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ON THE SPECTRAL PROPERTIES OF SOME CLASSES OF OPERATORS

PIETRO AIENA

ABSTRACT. This article concerns the spectral theory of many classes of operators defined by means of some inequalities. Particular emphasis is given to the Fredholm theory and local spectral theory of these classes of operators.

1. INTRODUCTORY REMARKS

It is well-known that the class of normal operators on Hilbert spaces possesses a very nice spectral theory. Many classes of operators defined on Hilbert spaces, and studied more recently, are defined by means of some (order) inequalities. These inequalities may be though obtained by relaxing the condition of normality. In this expository article we shall consider some of the spectral properties of these classes of operators, showing that these operators share with the normal operators on Hilbert spaces, many spectral properties, mostly of them concerning Fredholm theory and local spectral theory. More precisely, our main interest concerns the isolated points of the spectra of these operators, as well as the isolated points of the approximate point spectra. These properties lead to the concept of polaroid operator, and this concept together with SVEP, an important property in local spectral theory, produce a general framework from which we can state that the several versions of Weyl type theorems, in the classical form or in the generalized form, hold for all these operators.

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2. LOCAL SPECTRAL THEORY

For many reasons the most satisfactory generalization to the general Banach space setting of the normal operators on a Hilbert space is the concept of decomposable operator. In fact the class of this operators possesses a spectral theorem and a rich lattice structure for which it is possible to develop what it is called a *local spectral theory*, i.e. a local analysis of their spectra. Decomposability may be defined in several ways, for instance by means of the concept of *glocal spectral subspace.* For an arbitrary bounded linear operator on a Banach space $T \in L(X)$ and a closed subset F of $\mathbb{C}$ the glocal spectral subspace $\mathcal{X}_T(F)$ defined as the set of all $x \in X$ such that there is an analytic X-valued function $f : \mathbb{C} \setminus F \to X$ for which $(\lambda I - T)f(\lambda) = x$ for all $\lambda \in \mathbb{C} \setminus F$. A bounded operator $T \in L(X)$ is said to have the *Dunford property* (C) if every glocal spectral subspace is closed. A bounded operator T is said to be *decomposable* if T has property both properties (C) and *property* (δ), where the last property means that for every open covering (U,V) of $\mathbb{C}$ we have $X = \mathcal{X}_T(\overline{U}) + \mathcal{X}_T(\overline{V})$. Standard examples of decomposable operators are normal operators on Hilbert spaces and operators which have totally disconnected spectra, as for instance compact operators. An other important class of decomposable operator is defined as follows (see [41] for details):

Definition 2.1. *An operator $T \in L(X)$, X a Banach space, is said to be* generalized scalar *if there exists a continuous algebra homomorphism $\Psi : C^\infty(\mathbb{C}) \to L(X)$ such that $\Psi(1) = I$ and $\Psi(Z) = T$, where $C^\infty(\mathbb{C})$ denote the Fréchet algebra of all infinitely differentiable complex-valued functions on $\mathbb{C}$, and Z denotes the identity function on $\mathbb{C}$.*

Two important properties in local spectral theory related to property (C) are the so-called *property* (β) and the *single valued extension property.* Property (β) has been introduced by Bishop ([17]) and is defined as follows. Let U be an open subset of $\mathbb{C}$ and denote by $\mathcal{H}(U,X)$ the Fréchet space of all analytic functions $f : U \to X$ with respect the pointwise vector space operations and the topology of locally uniform convergence. $T \in L(X)$ has Bishop's *property* (β) if for every open $U \subseteq \mathbb{C}$ and every sequence $(f_n) \subseteq \mathcal{H}(U,X)$ for which $(\lambda I - T)f_n(\lambda)$ converges to 0 uniformly on every compact subset of U, then also $f_n \to 0$ in $\mathcal{H}(U,X)$. Let T' denote the dual of T. Property (β) and property (δ) are dual each other, i.e. $T \in L(X)$ satisfies (β) (respectively (δ)) if and only if T' satisfies (δ) (respectively, (β)), see [41]. Examples of operators satisfying property (β) but not decomposable may be found among multipliers of semi-simple commutative Banach algebras, see [41]. Other examples will be given in this note.

An operator $T \in L(X)$ is said to have *the single valued extension property* at $\lambda_0 \in \mathbb{C}$ (abbreviated SVEP at λ_0), if for every open neighborhood U of λ_0, the only analytic function $f : U \to X$ which satisfies the equation $(\lambda I - T)f(\lambda) = 0$ for all $\lambda \in U$ is the function $f \equiv 0$. The operator T is said to have SVEP if it has SVEP at every $\lambda \in \mathbb{C}$.

Two important subspaces in local spectral theory, as well as in Fredholm theory, are $\mathcal{X}_T(\{\lambda\})$, the glocal spectral subspace associated with the singleton set $\{\lambda\}$ and $\mathcal{X}_T(\mathbb{C} \setminus \{\lambda\})$. The subspace $\mathcal{X}_T(\{\lambda\})$ coincides with the *quasi-nilpotent part* of $\lambda I - T$, defined as

$$H(\lambda I - T) := \{x \in X : \lim_{n\to\infty} \|(\lambda I - T)^n x\|^{\frac{1}{n}} = 0\},$$

while $\mathcal{X}_T(\mathbb{C} \setminus \{\lambda\})$ coincides with the *analytic core* defined as the set $K(\lambda I - T)$ of all $x \in X$ such that there exist $c > 0$ and a sequence (x_n) in X for which

$$(\lambda I - T)x_1 = x, (\lambda I - T)x_{n+1} = x_n \text{ and } \|x_n\| \le c^n\|x\| \text{ for all } n \in \mathbb{N},$$

see [1]. Note that $H(\lambda I - T)$ and $K(\lambda I - T)$ are in general not closed. Moreover, $H_0(\lambda I - T)$ contains the kernels $\ker(\lambda I - T)^n$ for all $n \in \mathbb{N}$, while $K(\lambda I - T) \subseteq (\lambda I - T)^n(X)$ for all $n \in \mathbb{N}$ and $(\lambda I - T)(K(\lambda I - T) = K(\lambda I - T)$. We also have

$$H(\lambda I - T) \text{ closed} \Rightarrow T \text{ has SVEP at } \lambda. \tag{1}$$

Definition 2.2. *A bounded operator $T \in L(X)$ is said to have* property (Q) *if $\mathcal{X}_T(\{\lambda\}) = H(\lambda I - T)$ is closed for all $\lambda \in \mathbb{C}$.*

We have,

$$\text{property } (\beta) \Rightarrow \text{property } (C) \Rightarrow \text{property } (Q) \Rightarrow \text{SVEP},$$

see [41]. Although the condition (Q) seems to be rather strong, the class of operators having property (Q) is considerably large. A first example of operators which satisfy this property is given by convolution operators of the group algebra $L_1(G)$, G an Abelian locally compact group, see Theorem 1.8 of [5]. Another example is given by *transaloid* operators on Banach spaces, i.e. the operators for which the spectral radius $r(\lambda I - T)$ is equal to $\|\lambda I - T\|$ for every $\lambda \in \mathbb{C}$, see [25].

