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On a new construction of geometric mean of n-operators

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ABSTRACT

For n positive definite operators $A_1, \cdots, A_n$, Ando-Li-Mathias defined geometric mean of n-operators $\mathfrak{G}(A_1, \cdots, A_n)$ by symmetric procedure. It has many nice properties, and is studied by many authors. But the process is so complicated to compute. In this paper, we shall attempt to make a new construction of geometric mean of n-operators which we can compute it easier than geometric mean by Ando-Li-Mathias.

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[JLY] C. Jung, H. Lee and T. Yamazaki, *On a new construction of geometric mean of n-operators*, Linear Algebra Appl., **431** (2009), 1477–1488.

1. INTRODUCTION

In 1975, theory of operator means has been introduced in [14], where operator means a bounded linear operators on a complex Hilbert space $\mathcal{H}$. In the operator case, arithmetic and harmonic means are easily defined (whose definitions will be introduced later), but since operators are not commutative, geometric mean is not easy to define. In [14], geometric mean of two operators is defined as follows: Let A and B be positive invertible operators. Then the geometric mean $A\sharp B$ between A and B is defined by

$$A\sharp B=A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}.$$

If A and B are not invertible, we consider geometric mean $A\sharp B$ as $\lim_{\varepsilon\searrow 0}(A+\varepsilon I)\sharp(B+\varepsilon I)$, strongly. As a more important result, Kubo-Ando [10] obtained that every operator mean of two positive operators has one to one connection with an operator monotone function. Hence theory of operator means is closely related to one of operator monotone function.

To extend operator means of two operators to more than three operators case is quite natural, and many authors have discussed the problem. Of course, arithmetic and harmonic means of n-operators are easily defined as follows: Let $A_1, \cdots, A_n$ be positive operators. Then arithmetic mean $\mathfrak{A}(A_1, \cdots, A_n)$ of $A_1, \cdots, A_n$ is defined as follows:

$$\mathfrak{A}(A_1,\cdots,A_n)=\frac{A_1+\cdots+A_n}{n}.$$

If $A_1, \cdots, A_n$ are all invertible, we can define harmonic mean $\mathfrak{H}(A_1, \cdots, A_n)$ by

$$\mathfrak{H}(A_1, \cdots, A_n) = \mathfrak{A}(A_1^{-1}, \cdots, A_n^{-1})^{-1}.$$

But to define geometric mean of n-operators is not easy. Recently, some authors have defined it by several way, for example [1, 16] and also see [3], especially, Ando-Li-Mathias [3] have given a very good definition of geometric mean of n-operators. It needs so-called symmetric procedure as follows:

$n = 2$ **case.** Define geometric mean $\mathfrak{G}(A, B)$ by

$$\mathfrak{G}(A, B) = A\sharp B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}.$$

$n = 3$ **case.** Let $A_n = B_{n-1}\sharp C_{n-1}$, $B_n = C_{n-1}\sharp A_{n-1}$, $C_n = A_{n-1}\sharp B_{n-1}$. Then there exist $\lim_{n\to\infty} A_n$, $\lim_{n\to\infty} B_n$, $\lim_{n\to\infty} C_n$ in the Thompson metric (Thompson metric will be introduced later), and all the same. Hence we can define the geometric mean $\mathfrak{G}(A, B, C)$ by

$$\mathfrak{G}(A, B, C) = \lim_{n\to\infty} A_n = \lim_{n\to\infty} B_n = \lim_{n\to\infty} C_n.$$

$n = 4$ **case.** Let $A_n = G(B_{n-1}, C_{n-1}, D_{n-1})$, $B_n = G(A_{n-1}, C_{n-1}, D_{n-1})$, $C_n = G(A_{n-1}, B_{n-1}, D_{n-1})$, $D_n = G(A_{n-1}, B_{n-1}, C_{n-1})$. Then there exist all limits of operator sequences $\{A_n\}, \{B_n\}, \{C_n\}, \{D_n\}$ in the Thompson metric, and all the same. We define the geometric mean $\mathfrak{G}(A, B, C, D)$ by

$$\mathfrak{G}(A, B, C, D) = \lim_{n\to\infty} A_n = \lim_{n\to\infty} B_n = \lim_{n\to\infty} C_n = \lim_{n\to\infty} D_n.$$

We can define $\mathfrak{G}(A_1, \cdots, A_n)$ in the case $n \geq 5$ by the same way.

