ if there exists $k = 1, \cdots, n$ such that $\tilde{x}_i = \tilde{y}_i$ $(i = 1, \cdots, k-1)$ and $\tilde{x}_k > \tilde{y}_k$. For a submodular game (N, c) the *egalitarian allocation* is the unique vector x in the core which lexicographically maximizes $\tilde{x}$ over the core. The concept of egalitarian allocation for general cooperative games was introduced in [3] and that for concave games was studied in [6].

For a cooperative game (N, c) and a vector x such that $x(N) = c(N)$, the *excess* $e(S, x)$ of x for subset $S \subseteq N$ is defined as

$$e(S, x) = c(S) - x(S). \tag{4}$$

Given a vector x with $x(N) = c(N)$ let us denote by $\theta(x)$ the sequence of components $e(S, x)$ $(\emptyset \subset S \subset N)$ arranged in order of nondecreasing magnitude. The *nucleolus* [16] of game (N, c) is defined to be the unique vector x which lexicographically maximizes $\theta(x)$ over all the vectors x with $x(N) = c(N)$.

2.2 (Ultrametric) MCST games

All graphs we consider in this paper are simple undirected graphs (without self-loop and parallel edges). Therefore, an edge a of a graph $G = (V, A)$ is an unordered pair of distinct vertices $u, v \in V$ but we write $a = (u, v)$ instead of $a = \{u, v\}$. A graph $G = (V, A)$ is *complete* if $A = \{(u, v) \mid u, v \in V, u \neq v\}$ and we denote such a complete graph by K_V.

A graph $G = (V, A)$ is called a *tree* if it is connected and contains no cycle. For a tree $T = (V, A)$, a vertex $v \in V$ is called a *leaf* if there exists exactly one edge incident to v. For a graph $G = (V, A)$ a subgraph $H = (W, B)$ is called a *spanning tree* if $V = W$ and H is a tree. We also say that B is a spanning tree of $G = (V, A)$ if $H = (W, B)$ is a spanning tree of G.

Let K_{N_0} be the complete graph with vertex set $N_0 = \{0, 1, \cdots, n\}$ and let $w: N_0 \times N_0 \to \mathbb{R}_+$ be a function such that $w(i, i) = 0$ for all $i \in N_0$ and $w(i, j) = w(j, i)$ for all $i, j \in N_0$. We call such a pair (K_{N_0}, w) a *network*. For each subset Γ of edges of K_{N_0}, we define the *cost* $w(\Gamma)$ of Γ by

$$w(\Gamma) = \sum_{(i,j)\in\Gamma} w(i, j). \tag{5}$$

For each $S \subseteq N$ we write $S_0 = S \cup \{0\}$. The *minimum cost spanning tree game* (or *MCST game* for short) associated with network (K_{N_0}, w) is a cooperative game (N, c_w) defined by

$$c_w(S) = \min\{w(\Gamma) \mid \Gamma \text{ is a spanning tree of } \mathrm{K}_{S_0}\} \quad (S \subseteq N), \tag{6}$$

where K_{S_0} is the complete subgraph of K_{N_0} with vertex set S_0. The core of an MCST game is always nonempty. Indeed, a vector called a Bird allocation [2] is in the core (see [8]). It is easy to see that an MCST game is subadditive. However, an MCST game is not submodular in general even if w is a metric.

A function $w: N_0 \times N_0 \to \mathbb{R}_+$ is called an *ultrametric* if for each distinct $i, j, k \in N_0$ we have

$$w(i, k) \leq \max\{w(i, j), w(j, k)\}. \tag{7}$$

Equivalently, w is an ultrametric if and only if for each distinct $i, j, k \in N_0$ the maximum of $w(i, j), w(j, k), w(i, k)$ is attained by at least two pairs. An MCST game (N, c_w) is called *ultrametric* if w is an ultrametric.

In the rest of this section, we show that every ultrametric MCST game is submodular. The statement of the following lemma can be found in [2].

Lemma 2.1 *Suppose that (N, c_w) is an ultrametric MCST game associated with network (K_{N_0}, w). For $S \subseteq N$ and $i \notin S$ we have*

$$c_w(S \cup \{i\}) = c_w(S) + w(i, j^*), \tag{8}$$

where $j^* \in S_0$ is such that $w(i, j^*) = \min\{w(i, j) \mid j \in S_0\}$.