The following class of operators has been introduced by Oudghiri [45].

Definition 2.3. *A bounded operator $T \in L(X)$ is said to belong to the class $H(p)$ if there exists a natural $p := p(\lambda)$ such that:*

$$H_0(\lambda I - T) = \ker(\lambda I - T)^p \quad \textit{for all } \lambda \in \mathbb{C}. \tag{2}$$

Clearly, every operator T which belongs to the class $H(p)$ has property (Q). In the case that $p = p(\lambda) = 1$ for every $\lambda \in \mathbb{C}$ we shall say that T belongs to the class $H(1)$. Every convolution operator of the group algebra $L^1(G)$ is $H(1)$. In the sequel we see that other important classes of operators are $H(1)$ [9].

In the sequel, if $T \in L(H)$, H a Hilbert space, we denote by T^* the adjoint of T.

(a) **Paranormal operators.** Recall that $T \in L(X)$ is said *paranormal* if $\|Tx\| \leq \|T^2x\|\|x\|$ for all $x \in X$. The property of being paranormal is not translation-invariant. $T \in L(X)$ is called *totally paranormal* if $\lambda I - T$ is paranormal for all $\lambda \in \mathbb{C}$. Every totally paranormal operator has property $H(1)$ ([40]). In fact, if $x \in H_0(\lambda I - T)$ then $\|(\lambda I - T)^n x\|^{1/n} \to 0$ and since T is totally paranormal then $(\lambda I - T)^n x\|^{1/n} \geq \|(\lambda I - T)x\|$. Therefore, $H_0(\lambda I - T) \subseteq \ker(\lambda I - T)$ and since the reverse inclusion holds for every operator then $H_0(\lambda I - T) = \ker(\lambda I - T)$. Recall every paranormal operator T is *normaloid*, i.e. $\|T\|$ is equal to the spectral radius of T.

Clearly, every totally paranormal operator has property (Q). We can say much more: every totally paranormal operator has property (C), see for instance Proposition 4.14 of [40], but it is not known if every totally paranormal operator has property (β).

(b) **Hyponormal operators.** A bounded operator $T \in L(H)$ on a Hilbert space is said to be *hyponormal* if $\|T^*x\| \leq \|Tx\|$ for all $x \in H$, or equivalently $T^*T \geq TT^*$. It is easily seen that every hyponormal operator is totally paranormal, hence $H(1)$. The class of totally paranormal operators includes also subnormal operators and quasi-normal operators, since these operators are hyponormal, see [24] or [33].

Two operators $T \in L(X)$, $S \in L(Y)$, X and Y Banach spaces, are said to be *intertwined* by $A \in L(X, Y)$ if $SA = AT$; and A is said to be a *quasi-affinity* if it has a trivial kernel and dense range. If T and S are intertwined by a quasi-affinity then T is called a *quasi-affine transform* of S, and we write $T \prec S$. If both $T \prec S$ and $S \prec T$ hold then T and S are said to be *quasi-similar*.
The next result shows that property $H(1)$ is preserved by quasi-affine transforms.

Theorem 2.4. *Suppose that $S \in L(Y)$ has property $H(1)$ and $T \prec S$. Then T has property $H(1)$. Analogously, if $S \in L(Y)$ has property $H(p)$ and $T \prec S$, then T has property $H(p)$*

Proof. Suppose S has property $H(1)$, $SA = AT$, with A injective. If $\lambda \in \mathbb{C}$ and $x \in H_0(\lambda I - T)$ then

$$\|(\lambda I - S)^n Ax\|^{1/n} = \|A(\lambda I - T)^n x\|^{1/n} \leq \|A\|^{1/n}\|(\lambda I - T)^n x\|^{1/n},$$

from which it follows that $Ax \in H_0(\lambda I - S) = \ker\,(\lambda I - S)$. Hence $A(\lambda I - T)x = (\lambda I - S)Ax = 0$ and, since A is injective, this implies that $(\lambda I - T)x = 0$, i.e. $x \in \ker\,(\lambda I - T)$. Therefore $H_0(\lambda I - T) = \ker\,(\lambda I - T)$ for all $\lambda \in \mathbb{C}$.
The more general case of $H(p)$-operators is proved by a similar argument. ∎

For $T \in L(H)$ let $T = W|T|$ be the polar decomposition of T. Then $R := |T|^{1/2}W|T|^{1/2}$ is said the *Aluthge transform* of T. If $R = V|R|$ is the polar decomposition of R, define $\widetilde{T} := |R|^{1/2}V|R|^{1/2}$.

(c) **Log-hyponormal operators.** An operator $T \in L(H)$ is said to be be *log-hyponormal* if T is invertible and satisfies $\log\,(T^*T) \geq \log\,(TT^*)$. If T is log-hyponormal then $\widetilde{T}$ is hyponormal and $T = K\widetilde{T}K^{-1}$, where $K := |R|^{1/2}|T|^{1/2}$, see ([50], [21]). Hence T is similar to a hyponormal operator and therefore, by Theorem 2.4, has property $H(1)$.

(d) **p-hyponormal operators.** An operator $T \in L(H)$ is said to be be *p-hyponormal,* with $0 < p \leq 1$, if $(T^*T)^p \geq (TT^*)^p$. Every p-hyponormal operator is paranormal, see [10] or [19]. In [20] M. Chō and J. I. Lee have given an example of p-hyponormal operator which is not translation-invariant. Every invertible p-hyponormal T is quasi-similar to a log-hyponormal operator and consequently, by Theorem 2.4, it has property $H(1)$(([12], [27]). This is also true for p-hyponormal operators which are not invertible, see [28]. It is well known that p-hyponormal operators have property (β), see [18].

Theorem 2.5. [45] *For a bounded operator $T \in L(X)$ the following assertions are equivalent:*

(i) *T has the property $H(p)$;*

(ii) *$f(T)$ has the property $H(p)$ for every $f \in \mathcal{H}(\sigma(T))$;*

(iii) *There exists an analytic function h defined in an open neighborhood $\mathcal{U}$ of $\sigma(T)$, non identically constant in any component of $\mathcal{U}$, such that $h(T)$ has the property $H(p)$.*

An obvious consequence of Theorem 2.5 is that $T \in L(H)$ is *algebrically hyponormal* (i.e. there exists a non-trivial polynomial h for which $h(T)$ is hyponormal) then T is $H(p)$. In [33, §2.72] is given an example of a hyponormal operator T for which T^2 is not hyponormal. However, an important consequence of Theorem 2.5 is that T^2 inherits from T the property of being $H(1)$.