It is a very natural definition and interesting. But it is not good for concrete computation since it requires an enormous calculation. In this paper, we shall discuss a new construction of geometric mean of n-operators which can be obtained easier than the geometric mean by Ando-Li-Mathias. This paper consists the following sections; In section 2, we shall introduce some properties of geometric mean by Ando-Li-Mathias and Thompson metric, briefly. In section 3, we shall introduce a new idea for construction of geometric mean of n-operators. In section 4, we shall discuss relations between arithmetic mean and our idea defined in section 3. Lastly, we will construct a new geometric mean of 4-operators which can be calculate easier than that of Ando-Li-Mathias.

2. Primarily

In what follows, a capital letter means a bounded linear operators on a complex Hilbert space $\mathcal{H}$. An operator is said to be positive (resp. strictly positive) if and only if $\langle Ax, x\rangle \geq 0$ (resp. $\langle Ax, x\rangle > 0$) for all $x \in \mathcal{H}$. For self-adjoint operators A and B, $A \geq B$ means that $A - B$ is positive.

Firstly, we shall introduce some basic properties of geometric mean by Ando-Li-Mathias as follows: Let $A_1, \cdots, A_n$ be positive operators. Then the following properties (P1)–(P10) hold [3]:

(P1) If $A_1, \cdots, A_n$ commute with each other, then

$$\mathfrak{G}(A_1, \cdots, A_n) = (A_1 \cdots A_n)^{1/n}.$$

(P2) Joint homogeneity.

$$\mathfrak{G}(a_1 A_1, \cdots, a_n A_n) = (a_1 \cdots a_n)^{1/n} \mathfrak{G}(A_1, \cdots, A_n)$$

for positive numbers $a_i > 0$ $(i = 1, \cdots, n)$.

(P3) Permutation invariance. For any permutation π,

$$\mathfrak{G}(A_1, \cdots, A_n) = \mathfrak{G}(A_{\pi(1)}, \cdots, A_{\pi(n)}).$$

(P4) Monotonicity. For each $i = 1, 2, \cdots, n$, if $B_i \leq A_i$, then

$$\mathfrak{G}(B_1, \cdots, B_n) \leq \mathfrak{G}(A_1, \cdots, A_n).$$

(P5) Continuity from above. For each $i = 1, 2, \cdots, n$, if operator sequences $\{A_i^{(k)}\}_{k=1}^{\infty}$ are monotone decreasing with $A_i^{(k)} \searrow A_i$ as $k \to \infty$, then

$$\mathfrak{G}(A_1^{(k)}, \cdots, A_n^{(k)}) \searrow \mathfrak{G}(A_1, \cdots, A_n) \quad \text{as } k \to \infty.$$

(P6) Congruence invariance. For any invertible operator S,

$$\mathfrak{G}(S^* A_1 S, \cdots, S^* A_n S) = S^* \mathfrak{G}(A_1, \cdots, A_n) S.$$

(P7) Joint concavity.

$$\begin{aligned}&\mathfrak{G}(\lambda A_1 + (1-\lambda)A_1', \cdots, \lambda A_n + (1-\lambda)A_n') \\ &\qquad \geq \lambda \mathfrak{G}(A_1, \cdots, A_n) + (1-\lambda)\mathfrak{G}(A_1', \cdots, A_n') \quad \text{for } 0 \leq \lambda \leq 1.\end{aligned}$$

(P8) Self-duality.

$$\mathfrak{G}(A_1^{-1}, \cdots, A_n^{-1})^{-1} = \mathfrak{G}(A_1, \cdots, A_n).$$

(P9) Determinant identity.

$$\det\left(\mathfrak{G}(A_1, \cdots, A_n)\right) = \left\{(\det A_1) \cdots (\det A_n)\right\}^{1/n}.$$

(P10) Arithmetic-geometric-harmonic means inequality.

$$\mathfrak{H}(A_1, \cdots, A_n) \leq \mathfrak{G}(A_1, \cdots, A_n) \leq \mathfrak{A}(A_1, \cdots, A_n).$$

We shall define geometric mean which satisfies the two conditions: (i) not require an enormous calculation, and (ii) satisfying all properties as above.