(Proof) Let Γ be a minimum cost spanning tree of K_{S_0}. It suffices to show that $\Gamma \cup \{(i, j^*)\}$ is a minimum cost spanning tree of $\mathrm{K}_{S_0 \cup \{i\}}$. For $j \in S_0$ with $j \neq j^*$ let us consider the path

$$j^* = j_0, j_1, \cdots, j_k = j \tag{9}$$

from j^* to j in Γ. By the definition of j^*, we have $w(i, j^*) \leq w(i, j)$. Then, since w is an ultrametric, we must have $w(j, j^*) \leq w(i, j)$. Since Γ is a minimum cost spanning tree of K_{S_0} we must have

$$w(j_{p-1}, j_p) \leq w(j, j^*) \quad (p = 1, \cdots, k). \tag{10}$$

Therefore, we have

$$w(j_{p-1}, j_p) \leq w(i, j) \quad (p = 1, \cdots, k). \tag{11}$$

Hence, it follows from the optimality condition of the minimum cost spanning tree [13, Theorem 6.2] that $\Gamma \cup \{(i, j^*)\}$ is a minimum cost spanning tree of $K_{S_0 \cup \{i\}}$ as required. □

Proposition 2.2 (Kuipers [14]) *Every ultrametric MCST game is submodular.*

(Proof) Supposer that (N, c_w) is an ultrametric MCST game associated with network (K_{N_0}, w). It suffices to prove that $S \subseteq T \subseteq N$ and $i \in N - T$ imply the following inequality:

$$c_w(S \cup \{i\}) - c_w(S) \geq c_w(T \cup \{i\}) - c_w(T). \tag{12}$$

However, inequality (12) follows from Lemma 2.1. □

2.3 Cost allocation game on rooted trees

Let $T = (V, A)$ be a tree with a distinguished vertex r and the set of leaves being $N = \{1, \ldots, n\}$. We call the vertex r the *root* of T and do not consider r to be a leaf. Let $l: A \to \mathbb{R}_+$ be a function on A. We call such a pair (T, l) a *rooted tree.*

Denote by A_i the set of edges on the unique path from i to r and for each $S \subseteq N$ define A_S by $A_S = \bigcup_{i \in S} A_i$. Then, the *cost allocation game* (N, t_l) on a rooted tree (T, l) is defined by

$$t_l(S) = \sum_{a \in A_S} l(a) \quad (S \subseteq N). \tag{13}$$

It is easy to see that any cost allocation game (N, t_l) on a rooted tree is submodular. Megiddo [15] showed that the Shapley value and the nucleolus of any cost allocation game on a rooted tree can be found in $\mathrm{O}(n)$ and $\mathrm{O}(n^3)$, respectively. Galil [7] improved the latter time bound to $\mathrm{O}(n \log n)$. Iwata and Zuiki [12] gave $\mathrm{O}(n \log n)$ algorithms for computing the nucleolus and the egalitarian allocation of cost allocation games on rooted trees. Summarizing, we have the following lemma.

Lemma 2.3 (Megiddo [15], Galil [7], Iwata and Zuiki [12]) *For each cost allocation game (N, t_l) on a rooted tree the Shapley value, the nucleolus and the egalitarian allocation can be computed in $\mathrm{O}(n)$, $\mathrm{O}(n \log n)$ and $\mathrm{O}(n \log n)$ time, respectively.*

3 Equidistant Representation of Ultrametrics

Let $(T = (V, A), l)$ be a rooted tree with root r and the set of leaves being M. For each pair (u, v) of vertices of T, let us denote by $d_l(u, v)$ the length of the path from u to v with respect to the function $l: A \to \mathbb{R}_+$. We call a rooted tree (T, l) *equidistant* if for all $i, j \in M$ we have $d_l(i, r) = d_l(j, r)$. A rooted tree (T, l) with the set of leaves being M is said to *represent* a function $w: M \times M \to \mathbb{R}_+$ if

$$w(i, j) = d_l(i, j) \quad (i, j \in M). \tag{14}$$

Let $(T = (V, A), l)$ be a rooted tree and let r be the root of T. The rooted tree naturally induces a partial order $\preceq$ on V: for $u, v \in V$, $v \preceq u$ if and only if u is on the unique path from v to r. If $v \preceq u$, we say that u is an *ancestor* of v and that v is a *descendant* of u. For $u, u' \in V$, v is called the *least common ancestor* if v is a common ancestor (i.e. $u \preceq v$ and $u' \preceq v$) and every common ancestor of u and u' is an ancestor of v. We denote by $\mathrm{lca}(u, u')$ the least common ancestor of u and u'.