Lemma 2.6. *Let $T \in L(X)$ be a bounded operator on a Banach space X. If T has the property $H(p)$ and Y is a closed T-invariant subspace of X then $T|Y$ has the property $H(p)$.*

Proof. If $H_0(\lambda I - T) = \ker(\lambda I - T)^p$ then

$$H_0((\lambda I - T)|Y) \subseteq \ker(\lambda I - T)^p \cap Y = \ker((\lambda I - T)|Y)^p,$$

from which we obtain $H_0((\lambda I - T)|Y) = \ker((\lambda I - T)|Y)^p$. ∎

An operator similar to a restriction of a generalized scalar operator to one of its closed invariant subspaces is called *subscalar*. The interested reader can find a well organized study of these operators in the Laursen and Neumann book [41]. Note that every quasi-nilpotent generalized scalar operator is nilpotent, [41, Proposition 1.5.10].

Theorem 2.7. *Every subscalar operator $T \in L(X)$ is $H(p)$.*

Proof. By Lemma 2.6 and Theorem 2.4 we may assume that T is generalized scalar. Consider a continuous algebra homomorphism $\Psi : \mathcal{C}^\infty(\mathbb{C}) \to L(X)$ such that $\Psi(1) = I$ and $\Psi(Z) = T$. Let $\lambda \in \mathbb{C}$. Since every generalized scalar operator is decomposable and hence has the property (C), then $H_0(\lambda I - T) = \mathcal{X}_T(\{\lambda\})$ is closed. On the other hand, if $f \in \mathcal{C}^\infty(\mathbb{C})$ then $\Psi(f)(H_0(\lambda I - T)) \subseteq H_0(\lambda I - T)$ because $T = \Psi(Z)$ commutes with $\Psi(f)$. Define $\tilde{\Psi} : \mathcal{C}^\infty(\mathbb{C}) \to L(H_0(\lambda I - T))$ by

$$\tilde{\Psi}(f) = \Psi(f)|H_0(\lambda I - T) \quad \text{for every } f \in \mathcal{C}^\infty(\mathbb{C}).$$

Clearly, $T|H_0(\lambda I - T)$ is generalized scalar and quasi-nilpotent, so it is nilpotent. Thus there exists $p \geq 1$ for which $H_0(\lambda I - T) = \ker(\lambda I - T)^p$. ∎

Therefore, we have

$$\text{subscalarity} \Rightarrow \text{property } H(p) \Rightarrow \text{property } (Q) \Rightarrow \text{SVEP}.$$

Classical example of subscalar operators are hyponormal operators. Theorem 2.7 implies that some other important classes of operators are $H(p)$.

(e) **M-hyponormal operators.** Recall that $T \in L(H)$ is said to be *M-hyponormal* if there exists $M > 0$ such that $TT^* \leq MT^*T$. Every M-hyponormal operator is subscalar ([41, Proposition 2.4.9]) and hence $H(p)$.

(f) **w-hyponormal operators.** If $T \in L(H)$ and $T = U|T|$ is the polar decomposition, define $\hat{T} := |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$. $T \in L(H)$ is said to be *w-hyponormal* if $|\hat{T}| \geq |T| \geq |\hat{T}^*|$. Examples of w-hyponormal operators are p-hyponormal operators and log-hyponormal operators. All w-hyponormal operators are subscalar (together with its Aluthge transformation, see [44]), and hence $H(p)$. In [37, Theorem 2.5] it is shown that for every isolated point λ of the spectrum of a w-hyponormal operator T we have $H_0(\lambda I - T) = \ker(\lambda I - T)$ and hence λ is a simple pole of the resolvent.

(g) **p-quasihyponormal operators.** A Hilbert space operator $T \in L(H)$ is said to be *p-quasihyponormal* for some $0 < p \leq 1$ if

$$T^*|T^*|^{2p}T \leq T * |T|^{2p}T.$$

Every p-quasi-hyponormal is paranormal [42].

Let us denote by $p_* - QH$ the class of all p-quasihyponormal operators T for which $\ker T \subseteq \ker T*$. Decompose T into its normal and pure parts $T = T_n \oplus T_p$, with respect to some decomposition $H = H_n \oplus H_p$. Since non-zero eigenvalues of T are normal, see [51, Lemma 3], then T_p has no eigenvalues from which it follows that $\ker(\lambda I - T) \subseteq \ker(\overline{\lambda} I - T^*)$ and $p(\lambda I - T) \leq 1$ for all $\lambda \in \mathbb{C}$. The following result is due to Duggal and Jeon ([30, Theorem 2.2 and Theorem 2.12]).

Theorem 2.8. *Every $p_* - QH$ operator is $H(1)$ and has property (β)*

Proof. Let $T = T_n \oplus T_p$ be the decomposition of T into its normal and pure parts, where $H = H_n \oplus H_p$, $T_n = T|H_n$ and $T_p = T|H_p$. Clearly, $H_0(\lambda I - T) = H_0(\lambda I - T_n) \oplus H_0(\lambda I - T_p)$ and T_n is $H(1)$, since is normal, hence $H_0(\lambda I - T_n) = \ker(\lambda I - T_n)$. Let $x \in H_0(\lambda I - T)$ and write $= x = x_1 \oplus x_2$ with $x_1 \in H_0(\lambda I - T_n)$ and $x_2 \in H_0(\lambda I - T_p)$. If $A_p := U_p|T_p|$ is the polar decomposition of T_p, then $|T_p|T_p = B_p|T_p|$ where $B_p := |T_p|U_p$ and since T_p is injective then $|T_p|$ is a quasi-affinity. The operator B_p is p-hyponormal, hence $H(1)$ from which we conclude, by Theorem 2.4, that T_p is $H(1)$, i.e $H_(\lambda I - T_p) = \ker(\lambda I - T_p)$. Therefore, $x \in \ker(\lambda I - T_n) \oplus \ker(\lambda I - T_p) = \ker(\lambda I - T)$, thus T is $H(1)$.

To show property (β), observe first that in the decomposition $T = T_n \oplus T_p$ the property (β) for T is equivalent to the property (β) for T_p. As observed B_p is p-hyponormal, hence satisfies property (β). Since $|A_p|$ and U_p are injective, by [18, Theorem 5] then $A_p = U_p|A_p|$ has property (β). ∎

3. Polaroid operators

It is well-known that every isolated point of the spectrum of a normal operator on a Hilbert space is a (simple) pole of the resolvent. In this section we extend this property to some other classes of operators.

Recall that the *ascent* of an operator $T \in L(X)$ is defined as the smallest non-negative integer $p := p(T)$ such that $\ker T^p = \ker T^{p+1}$. If such integer does not exist we put $p(T) = \infty$. Analogously, the *descent* of T is defined as the smallest non-negative integer $q := q(T)$ such that $T^q(X) = T^{q+1}(X)$, and if such integer does not exist we put $q(T) = \infty$. It is well-known that if $p(T)$ and $q(T)$ are both finite then $p(T) = q(T)$, see [1, Theorem 3.3]. Moreover, if $\lambda \in \mathbb{C}$ then $0 < p(\lambda I - T) = q(\lambda I - T) < \infty$ if and only if λ is a pole of the resolvent of T. In this case λ is an eigenvalue of T and an isolated point of the spectrum $\sigma(T)$, see [38, Prop. 50.2]. We also have (see [1, Theorem 3.8])

$$p(\lambda I - T) < \infty \Rightarrow T \text{ has SVEP at } \lambda, \tag{3}$$

and dually

$$q(\lambda I - T) < \infty \Rightarrow T^* \text{ has SVEP at } \lambda. \tag{4}$$

The converse of these implications holds if $\lambda I - T$ is semi-Fredholm.