Next, we shall introduce an important theory of the cone of positive operators, briefly. For positive operators A and B, Thompson metric $d(A, B)$ ([15]) between A and B is defined by

$$d(A, B) = \max\{\log M(A \backslash B), \log M(B \backslash A)\},$$

where $M(A \backslash B) = \inf\{\lambda > 0;\ A \leq \lambda B\} = \|B^{-1/2} A B^{-1/2}\|$. We note that the cone of positive operators will be complete in Thompson metric ([15]). By the definition of Thompson metric, we can obtain

(2.1) $\qquad d(X^* A X, X^* B X) = d(A, B) \quad$ for any invertible operator X.

The following properties are important [4, 11]:

$$d(A_1\sharp_t A_2, B_1\sharp_t B_2) \le (1-t)d(A_1,B_1) + td(A_2,B_2), \tag{2.2}$$

where $A\sharp_t B$ means weighted geometric mean defined by

$$A\sharp_t B = A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2}.$$

3. A new construction of geometric mean

In this section, we shall consider an operator mean of n-operators which is defined by only using geometric mean of 2-operators. Throughout the paper, we will consider two operators as follows: Let $A_1, \cdots, A_n$ be positive operators on a Hilbert space $\mathcal{H}$, and $\mathcal{K}$ be a its direct sum, that is,

$$\mathcal{K} = \cdots \oplus \mathcal{H} \oplus \boxed{\mathcal{H}} \oplus \mathcal{H} \oplus \cdots.$$

Let U be a bilateral shift and P be a positive operator on $\mathcal{K}$ defined by

$$U = \begin{pmatrix} \ddots & & & & \\ \ddots & \ddots & & & \\ & I & 0 & & \\ & & I & \boxed{0} & \\ & & & I & 0 \\ & & & & \ddots \quad \ddots \end{pmatrix} \quad \text{and} \quad P = \begin{pmatrix} \ddots & & & & \\ & A_{n-1} & & & \\ & & A_n & & \\ & & & \boxed{A_1} & \\ & & & & A_2 \\ & & & & & \ddots \end{pmatrix} \tag{3.1}$$

on $\mathcal{K} = \cdots \oplus \mathcal{H} \oplus \boxed{\mathcal{H}} \oplus \mathcal{H} \oplus \cdots$, where $\boxed{X}$ means the $(0,0)$ element in the operators.

Theorem 1. *Let $A_1, \cdots, A_n$ be positive operators on a Hilbert space $\mathcal{H}$, and let U and P be defined in* (3.1). *Assume*

$$P_i = P_{i-1}\sharp U P_{i-1} U^* \quad \textit{and} \quad P_0 = P.$$

Then there exists a positive operator L on $\mathcal{H}$ such that

$$\lim_{i\to\infty} P_i = I \otimes L.$$

in the Thompson metric.

To prove Theorem 1, we prepare the following notion of a kind of convex set.

Definition 1 (Convex set under geometric mean). Let $\mathcal{M}$ be a subset of all positive operators. $\mathcal{M}$ is said to be a *convex set under geometric mean* if

$$A, B \in \mathcal{M} \text{ implies } A\sharp_t B \in \mathcal{M} \text{ for all } t \in [0,1].$$

For positive operators A and B, $[A,B] = \{A\sharp_t B;\ t \in [0,1]\}$ is a typical example of convex set under geometric mean. For positive operators $A_1, \cdots, A_n$, $[A_1, \cdots, A_n]$ means a convex set under geometric mean which is generated by $\{A_1, \cdots, A_n\}$.

Proof of Theorem 1. Noting that by concrete computation, we have

$$UPU^* = \mathrm{diag}(\cdots, A_{n-1}, \boxed{A_n}, A_1, \cdots, A_n, A_1, \cdots).$$

Then we have $P = U^n P U^{n*}$.

By the definition of P_i, we have

$$[P_1, UP_1U^*, \cdots, U^{n-1}P_1U^{n-1^*}] \subset [P, UPU^*, \cdots, U^{n-1}PU^{n-1^*}].$$

Hence there exists a convex set under geometric mean $\mathcal{M}$ such that

$$\mathcal{M} = \bigcap_{i=0}^{\infty}[P_i, UP_iU^*, \cdots, U^{n-1}P_iU^{n-1^*}].$$

Here we shall prove that $\mathcal{M}$ is a singleton of a positive operator. To prove this, we have to prove

$$\lim_{i\to\infty} d(P_i, U^kP_iU^{k^*}) = 0 \quad \text{for all } k = 1, 2, \cdots, n-1,$$

since the cone of positive definite operators is complete under the Thompson metric.