Lemma 3.1 *Let* (K_M, w) *be a network, where* $w: M \times M \to \mathbb{R}_+$ *is an ultrametric. Suppose that* Γ *is a minimum cost spanning tree of* (K_M, w). *Then, we have*

$$w(i,j) = \max\{w(k,l) \mid (k,l) \text{ is an edge on the path from } i \text{ to } j \text{ in } \Gamma\}. \tag{15}$$

(Proof) Let

$$P : i = j_0, j_1, \cdots, j_s = j \tag{16}$$

be the path from i to j in Γ. Since w is an ultrametric, we have

$$w(i,j) \leq \max\{w(j_{p-1}, j_p) \mid p = 1, \cdots, s\}. \tag{17}$$

However, by the optimality condition of the minimum cost spanning tree [13, Theorem 6.2], we must have the equality in (17). □

Lemma 3.2 (cf. Semple and Steel [17] and Gusfield [10]) *For a function* $w: M \times M \to \mathbb{R}_+$, w *is an ultrametric if and only if there exists an equidistant rooted tree which represents* w.

(Proof) [The "if" part:] Suppose that $w: M \times M \to \mathbb{R}_+$ is represented by an equidistant rooted tree $(T = (V, A), l)$. Let $i, j, k \in M$ be distinct three elements of M. We will show the inequality (7). Since both of $\mathrm{lca}(i,j)$ and $\mathrm{lca}(j,k)$ are on the path from j to the root in $T = (V, A)$, we have $\mathrm{lca}(i,j) \preceq \mathrm{lca}(j,k)$ or $\mathrm{lca}(i,j) \succeq \mathrm{lca}(j,k)$. We only consider the former case since the other case is treated similarly. Then, since $i \preceq \mathrm{lca}(i,j) \preceq \mathrm{lca}(j,k)$ and $k \preceq \mathrm{lca}(j,k)$, we have $\mathrm{lca}(i,k) \preceq \mathrm{lca}(j,k)$. Therefore, we have

$$w(i,k) = d_l(i,k) \leq d_l(j,k) = w(j,k) = \max\{w(i,j), w(j,k)\}, \tag{18}$$

where the last equation follows from $\mathrm{lca}(i,j) \preceq \mathrm{lca}(j,k)$.

[The "only if" part:] Suppose that w is an ultrametric. We proceeds by the induction on $m = |M|$. For $m = 1, 2$ it is trivial to see that there exists an equidistant rooted tree that represents w. Let $m > 2$.

Suppose that Γ is a minimum cost spanning tree of (K_M, w) and let $(i^*, j^*) \in \Gamma$ be such that

$$w(i^*, j^*) = \max\{w(i,j) \mid (i,j) \in \Gamma\}. \tag{19}$$

Since Γ is a spanning tree, $\Gamma - \{(i^*, j^*)\}$ has exactly two connected components. Let M_1 and M_2 be the vertex sets of these components. Note that we have from Lemma 3.1 that

$$w(i,j) = w(i^*, j^*) \quad (i \in M_1, j \in M_2). \tag{20}$$

Let us denote by $w|M_p$ the restriction of w to M_p $(p = 1, 2)$. Since $|M_p| < m$, we have by the induction hypothesis that there exists an equidistant rooted tree $(T_p = (V_p, A_p), l_p)$ which represents $w|M_p$ for $p = 1, 2$.

For $p = 1, 2$, let r_p be the root of T_p $(p = 1, 2)$ and let us denote by δ_p the distance $d_{l_p}(r_p, i)$ between r_p and $i \in M_p$. Let $\hat{v}$ be a new vertex which is not in $V_1 \cup V_2$. Define a rooted tree $(T = (V, A), l)$ with root $\hat{v}$ as follows.

$$V = V_1 \cup V_2 \cup \{\hat{v}\}, \tag{21}$$

$$A = A_1 \cup A_2 \cup \{(\hat{v}, r_1), (\hat{v}, r_2)\}, \tag{22}$$

$$l(u, v) = \begin{cases} \frac{1}{2}w(i^*, j^*) - \delta_1 & \text{if } (u, v) = (\hat{v}, r_1), \\ \frac{1}{2}w(i^*, j^*) - \delta_2 & \text{if } (u, v) = (\hat{v}, r_2), \\ l_1(u, v) & \text{if } (u, v) \in A_1, \\ l_2(u, v) & \text{if } (u, v) \in A_2 \end{cases} \quad ((u, v) \in A). \tag{23}$$