Definition 3.1. *An operator $T \in L(X)$ is said to be* polaroid *if every isolated point of the spectrum $\sigma(T)$ is a pole of the resolvent of T.*

Note that

$$T \text{ is polaroid} \Leftrightarrow T' \text{ is polaroid},$$

and in the case of Hilbert space operators

$$T \text{ is polaroid} \Leftrightarrow T^* \text{ is polaroid},$$

see [3]. In the sequel by $\operatorname{iso} K$ we denote the set of all isolated points of $K \subseteq \mathbb{C}$. The condition of being polaroid may be characterized by means of the quasi-nilpotent part:

Theorem 3.2. *$T \in L(X)$ is polaroid if and only if there exists $p := p(\lambda I - T) \in \mathbb{N}$ such that*

$$H_0(\lambda I - T) = \ker(\lambda I - T)^p \quad \textit{for all } \lambda \in \textit{iso}\,\sigma(T). \tag{5}$$

Proof. Suppose T satisfies (5) and that λ is an isolated point of $\sigma(T)$. Since λ is isolated in $\sigma(T)$ then, by [1, Theorem 3.74],

$$X = H_0(\lambda I - T) \oplus K(\lambda I - T) = \ker(\lambda I - T)^p \oplus K(\lambda I - T),$$

from which we obtain

$$(\lambda I - T)^p(X) = (\lambda I - T)^p(K(\lambda I - T)) = K(\lambda I - T).$$

So $X = \ker(\lambda I - T)^p \oplus (\lambda I - T)^p(X)$, which implies, by [1, Theorem 3.6], that $p(\lambda I - T) = q(\lambda I - T) \le p$, hence λ is a pole of the resolvent, so that T is polaroid.

Conversely, suppose that T is polaroid and λ is an isolated point of $\sigma(T)$. Then λ is a pole, and if p is its order then $H_0(\lambda I - T) = \ker(\lambda I - T)^p$, see Theorem 3.74 of [1]. ∎

Let $\mathcal{H}_{nc}(\sigma(T))$ denote the set of all analytic functions, defined on an open neighborhood of $\sigma(T)$, such that f is non constant on each of the components of its domain. Define, by the classical functional calculus, $f(T)$ for every $f \in \mathcal{H}_{nc}(\sigma(T))$.

Theorem 3.3. *Let $f \in \mathcal{H}_{nc}(\sigma(T))$. If T is polaroid then $f(T)$ is polaroid.*

Proof. Let $\lambda_0 \in \operatorname{iso}\sigma(f(T))$. The spectral mapping theorem implies $\lambda_0 \in \operatorname{iso} f(\sigma(T))$. Let us show that $\lambda_0 \in f(\operatorname{iso}\sigma(T))$.

Select $\mu_0 \in \sigma(T)$ such that $f(\mu_0) = \lambda_0$. Denote by Ω the connected component of the domain of f which contains μ_0 and suppose that μ_0 is not isolated in $\sigma(T)$. Then there exists a sequence $(\mu_n) \subset \sigma(T) \cap \Omega$ of distinct scalars such that $\mu_n \to \mu_0$. Since $K := \{\mu_0, \mu_1, \mu_2, \dots\}$ is a compact subset of Ω, the principle of isolated zeros of analytic functions says to us that f may assume the value $\lambda_0 = f(\mu_0)$ only a finite number of points of K; so for n sufficiently large $f(\mu_n) \neq f(\mu_0) = \lambda_0$, and since $f(\mu_n) \to f(\mu_0) = \lambda_0$ it then follows that λ_0 is not an isolated point of $f(\sigma(T))$, a contradiction. Hence $\lambda_0 = f(\mu_0)$, with $\mu_0 \in \operatorname{iso}\sigma(T)$. Since T is polaroid, μ_0 is a pole of T and by [8, Theorem 2.9]; hence λ_0 is a pole for $f(T)$, which proves that $f(T)$ is polaroid. ∎

In the sequel the part of an operator T means the restriction of T to a closed T-invariant subspace.

Definition 3.4. *An operator $T \in L(X)$ is said to be* hereditarily polaroid *if every part of T is polaroid.*

It is easily seen that the property of being hereditarily polaroid is similarity invariant, but is not preserved by a quasi-affinity. Every hereditarily polaroid operator has SVEP, see [31, Theorem 2.8]

Corollary 3.5. *Every $H(p)$ operator T is hereditarily polaroid. If T is $H(1)$ then every isolated point of the spectrum is a simple pole of the resolvent.*

Proof. Evidently, every $H(p)$ operator is polaroid and hence by Theorem 2.6 is hereditarily polaroid. If T is $H(1)$ and $\lambda \in \operatorname{iso}\sigma(T)$ then $\ker(\lambda I - T)^2 \subseteq H_0(\lambda I - T) = \ker(\lambda I - T)$, so $p(\lambda I - T) = 1$. ∎

A paranormal operator need not to be $H(p)$:

Example 3.6. For instance if $T \in L(\ell_2(\mathbb{N}))$ is defined by

$$T(x_1, x_2, ..) := (x_1+x_2, x_1+x_2+x_3, \dots, x_n+x_{n+1}+x_{n+2}, \dots) \quad \text{for all } (x_n) \in \ell_2(\mathbb{N}),$$

then T is paranormal, while the operator

$$(I-T)(x_1, x_2, ..) := (x_2, x_1+x_3, \dots, x_n+x_{n+2}, \dots) \quad \text{for all } (x_n) \in \ell_2(\mathbb{N}),$$

has ascent $p(I-T) = \infty$. Consequently, $H_0(I-T)$ properly contains $\ker (I-T)^n$ for all $n \in \mathbb{N}$.

However, the next result shows every paranormal operator is polaroid. Recall first that given a class of operators $\mathbf{L}$, an operator T is said to be *algebraically* $\mathbf{L}$ if there exists a non-trivial polynomial h for which $h(T)$ belongs to $\mathbf{L}$.

Theorem 3.7. *Every algebraically paranormal $T \in L(H)$ is polaroid. Fuerthermore, T has SVEP.*

Proof. Note first that every quasi-nilpotent algebraically paranormal operator T is nilpotent. In fact, suppose that h is a polynomial for which $h(T)$ is paranormal. From the spectral mapping theorem we have $\sigma(h(T)) = h(\sigma(T)) = \{h(0)\}$, so $h(0)I - h(T)$ is quasinilpotent. Since $h(0)I - h(T)$ is paranormal then $h(0)I - h(T) = 0$, and hence there are some $n \in \mathbb{N}$ and $\mu \in \mathbb{C}$, such that

$$0 = h(0)I - h(T) = \mu T^m \prod_{i=1}^{n} (\lambda_i I - T) \quad \text{with } \lambda_i \neq 0.$$

Since all $\lambda_i I - T$ are invertible it then follows that $T^m = 0$.