Since U is unitary, (2.1) and (2.2), we have

$$\begin{aligned}
\sum_{k=1}^{n-1} \alpha_k d(P_1, U^kP_1U^{k^*}) &= \sum_{k=1}^{n-1} \alpha_k d(P \sharp UPU^*, U^kPU^{k^*} \sharp U^{k+1}PU^{k+1^*}) \\
&\leq \sum_{k=1}^{n-1} \frac{\alpha_k}{2}\left\{d(P, U^{k+1}PU^{k+1^*}) + d(UPU^*, U^kPU^{k^*})\right\} \\
&= \sum_{k=1}^{n-1} \frac{\alpha_k}{2}\left\{d(P, U^{k+1}PU^{k+1^*}) + d(P, U^{k-1}PU^{k-1^*})\right\} \\
&= \frac{\alpha_2}{2} d(P, UPU^*) + \sum_{k=2}^{n-2} \frac{\alpha_{k-1} + \alpha_{k+1}}{2} d(P, U^kPU^{k^*}) \\
&\quad + \frac{\alpha_{n-2}}{2} d(P, U^{n-1}PU^{n-1^*}),
\end{aligned}$$

for positive numbers $\alpha_1, \cdots, \alpha_{n-1}$.

By this procedure, the $n-1$-tuple of coefficients $(\alpha_1, \cdots, \alpha_{n-1})$ changes into

$$\left(\frac{\alpha_2}{2}, \frac{\alpha_1 + \alpha_3}{2}, \frac{\alpha_2 + \alpha_4}{2}, \cdots, \frac{\alpha_{n-3} + \alpha_{n-1}}{2}, \frac{\alpha_{n-2}}{2}\right).$$

This operation can be represented by an $n-1$-by-$n-1$ matrix A as follows:

$$A = \frac{1}{2}\begin{pmatrix} 0 & 1 & & & & \\ 1 & 0 & 1 & & & \\ & 1 & 0 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & 1 \\ & & & & 1 & 0 \end{pmatrix}.$$

Hence we only prove $\lim_{i\to\infty} A^i = 0$.

Define an $n-1$-by-$n-1$ matrix T by

$$T = \begin{pmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & 0 & 1 & & \\ & & & \ddots & \ddots & \\ & & & & \ddots & 1 \\ & & & & & 0 \end{pmatrix}.$$

We note that the numerical radius $w(T)$ of T is known as $w(T) = \cos\frac{\pi}{n+1} < 1$ (see [13], also [8, p. 8, Example]). Moreover,

$$w(A) \leq \frac{1}{2}(w(T) + w(T^*)) = w(T) = \cos\frac{\pi}{n+1}.$$

Hence we have

$$\frac{1}{2}\|A^i\| \leq w(A^i) \leq (w(A))^i \leq \cos^i\frac{\pi}{n+1} \longrightarrow 0 \text{ (as } i \to \infty),$$

that is, $\lim_{i\to\infty} A^i = 0$. Hence $\mathcal{M}$ is a singleton.

Next, we shall prove $\lim_{i\to\infty} P_i = I \otimes L$. Since $\mathcal{M}$ is a singleton, there exists a positive operator X on $\mathcal{K}$ such that $\mathcal{M} = \{X\}$ and

$$\lim_{i\to\infty} P_i = \lim_{i\to\infty} UP_iU^* = \cdots = \lim_{i\to\infty} U^{n-1}P_iU^{n-1^*} = X.$$

Since U is a bilateral shift and every $U^kP_iU^{k^*}$ is diagonal for $k = 0, 1, 2, \cdots, n-1$, X must be the form $X = I \otimes L$. It completes the proof. $\square$

As in the proof, Theorem 1 can be rewritten as the following form:

Theorem 1'. *Let $A_1, \cdots, A_n$ be positive operators on a Hilbert space $\mathcal{H}$. Assume*

$$A_k^{(i)} = A_k^{(i-1)} \sharp A_{k+1}^{(i-1)} \quad \textit{and} \quad A_n^{(i)} = A_n^{(i-1)} \sharp A_1^{(i-1)}.$$

Then there exists a positive operator L on $\mathcal{H}$ such that

$$\lim_{i\to\infty} A_k^{(i)} = L \quad \textit{for all } k = 1, 2, \cdots, n$$

in the Thompson metric.