By the definitions (21)–(23), $(T = (V, A), l)$ is equidistant. To see that $(T = (V, A), l)$ represents w, let $i, j \in M$. For $p = 1, 2$, if $i, j \in M_p$, then we have

$$w(i, j) = d_{l_p}(i, j) = d_l(i, j) \tag{24}$$

since (T_p, l_p) is a representation of w_p and the path from i to j in T is in T_p. If $i \in M_1$ and $j \in M_2$, we have by (20) and the definition of (T, l) that

$$w(i, j) = w(i^*, j^*) = d_l(i, j). \tag{25}$$

□

Gusfield [10] gave an algorithm for finding an equidistant rooted tree which represents an ultrametric $w: M \times M \to \mathbb{R}_+$. Heun [11] showed that a modification of Gusfield's algorithm achieves the optimal time bound $O(m^2)$, where $m = |M|$. We give an alternative time-optimal algorithm for finding an equidistant rooted tree which represents a given ultrametric. The algorithm is shown in Algorithm 1.

Algorithm 1 maintains a forest $F = (V, A)$ consists of rooted trees which is initialized to $F = (M, \emptyset)$. That is, initially there are m rooted trees. At each iteration, the algorithm merges two rooted trees into a rooted tree.

Lemma 3.3 *Let $(e_1, \cdots, e_{m-1})$ be an ordering of the edges of a minimum cost spanning tree Γ of (K_M, w) in the Algorithm 1. For $s = 0, 1, \cdots, m-1$, let $F_s = (V_s, A_s)$ be the forest obtained after the s-th iteration of the for-loop in Algorithm 1 and let us define $G_s = (M, \Gamma_s)$ by*

$$\Gamma_s = \{e_1, \cdots, e_s\}. \tag{26}$$

Then, for all $i, j \in M$ i and j are in a connected component of G_s if and only if they are leaves of a rooted tree of F_s.

(Proof) We proceed by induction on s. For $s = 0, 1$ the statement is obviously true. Let $s > 1$.

Let $e_s = (i, j)$. Let C_i and C_j be the connected components of $G_{s-1} = (M, \Gamma_{s-1})$ which contain i and j, respectively. Let M_i and M_j be the vertex sets of C_i and C_i, respectively. By the induction hypothesis, for $k = i, j$ the leave set of the rooted tree T_k of F_{s-1} containing k is M_k.

Input: An ultrametric w: $M \times M \to \mathbb{R}_+$.
Output: An equidistant rooted tree which represent w.
Find a minimum cost spanning tree Γ of (K_M, w);
Let $(e_1, \cdots, e_{m-1})$ be an ordering of the edges in Γ arranged in nondecreasing magnitude of their costs;
Let $F = (V, A)$ be the initial forest where $V = M$ and $A = \emptyset$;
for $s = 1$ **to** $m - 1$ **do**
  Let $(i, j) = e_s$;
  Find the roots r_i and r_j of the rooted trees of F containing i and j respectively;
  Let $\hat{v}$ be a new vertex which does not belong to the vertex set V of the current forest $F = (V, A)$;
  Let $V = V \cup \{\hat{v}\}$, $A = A \cup \{(r_i, \hat{v}), (r_j, \hat{v})\}$ and

$$l(r_i, \hat{v}) = \frac{1}{2}w(i, j) - d_l(r_i, i),$$
$$l(r_j, \hat{v}) = \frac{1}{2}w(i, j) - d_l(r_j, j);$$

  Let $\hat{v}$ be the root of the merged tree;
end
Output the equidistant rooted tree $(F = (V, A), l)$.

Algorithm 1: Algorithm for finding an equidistant representation of an ultrametric.

At the s-the iteration of the for-loop, the rooted trees T_i and T_j are merged into one rooted tree whose set of leaves is $M_i \cup M_j$. On the other hand, in G_s the two components containing i and j are merged into one component whose vertex set is $M_i \cup M_j$.

Connected components of G_{s-1} other than C_i and C_j are those of G_{s-1} and rooted trees of F_{s-1} other than T_i and T_j are those of G_s. This completes the proof of the present lemma. □

Theorem 3.4 *Given an ultrametric w: $M \times M \to \mathbb{R}_+$, Algorithm 1 terminates in $O(m^2)$ time and outputs an equidistant rooted tree $(T = (V, A), l)$ which represents w, where $m = |M|$.*

(Proof) First, we prove the validity of the algorithm. We proceed by induction on $m = |M|$. For $m = 1, 2$, the validity of Algorithm 1 is obvious.