We show now that for every isolated point λ of $\sigma(T)$ we have $p(\lambda I - T) = q(\lambda I - T) < \infty$, i.e. λ is a pole of the resolvent. If $\lambda \in \operatorname{iso} \sigma(T)$, let P denote the spectral projection associated with $\{\lambda\}$, $M := K(\lambda I - T) = \ker P$ and $N := H_0(\lambda I - T) = P(X)$. Then, by the classical spectral decomposition, (M, N) is a GKD for $\lambda I - T$. Since $\lambda I - T|N$ is quasi-nilpotent and algebraically paranormal then $\lambda I - T|N$ is nilpotent and hence $\lambda I - T$ is of Kato type. The SVEP for T and T^* at λ then implies by Theorem 3.16 and Theorem 3.17 of [1] that both $p(\lambda I - T)$ and $q(\lambda I - T)$ are finite, hence λ is a pole of the resolvent.

To show the SVEP for T has SVEP first we show the SVEP for paranormal operators. If $\lambda \neq 0$ and $\lambda \neq \mu$ then, by Theorem 2.6 of [22], we have $\|x+y\| \geq \|y\|$ whenever $x \in \ker (\mu I - T)$ and $y \in \ker (\lambda I - T)$. It then follows that if U is an open disc and $f : U \to X$ is an analityc function such that $0 \neq f(z) \in \ker (zI - T)$ for all $z \in U$, then f fails to be continuous at every $0 \neq \lambda \in U$. Finally, if T is algebraically paranormal then $h(T)$ is paranormal for some non-trivial polynomial h, and hence $h(T)$ has SVEP. This implies that T has SVEP, see [1, Theorem 2.40]. ∎

(h) **Class A operators** An operator $T \in L(H)$ is said to be *a class A operator* if $|T^2| \geq |T|^2$. Every log-hyponormal operator is a class A operator [34] but the converse is no true, see [33, p. 176]. Every class A operator is paranormal (an example of a paranormal operator which is not a class A operator can be found in [33, p. 177]). Therefore every class A operator, as well as every algebraically class A operator is polaroid.

(i) **Quasi-class A operators** An operator $T \in L(H)$ is said to be *a quasi-class A operator* if $T^*|T^2|T \geq T^*|T|^2T$. The quasi-class A operators contains the class of al p-quasinormal operators and the class of all class A operators. In [29] it is given an example of a quasi-class A operator which is not paranormal. Every quasi-class A operator has SVEP, since $p(\lambda I - T) \leq 1$ for all $\lambda \in \mathbb{C}$, while every non-zero λ_0 isolated point of the spectrum is a simple pole of T and $H_0(\lambda_0 I - T) = \ker(\lambda_0 I - T)$, see [29].

(h) **$*$-paranormal operators** A bounded operator $T \in L(H)$ is said to be *$*$-paranormal* if $\|T^*x\|^2 \leq \|T^2x\|$ for every unit vector $x \in H$. Paranormality is independent of $*$-paranormality and, evidently, hyponormal operators are both paranormal and $*$-paranormal. It is known ([11] that

$$T \text{ is } *\text{-paranormal} \Leftrightarrow T^{*2}T^2 - 2\lambda TT^* + \lambda^2 \geq 0 \text{ for each } \lambda > 0.$$

Every $*$-paranormal operator T is normaloid, in the sense that $\|T\|$ is equal to the spectral radius $r(T)$. Moreover, $\ker(\lambda I - T) \subseteq \ker(\overline{\lambda} I - T^*)$ for all $\lambda \in \mathbb{C}$, from which it easily follows that $p(\lambda I - T) < \infty$ for all $\lambda \in \mathbb{C}$, thus T has SVEP.

The operator $T \in L(H)$ is said to be *totally $*$-paranormal* if $\lambda I - T$ is $*$-paranormal for every $\lambda \in \mathbb{C}$. An example of a $*$-paranormal operator which is not totally $*$-paranormal may be found in [36]. It is not known to the author if every totally $*$-paranormal operator has property (C).

Theorem 3.8. [36, Lemma 2.2] *Every totally algebraically $*$-paranormal operator is $H(1)$ and hence hereditarily polaroid.*

Proof. $\mu I - T$ is normaloid for all $\mu \in \mathbb{C}$, so $\|(\lambda I - T)x\| \leq \|(\lambda I - T)^n x\|^{\frac{1}{n}}$ for all $x \in X$ and $\lambda \in \mathbb{C}$, so that $H_0(\lambda I - T) \subseteq \ker(\lambda I - T)$ for all $\lambda \in \mathbb{C}$. ∎

The class of p-quasihyponormal may be extended as follows:

(e) **(p,k)-quasihyponormal operators.** $T \in L(H)$ is said to be *(p,k)-quasihyponormal* for some $0 < p \leq 1$ and $k \in \mathbb{N}$ if

$$T^{*k}|T^*|^{2p}T^k \leq (T^{*k}|T|^{2p}T^k.$$

Evidently,
(I) a $(1,1)$-quasihyponormal operator is quasihyponormal;
(II) a $(p,1)$-quasihyponormal operator is p-quasihyponormal;
(III) a $(p,0)$-quasihyponormal operator is p-hyponormal if $0 < p < 1$ and hyponormal if $p = 1$.

The classes of (p,k)-quasihyponormal operators provide examples of hereditarily polaroid operators which are not $H(p)$:

Theorem 3.9. [51, Theorem 6] *Every (p,k)-quasihyponormal operator $T \in L(H)$ is hereditarily polaroid.*

It should be noted that the class of totally $*$-paranormal operators, as well as the class of M-hyponormal operators, are independent of the classes (p,k)-quasihyponormal.

4. Weyl type theorems

In this section we show that the classes of operators defined in the previous sections have a very nice spectral structure similar to that of a normal operator.