In what follows, for positive operators $A_1, \cdots, A_n$, we denote the above limit L by $\mathfrak{L}(A_1, \cdots, A_n)$. Of course, for positive operators A, B, C, $\mathfrak{G}(A, B, C) = \mathfrak{L}(A, B, C)$. Next, we shall check that $\mathfrak{L}(A_1, \cdots, A_n)$ satisfies properties (P1) – (P10) which is introduced in the second section. Obviously, $\mathfrak{L}(A_1, \cdots, A_n)$ satisfies properties (P4) – (P8). We obtain that $\mathfrak{L}(A_1, \cdots, A_n)$ satisfies (P1), (P2) and (P9) by the following proposition:

Proposition 2. *Let $A_1, \cdots, A_n$ be positive operators such that they commute with each other. Then $\mathfrak{L}(A_1, \cdots, A_n) = (A_1 \cdots A_n)^{1/n}$.*

Proof. Let P and U be defined in (3.1). Since $P = \mathrm{diag}(\cdots A_n, \boxed{A_1}, A_2, \cdots, A_n, \cdots)$, $UPU^* = \mathrm{diag}(\cdots A_{n-1}, \boxed{A_n}, A_1, \cdots, A_n, \cdots)$. Hence we have

$$\begin{aligned} P_1 &= \mathrm{diag}(\cdots, A_n \sharp A_{n-1}, \boxed{A_1 \sharp A_n}, A_2 \sharp A_1, \cdots, A_n \sharp A_{n-1}, \cdots) \\ &= \mathrm{diag}(\cdots, \sqrt{A_n A_{n-1}}, \boxed{\sqrt{A_1 A_n}}, \sqrt{A_2 A_1}, \cdots, \sqrt{A_n A_{n-1}}, \cdots). \end{aligned}$$

Here we note that $\sqrt{A_1 A_n}\sqrt{A_2 A_1} \cdots \sqrt{A_n A_{n-1}} = A_1 \cdots A_n$ holds. Then, for

$$\lim_{i \to \infty} P_i = \mathrm{diag}(\cdots, \mathfrak{L}(A_1, \cdots, A_n), \boxed{\mathfrak{L}(A_1, \cdots, A_n)}, \mathfrak{L}(A_1, \cdots, A_n), \cdots),$$

we have

$$\mathfrak{L}(A_1, \cdots, A_n)^n = A_1 \cdots A_n,$$

that is, $\mathfrak{L}(A_1, \cdots, A_n) = (A_1 \cdots A_n)^{1/n}$. □

We shall discuss (P3) and (P10) in the later.

4. Arithmetic and harmonic means

In the previous section, we consider a kind of operator mean via geometric mean of 2-operators. But we have not known whether it is the same of geometric mean by Ando-Li-Mathias or not. In this section, we will give a new construction of arithmetic mean of n-operators by using the same method of the previous section.

Theorem 3. *Let $A_1, \cdots, A_n$ be positive operators on a Hilbert space $\mathcal{H}$. Let U and P be defined in* (3.1). *Assume*

$$P_i = \frac{P_{i-1} + UP_{i-1}U^*}{2} \quad \text{and} \quad P_0 = P.$$

Then

$$\lim_{i \to \infty} P_i = I \otimes \frac{A_1 + A_2 + \cdots + A_n}{n}$$

in the norm topology.

Proof. Noting that by concrete computation, we have

$$UPU^* = \mathrm{diag}(\cdots, A_{n-1}, \boxed{A_n}, A_1, \cdots, A_n, A_1, \cdots).$$

Hence we have $P = U^n P U^{n*}$.