Let $m > 2$ and let $(i, j) = e_{m-1}$. G_{m-2} has exactly two connected components C_i and C_j which contain i and j respectively. Let the vertex set of C_k be M_k $(k = i, j)$.

By Lemma 3.3, at the end of the $(m-2)$-th iteration of the for-loop, the forest F_{m-2} has exactly two rooted trees T_i and T_j and the sets of leaves of T_i and T_j are M_i and M_j, respectively.

Since the two connected components of G_{m-2} are minimum cost spanning trees of K_{M_i} and K_{M_j}, it follows from the induction hypothesis that $(T_i = (V_i, A_i), l|A_i)$ and $(T_j = (V_j, A_j), l|A_j)$ are representations of $w|M_i$ and $w|M_j$, respectively. Then, by the proof of the "only if" part of Lemma 3.2, after the $(m-1)$-th iteration, the finally obtained forest is an equidistant rooted tree which represents w. This completes the proof of the validity of the algorithm.

Let us consider the time complexity of the algorithm. By Prim's algorithm (see e.g. [13]), a minimum cost spanning tree Γ can be found in time $O(m^2)$ and the sorting of Γ is done in $O(m \log m)$. At each iteration of the for-loop, finding the roots takes $O(m)$ time and the other steps take $O(1)$ time. Hence, we have the claimed time bound $O(m^2)$. $\square$

4 The Reduction to Cost Allocation Games on Rooted Trees

We first show the following theorem, which is the main result of this paper.

Theorem 4.1 *For each ultrametric MCST game* (N, c_w) *there exists a cost allocation game* (N, t_l) *on a rooted tree* (T, l) *such that*

$$c_w(S) = t_l(S) \quad (S \subseteq N). \tag{27}$$

(Proof) Let (N, c_w) be an ultrametric MCST game, where $w: N_0 \times N_0 \to \mathbb{R}_+$ is an ultrametric. By Lemma 3.2, there exists an equidistant rooted tree $(T' = (V', A'), l')$ which represents w where the set of leaves of T' is N_0. Define $l: A' \to \mathbb{R}_+$ by

$$l(u, v) = \begin{cases} 0 & \text{if } (u, v) \text{ is on the path from 0 to the root,} \\ 2l'(u, v) & \text{otherwise} \end{cases} \quad ((u, v) \in A') \tag{28}$$

and let us consider the rooted tree (T', l).

It suffices to show that

$$c_w(S) = t_l(S_0) \quad (S \subseteq N) \tag{29}$$

since the desired rooted tree (T, l) can be derived by contracting all the edges on the path from 0 to the root of T', where we let the newly created vertex be the root of T, provided that we have (29).

We prove (29) by induction on $|S|$. For $S = \emptyset$ this is trivial. If $S = \{i\}$ for some $i \in N$, then we have

$$t_l(S_0) = d_{l'}(i, 0) = w(i, 0) = c_w(S) \tag{30}$$

since (T', l') represents w and (T', l') is equidistant.

Let $1 \le |S| < n$ and $i \in N - S$. We will show $c_w(S \cup \{i\}) = t_l((S \cup \{i\})_0)$. Let $j^* \in S_0$ be such that

$$w(i, j^*) = \min\{w(i, j) \mid j \in S_0\} \tag{31}$$

and let $v^* \in V$ be the least common ancestor of i and j^* in T'. Let

$$P : i = v_0, a_1, v_1, a_2, \cdots, v_{k-1}, a_k, v_k = v^* \tag{32}$$

be the path from i to v^* in T'. Then, we have

$$w(i, j^*) = d_{l'}(i, j^*) = d_l(i, v^*) = \sum_{p=1}^{k} l(a_p) \tag{33}$$

since (T', l') represents w and (T', l') is equidistant.

Claim. For all $p = 1, \cdots, k$, if $a_p \in A_{S_0}$, then we have $l(a_p) = 0$.
(Proof) Suppose that $a_p \in A_{S_0}$ and $l(a_p) > 0$ for some $p = 1, \cdots, k$. Since $a_p \in A_{S_0}$, vertex v_{p-1} is a common ancestor of i and some $\hat{j} \in S_0$. Then, since $l(a_p) > 0$ we must have $w(i, \hat{j}) < w(i, j^*)$, which contradicts the choice (31) of j^*. (End of the proof of the Claim)

It follows from the Claim, the induction hypothesis and Lemma 2.1 that

$$\begin{aligned} t_l((S \cup \{i\})_0) &= \sum_{a \in A_{S_0}} l(a) + \sum_{p=1}^{k} l(a_p) && (34) \\ &= t_l(S_0) + d_l(i, v^*) && (35) \\ &= c_w(S) + w(i, j^*) && (36) \\ &= c_w(S \cup \{i\}), && (37) \end{aligned}$$

which completes the proof of the present theorem. □

We have the following corollary from Theorem 4.1.