Let us before introduce some concepts from Fredholm theory. If $T \in L(X)$ let us denote by $\alpha(T)$ the dimension of the kernel $\ker T$ and by $\beta(T)$ the codimension of the range $T(X)$. Recall that the operator $T \in L(X)$ is said to be *upper semi-Fredholm*, $T \in \Phi_+(X)$, if $\alpha(T) < \infty$ and the range $T(X)$ is closed, while $T \in L(X)$ is said to be *lower semi-Fredholm*, $T \in \Phi_-(X)$, if $\beta(T) < \infty$. If either T is upper or lower semi-Fredholm then T is said to be a *semi-Fredholm operator*, while if T is both upper and lower semi-Fredholm then T is said to be a *Fredholm operator*. If T is semi-Fredholm then the *index* of T is defined by $\operatorname{ind}(T) := \alpha(T) - \beta(T)$. A bounded operator $T \in L(X)$ is said to be a *Weyl operator*, $T \in W(X)$, if T is a Fredholm operator having index 0. The classes of *upper semi-Weyl's* and *lower semi-Weyl's* operators are defined, respectively:

$$W_+(X) := \{T \in \Phi_+(X) : \operatorname{ind} T \le 0\},$$

$$W_-(X) := \{T \in \Phi_-(X) : \operatorname{ind} T \ge 0\}.$$

Clearly, $W(X) = W_+(X) \cap W_-(X)$. The *Weyl spectrum* and the *upper semi-Weyl spectrum* are defined, respectively, by

$$\sigma_{\mathrm{w}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin W(X)\}.$$

and

$$\sigma_{\mathrm{uw}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin W_+(X)\}.$$

In the sequel by $\sigma_{\mathrm{a}}(T)$ we denote the classical *approximate point spectrum*, while $\sigma_{\mathrm{s}}(T)$ denotes the *surjectivity spectrum*. It is well known that $\sigma_{\mathrm{a}}(T') = \sigma_{\mathrm{s}}(T)$ and $\sigma_{\mathrm{s}}(T') = \sigma_{\mathrm{a}}(T)$ for all $T \in L(X)$. Define

$$\pi_{00}(T) : \{\lambda \in \operatorname{iso} \sigma(T) : 0 < \alpha(\lambda I - T) < \infty\},$$

and

$$\pi^a_{00}(T) : \{\lambda \in \operatorname{iso} \sigma_{\mathrm{a}}(T) : 0 < \alpha(\lambda I - T) < \infty\}.$$

Let $p_{00}(T) := \sigma(T) \setminus \sigma_{\mathrm{b}}(T)$, i.e. $p_{00}(T)$ is the set of all poles of the resolvent of T having finite rank. Clearly, for every $T \in L(X)$ we have

$$p_{00}(T) \subseteq \pi_{00}(T) \subseteq \pi^a_{00}(T). \tag{6}$$

It should be noted that the condition $p_{00}(T) = \pi_{00}(T)$ is equivalent to saying that there exists $p := p(\lambda I - T) \in \mathbb{N}$ such that

$$H_0(\lambda I - T) = \ker (\lambda I - T)^p \quad \text{for all } \lambda \in \pi_{00}(T), \tag{7}$$

see [6, Theorem 2.2]). By Theorem 3.2 then every polaroid operator satisfies the equality $p_{00}(T) = \pi_{00}(T)$.

A classical result of H. Weyl [52] shows that for a normal operator T we have $\sigma(T) \setminus \sigma_{\mathrm{w}}(T) = \pi_{00}(T)$. For a normal operator T we know that T and T^* have SVEP. The SVEP for T^* entails that $\sigma(T) = \sigma_{\mathrm{a}}(T)$, see [1, Corollary 2.45], thus $\pi_{00}(T) = \pi^a_{00}(T)$. It is easily seen that the SVEP for T and T^* entails that $\sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{uw}}(T)$. The inclusion $\sigma_{\mathrm{uw}}(T) \subseteq \sigma_{\mathrm{w}}(T)$ holds for every operator.

Conversely if $\lambda \notin \sigma_{\mathrm{uw}}(T)$ then $\lambda I - T$ is semi-Fredholm and the SVEP of both

T and T^* implies that $p(\lambda I - T) = q(\lambda I - T) < \infty$, and by [1, Theorem 3.4] this implies that $\lambda I - T \in W(X)$, i.e. $\lambda \notin \sigma_{\mathrm{w}}(T)$. Therefore, for a normal operator we have:

$$\sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{uw}}(T) = \sigma(T) \setminus \sigma_{\mathrm{w}}(T) = \pi_{00}(T) = \pi_{00}^a(T).$$

These equalities motivate the following definitions. The symbols here used could generate a certain confusion, but these are the most used in literature.

Definition 4.1. *A bounded operator $T \in L(X)$ is said to satisfy* Weyl's theorem, *in symbol (W), if $\sigma(T) \setminus \sigma_{\mathrm{w}}(T) = \pi_{00}(T)$. $T \in L(X)$ is said to satisfy* a-Weyl's theorem, *in symbol (aW), if $\sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{uw}}(T) = \pi_{00}^a(T)$. $T \in L(X)$ is said to satisfy* property (w), *if $\sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{uw}}(T) = \pi_{00}(T)$.*

Weyl's theorem for T entails *Browder's theorem* for T, i.e. $\sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{b}}(T)$. Note that Browder's theorem for T and Browder's theorem for T^* are equivalent, since $\sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{w}}(T^*)$ and $\sigma_{\mathrm{b}}(T) = \sigma_{\mathrm{b}}(T^*)$. Furthermore, by [2, Theorem 3.1],

$$(W) \text{ holds for } T \iff \text{Browder's theorem holds for } T \text{ and } p_{00}(T) = \pi_{00}(T).$$

Either a-Weyl's theorem or property (w) entails Weyl's theorem. Property (w) and a-Weyl's theorem are independent, see [7]. It should be noted that Weyl's theorem for T in general does not imply that Weyl's theorem holds for $f(T)$. An example for which Weyl's theorem holds for T but not for T^2 may be found in [48].

The concept of semi-Fredholm operators has been generalized by Berkani ([13], [14]) in the following way: for every $T \in L(X)$ and a nonnegative integer n let us denote by $T_{[n]}$ the restriction of T to $T^n(X)$ viewed as a map from the space $T^n(X)$ into itself (we set $T_{[0]} = T$). $T \in L(X)$ is said to be *semi B-Fredholm* (resp. *B-Fredholm, upper semi B-Fredholm, lower semi B-Fredholm,*) if for some integer $n \geq 0$ the range $T^n(X)$ is closed and $T_{[n]}$ is a semi-Fredholm operator (resp. Fredholm, upper semi-Fredholm, lower semi-Fredholm). In this case $T_{[m]}$ is a semi-Fredholm operator for all $m \geq n$ ([14]). This enables one to define the index of a semi B-Fredholm as $\operatorname{ind} T = \operatorname{ind} T_{[n]}$. A bounded operator $T \in L(X)$ is said to be *B-Weyl* (respectively, *upper semi B-Weyl, lower semi B-Weyl*) if for some integer $n \geq 0$ $T^n(X)$ is closed and $T_{[n]}$ is Weyl (respectively, upper semi-Weyl, lower semi-Weyl). The classes of operators previously defined generate the *B-Weyl spectrum* $\sigma_{\mathrm{bw}}(T)$, the *upper B-Weyl spectrum* $\sigma_{\mathrm{usbw}}(T)$, and the *lower B-Weyl spectrum* $\sigma_{\mathrm{lsbw}}(T)$.

Remark 4.2. The implications (3), (4) and (1) are equivalences whenever $\lambda I - T$ is a quasi-Fredholm operator, in particular whenever $\lambda I - T$ is a semi B-Fredholm operator, see [8].

If $T \in L(X)$ define

$$E(T) := \{\lambda \in \operatorname{iso} \sigma(T) : 0 < \alpha(\lambda I - T)\},$$

and

$$E^a(T) := \{\lambda \in \operatorname{iso} \sigma_{\mathrm{a}}(T) : 0 < \alpha(\lambda I - T)\}.$$

Evidently, $E(T) \subseteq E^a(T)$ for every $T \in L(X)$.

Definition 4.3. *A bounded operator $T \in L(X)$ is said to satisfy* generalized Weyl's theorem, *in symbol (gW), if $\sigma(T) \setminus \sigma_{\mathrm{bw}}(T) = E(T)$. $T \in L(X)$ is said to satisfy* generalized a-Weyl's theorem, *in symbol (gaW), if $\sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{ubw}}(T) = E^a(T)$. $T \in L(X)$ is said to satisfy* generalized property (w), *in symbol (gw), if $\sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{ubw}}(T) = E(T)$.*

In the following diagram we resume the relationships between all Weyl type theorems:

$$(gw) \quad \Rightarrow (w) \Rightarrow (W)$$

$$(gaW) \quad \Rightarrow (aW) \Rightarrow (W),$$

see [15, Theorem 2.3], [7] and [16]. Generalized property (w) and generalized a-Weyl's theorem are also independent, see [15]. Furthermore,

$$(gw) \quad \Rightarrow (gW) \Rightarrow (W)$$

$$(gaW) \quad \Rightarrow (gW) \Rightarrow (W),$$

see [15] and [16]. The converse of all these implications in general does not hold.

Definition 4.4. *A bounded operator $T \in L(X)$ is said to be* left polaroid *if every isolated point of $\sigma_{\mathrm{a}}(T)$ is a left pole of the resolvent of T. $T \in L(X)$ is said to be* right polaroid *if every isolated point of $\sigma_{\mathrm{s}}(T)$ is a right pole of the resolvent of T. $T \in L(X)$ is said to be* a-polaroid *if every isolated point of $\sigma_{\mathrm{a}}(T)$ is a pole of the resolvent of T.*

If $T \in L(X)$ is both left and right polaroid then T is polaroid, but the converse in general does not hold. Clearly, every a-polaroid operator is both left polaroid and polaroid. Moreover T is left polaroid (respectively, right polaroid) if and only if T^* is right polaroid (respectively, left polaroid) ([3]).

Theorem 4.5. ([3]) *If $T \in L(X)$, X a Banach space, the following assertions hold:*

(i) *If T' has $SVEP$ then the properties of being polaroid, a-polaroid and left polaroid for T are all equivalent.*

(ii) *If T has $SVEP$ then the properties of being polaroid, a-polaroid and left polaroid for T' are all equivalent.*

Weyl type theorems and generalized Weyl type theorem are equivalent under some conditions:

Theorem 4.6. ([3]) *Let $T \in L(X)$. Then we have*

(i) *If T is left-polaroid then (aW) and (gaW) for T are equivalent. If T is right-polaroid then (aW) and (gaW) for T' are equivalent*

(ii) *If T is polaroid then (W), and (gW) for T are equivalent. Analogously, (W), and (gW) for T' are equivalent.*

(iii) *If $T \in L(X)$ is a-polaroid then (aW), (gaW), (w), (gw) for T are equivalent.*

The following result gives a very simple and useful framework for establishing Weyl type theorems for several classes of operators:

Theorem 4.7. *If $T \in L(X)$ is polaroid and either T or T' has SVEP then both $f(T)$ and $f(T')$ satisfy Weyl's theorem for all $f \in \mathcal{H}_{nc}(\sigma(T))$.*

Proof. We show first that both T and T' satisfy Weyl's theorem. The SVEP of either T or T' entails Browder's theorem for T, or equivalently Browder's theorem for T'. The polaroid condition for T entails that $p_{00}(T) = \pi_{00}(T)$, so Weyl's theorem holds for T. If T is polaroid then T' is polaroid and hence $p_{00}(T') = \pi_{00}(T')$, so Weyl's theorem holds also for T'. Let now $f \in \mathcal{H}_{nc}(\sigma(T))$. By Theorem 3.3 $f(T)$ and $f(T')$ are polaroid and by [1, Theorem 2.40] $f(T)$ (or $f(T')$) has SVEP, so $f(T)$ and $f(T')$ satisfy Weyl'theorem by the first part of the proof. ∎

As a consequence of Theorem 4.6 we then obtain:

Theorem 4.8. *Let $T \in L(X)$ be polaroid an suppose that $f \in \mathcal{H}_{nc}(\sigma(T))$. Then we have*

(i) *If T' has SVEP then (W), (aW), (w), (gW), (gaW) and (gw) hold for $f(T)$.*

(ii) *If T has SVEP then (W), (aW), (w), (gW), (gaW) and (gw) hold for $f(T')$.*

Remark 4.9. In the case of Hilbert space operators, in Theorem 4.6, Theorem 4.7 and Theorem 4.8, the condition that T' has SVEP may be replaced by the SVEP of the Hilbert adjoint T^*, while in the assertions concerning Weyl's type theorems and generalized Weyl's theorems for T' and $f(T')$ may be replaced by T^* and $f(T^*)$, respectively.

Theorem 4.10. [3] *Let $T \in L(X)$. Then we have*

(i) *If $T \in L(X)$ is left-polaroid and has SVEP then (aW) holds for $f(T)$, or equivalently (gaW) holds for $f(T)$ for all $f \in \mathcal{H}_{nc}(\sigma(T))$.*

(ii) *If $T \in L(X)$ is polaroid and has SVEP then (W) holds for $f(T)$, or equivalently (gW) holds for $f(T)$ for all $f \in \mathcal{H}_{nc}(\sigma(T))$.*

Remark 4.11. All Weyl type theorems, in their classical or in their generalized form, have been studied by a large number of authors. The results of the previous sections give us an unifying theoretical framework for establishing all Weyl type theorems for a large number of the commonly considered classes of operators, for instance all the operators cited in section 2 and section 3 of this note, excepted the quasi class A operators, for which only the non-zero isolated points of the spectrum are poles. It should be noted that for all these classes of operators, Weyl type theorems, or their generalized versions, have been proved, separately, in different papers.

5. Quasi-affinities

In the previous section it has been observed that the polaroid condition on T and the single-valued extension property entail Weyl's theorem for T. Moreover, if T' has SVEP (respectively, T has SVEP) then Weyl's theorem for T (respectively, for T') is equivalent to all the other variants of Weyl's theorem. It is easily seen that the SVEP is transmitted from S to T, if T and S are intertwined by an injective map. Therefore, in order to have the transmission of Weyl type theorems, it is useful to find conditions for which S polaroid implies that T is polaroid.