Corollary 4.2 *For any ultrametric MCST game the Shapley value, the nucleolus and the egalitarian allocation can be computed in* $O(n^2)$ *time.*

(Proof) By Lemma 3.4, we can construct the equidistant tree (T', l') which represents w in $O(n^2)$ time. Then, by Lemma 2.3, the Shapley value, the nucleolus and the egalitarian allocation of the game (N, t_l) can be found in time dominated by $O(n^2)$. Therefore, we have $O(n^2)$ time bound for computations of all these solutions. □

We have seen that any ultrametric MCST game can be represented as a cost allocation game on a rooted tree (T, l). The rooted tree (T, l) can be derived from an equidistant rooted tree (T', l') by compressing the path from 0 to the root. We call such a rooted tree nearly equidistant. More precisely, a rooted tree (T, l) is called *nearly equidistant* if for each immediate descendant v of the root of T, the subtree rooted at v is equidistant. Note that an equidistant rooted tree is nearly equidistant.

Theorem 4.3 *For each ultrametric MCST game* (N, c_w) *there exists a cost allocation game* (N, t_l) *on a nearly equidistant rooted tree* (T, l) *such that* $c_w = t_l$. *Conversely, for each cost allocation game* (N, t_l) *on a nearly equidistant rooted tree* (T, l), *there exists an ultrametric MCST game* (N, c_w) *such that* $c_w = t_l$.

(Proof) The first statement follows from Theorem 4.1.

We prove the second statement. Let $(T = (V, A), l)$ be a nearly equidistant rooted tree whose set of leaves is N. Let v_p $(p = 0, 1, \cdots, k)$ be the immediate descendants of the root r and let T_p be the equidistant subtree rooted at v_p $(p = 0, 1, \cdots, k)$. For each $p = 0, 1, \cdots, k$ let us denote by δ_p the distance $d_l(i, r)$ from a leaf i of T_p to the root r. We can assume without loss of generality that $\delta_0 \geq \delta_1 \geq \cdots \geq \delta_k$.

Suppose that $\{r_1, \cdots, r_k, 0\}$ is a set of new vertices such that $\{r_1, \cdots, r_k, 0\} \cap V = \emptyset$. Define a rooted tree $(T' = (V', A'), l')$ as follows.

$$V' = V \cup \{r_1, \cdots, r_k, 0\}, \tag{38}$$

$$A' = (A - \{(v_p, r) \mid p = 1, \cdots, k\}) \cup \{(v_p, r_p) \mid p = 1, \cdots, k\} \cup\{(r_p, r_{p-1}) \mid p = 2, \cdots, k\} \cup \{(r_1, r), (0, r_k)\}, \tag{39}$$

$$l'(a) = \begin{cases} l(v_p, r) & \text{if } a = (v_p, r_p) \text{ for some } p = 1, \cdots, k, \\ \delta_0 - \delta_1 & \text{if } a = (r_1, r), \\ \delta_{p-1} - \delta_p & \text{if } a = (r_p, r_{p-1}) \text{ for some } p = 2, \cdots, k, \\ \delta_k & \text{if } a = (0, r_k), \\ l(a) & \text{otherwise} \end{cases} \quad (a \in A'). \tag{40}$$

It is easy to see that rooted tree (T', l') is equidistant, and hence, it follows from Lemma 3.2 that there exists an ultrametric $w: N_0 \times N_0 \to \mathbb{R}_+$ which is represented by (T', l'). The construction of (T, l) in the proof of Theorem 4.1 shows that we have $c_w = t_l$. □

5 Conclusion

We showed that any ultrametric MCST game can be represented as a cost allocation game on a rooted tree and gave an $O(n^2)$ time algorithm to find such a representation, where n is the number of players. Using this representation theorem together with complexity results on the solutions of cost allocation games on rooted trees, we showed that the Shapley value, the egalitarian allocation and the nucleolus of an ultrametric MCST game can be computed in time $O(n^2)$.

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