If $T \prec S$ a classical result due to Rosenblum shows that $\sigma(S) \cap \sigma(T) \neq \emptyset$ ([47]). But quasi-similarity is, in general, not sufficient to preserve the spectrum. This happens only in some special cases, for instance if T and S are quasi-similar hyponormal operators [23], or whenever T and S have totally disconnected spectra, see [35, Corollary 2.5]. Therefore, it is not quite surprising that, if $T \prec S$, the preservation of "certain" spectral properties from S to T requires some other additional conditions.

Classical examples show that in general the polaroid property is not preserved if two bounded operators are intertwined by an injective map. For instance by [32] or [39], there exist bounded linear operators U, V, B on a Hilbert space H such that $BU = UV$, B and its Hilbert adjoint B^* are injective, V is quasi-nilpotent and the spectrum of U the unit disc $D(0,1)$. Let $T := V^*$, $S := U^*$ and $A := B^*$. Then $SA = AT$, so that T and S are intertwined by the injective operator A, S is polaroid, since $\sigma(S) = \overline{\sigma(U)} = D(0,1)$ has no isolated points, while T is also quasi-nilpotent and hence not polaroid.

Theorem 5.1. [4] *Suppose that $T \in L(X)$, $S \in L(Y)$ are intertwined by an injective map $A \in L(X,Y)$. If S is polaroid and $\mathrm{iso}\,\sigma(T) \subseteq \mathrm{iso}\,\sigma(S)$ then T is polaroid.*

If we assume that S satisfies property (C) the condition $\mathrm{iso}\,\sigma(T) \subseteq \mathrm{iso}\,\sigma(S)$ may be relaxed into the condition $\mathrm{iso}\,\sigma(T) \subseteq \sigma(S)$. In fact, in this case, by a result of Stampli ([49]) we have $\sigma(S) \subseteq \mathrm{iso}\,\sigma(T)$, and this easily implies that $\mathrm{iso}\,\sigma(T) \subseteq \mathrm{iso}\,\sigma(S)$.

Corollary 5.2. *Suppose that $T, S \in L(H)$, S totally paranormal, or $p^* - QH$, and $T \prec S$. If $\mathrm{iso}\,\sigma(T) \subseteq \sigma(S)$ then T is polaroid.*

Proof. As observed before, if T is totally paranormal then T is polaroid and has property (C). Analogously, if T is $p^* - QH$ then T is polaroid and has property (β). ∎

Clearly if $T \prec S$ then $T'A' = A'S'$, and A' is also injective, since A has dense range. As an immediate consequence of Theorem 5.1 we then obtain:

Corollary 5.3. *Suppose that $T \in L(X)$ and $S \in L(Y)$ are intertwined by a quasi-affinity $A \in L(X,Y)$ and $\mathrm{iso}\,\sigma(T) = \mathrm{iso}\,\sigma(S)$. Then T is polaroid if and only if S is polaroid.*

Taking into account the results of the previous section we also have:

Corollary 5.4. *Let $T \in L(X)$, $S \in L(Y)$ be intertwined by an injective map $A \in L(X,Y)$. Suppose that S is polaroid, has SVEP and $\mathrm{iso}\,\sigma(T) \subseteq \mathrm{iso}\,\sigma(S)$. Then we have:*

(i) *$f(T)$ satisfies (gW) for all $f \in \mathcal{H}_{nc}(\sigma(T))$.*

(ii) *$f(T')$ satisfies all Weyl type theorems for all $f \in \mathcal{H}_{nc}(\sigma(T))$.*

Note that quasi-similar operators may have unequal approximate point spectrum, for an example see [23].

Theorem 5.5. [4] *Let $T \in L(X)$, $S \in L(Y)$ be intertwined by an injective map $A \in L(X,Y)$ and suppose that $\mathrm{iso}\,\sigma_{\mathrm{a}}(T) \subseteq \mathrm{iso}\,\sigma_{\mathrm{a}}(S)$. If S is left polaroid then T is polaroid.*

Again, by Theorem 4.8, we deduce the following result:

Corollary 5.6. *Let $T \in L(X)$, $S \in L(Y)$ be intertwined by an injective map $A \in L(X,Y)$. Suppose that S is left polaroid operator which has SVEP and $\mathrm{iso}\,\sigma_{\mathrm{a}}(T) \subseteq \mathrm{iso}\,\sigma_{\mathrm{a}}(S)$. Then we have:*

(i) *$f(T)$ satisfies (W) or equivalently (gW) for all $f \in \mathcal{H}_{nc}(\sigma(T))$.*

(ii) *$f(T')$ satisfies all Weyl type theorems for all $f \in \mathcal{H}_{nc}(\sigma(T))$..*

A dual version of Corollary 5.6 is the following corollary

Corollary 5.7. *Let $S \in L(Y)$ and $T \in L(X)$ be intertwined by map $A \in L(Y,X)$ having dense range. Suppose that S is right-polaroid, S' has SVEP and $\mathrm{iso}\,\sigma_{\mathrm{s}}(T) \subseteq \mathrm{iso}\,\sigma_{\mathrm{s}}(S)$. Then we have:*

(i) *$f(T')$ satisfies (gW) for all $f \in \mathcal{H}_{nc}(\sigma(T))$.*

(ii) *$f(T)$ satisfies all Weyl type theorems for all $f \in \mathcal{H}_{nc}(\sigma(T))$.*

Under the stronger conditions of quasi-similarity and property (β), the assumption on the isolated points of the spectra of T and S in Theorem 5.1 may be omitted:

Theorem 5.8. *Let $T \in L(X)$, $S \in L(Y)$ be quasi-similar.*

(i) *If both T and S have property (β) then T is polaroid if and only if S is polaroid. In this case, T' is a-polaroid.*

(ii) *If both T and S are Hilbert spaces operators for which property (C) holds then T is polaroid if and only if S is polaroid. In this case, T' is a-polaroid.*

Consequently, under the assumptions (i) or (ii) on S and T, $f(T)$ satisfies (gW), while $f(T')$ satisfies all Weyl type theorems for all $f \in \mathcal{H}_{nc}(\sigma(T))$.

Proof. (i) By a result of Putinar [46] we have $\sigma(S) = \sigma(T)$, hence $\mathrm{iso}\,\sigma(T) = \mathrm{iso}\,\sigma(S)$. By Corollary 5.3 we then obtain that T is polaroid exactly when S is polaroid. Evidently, in this case T' is polaroid. Now, property (β) implies that S has SVEP and hence also T has SVEP. The SVEP for T, always by [1, Corollary 2.45], entails that $\sigma(T') = \sigma_{\mathrm{a}}(T')$, and hence T' is a-polaroid.

(ii) Also in this case, by a result of Stampfli [49], we have $\sigma(S) = \sigma(T)$, and property (C) entails SVEP, so the assertion follows by using the same argument of part (i).

The last assertion is clear from Corollary 5.4. ∎

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Dipartimento di Metodi e Modelli Matematici, Facoltà di Ingegneria, Università di Palermo, Viale delle Scienze, I-90128 Palermo (Italy), E-mail: paiena@unipa.it