Let

$$P_i = \alpha_1^{(i)} P + \alpha_2^{(i)} UPU^* + \cdots + \alpha_n^{(i)} U^{n-1} P U^{n-1*}.$$

Then

$$UP_iU^* = \alpha_n^{(i)} P + \alpha_1^{(i)} UPU^* + \cdots + \alpha_{n-1}^{(i)} U^{n-1} P U^{n-1*},$$

and we have

$$\begin{aligned} P_{i+1} &= \frac{P_i + UP_iU^*}{2} \\ &= \frac{\alpha_n^{(i)} + \alpha_1^{(i)}}{2} P + \frac{\alpha_1^{(i)} + \alpha_2^{(i)}}{2} UPU^* + \cdots + \frac{\alpha_{n-1}^{(i)} + \alpha_n^{(i)}}{2} U^{n-1} P U^{n-1*}. \end{aligned}$$

By this procedure, the n-tuple of coefficients $(\alpha_1^{(i)}, \cdots, \alpha_n^{(i)})$ changes into

$$\left(\frac{\alpha_n^{(i)} + \alpha_1^{(i)}}{2}, \frac{\alpha_1^{(i)} + \alpha_2^{(i)}}{2}, \cdots, \frac{\alpha_{n-1}^{(i)} + \alpha_n^{(i)}}{2}\right).$$

This operation can be represented by an n–by–n matrix A as follows:

$$A = \frac{1}{2}\begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 & 1 \\ 1 & 1 & 0 & \cdots & \cdots & 0 \\ & \ddots & \ddots & \ddots & & \vdots \\ & & \ddots & \ddots & \ddots & \vdots \\ & & & \ddots & \ddots & 0 \\ & & & & 1 & 1 \end{pmatrix} = \frac{I+N}{2},$$

where N is a unitary matrix such that

$$N = \begin{pmatrix} 0 & 0 & \cdots & \cdots & 0 & 1 \\ 1 & 0 & \cdots & \cdots & \cdots & 0 \\ & \ddots & \ddots & & & \vdots \\ & & \ddots & \ddots & & \vdots \\ & & & \ddots & \ddots & \vdots \\ & & & & 1 & 0 \end{pmatrix}.$$

Let U be an n–by–n unitary matrix with the following form:

$$U = \frac{1}{\sqrt{n}}\begin{pmatrix} 1 & \\ \vdots & * \\ 1 & \end{pmatrix}$$

such that $U^*NU = \mathrm{diag}(1, \omega, \cdots, \omega^{n-1})$, where ω means the n-th root of 1 with $\omega \neq 1$. Then

$$\begin{aligned} A^i &= \left(\frac{I+N}{2}\right)^i \\ &= U\begin{pmatrix} 1 & & & \\ & \left(\frac{1+\omega}{2}\right)^i & & \\ & & \ddots & \\ & & & \left(\frac{1+\omega^{n-1}}{2}\right)^i \end{pmatrix}U^* \\ &\longrightarrow U\begin{pmatrix} 1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}U^* = \frac{1}{n}\begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix} \qquad (\text{as } i \to \infty). \end{aligned}$$

Hence for each $k = 1, 2, \cdots, n$, we have

$$\lim_{i\to\infty} \alpha_k^{(i)} = \frac{\alpha_1^{(0)} + \alpha_2^{(0)} + \cdots + \alpha_n^{(0)}}{n}.$$

Here, by $P_0 = P$, we have $\alpha_k^{(0)} = \begin{cases} 1 & (k=1) \\ 0 & (k \neq 1) \end{cases}$, and

$$\lim_{i\to\infty} \alpha_k^{(i)} = \frac{1}{n} \quad \text{for all } k = 1, 2, \cdots, n.$$

Hence we have

$$\lim_{i\to\infty} P_i = \frac{1}{n}(P + UPU^* + U^2PU^{2^*} + \cdots + U^{n-1}PU^{n-1^*}) = I \otimes \frac{A_1 + \cdots + A_n}{n},$$

that is, the proof is complete. □

By the same way, we can define harmonic mean $\mathfrak{H}(A_1, \cdots, A_n)$ of n-operators. Moreover we can see that $\mathfrak{L}(A_1, \cdots, A_n)$ satisfies (P10) (arithmetic-geometric-harmonic means inequality) by using

$$\mathfrak{H}(A, B) \leq A\sharp B \leq \mathfrak{A}(A, B)$$

for all positive invertible operators A and B.

Theorem 3 can be rewritten as the following form, too:

Theorem 3'. *Let $A_1, \cdots, A_n$ be positive operators on a Hilbert space $\mathcal{H}$. Assume*

$$A_k^{(i)} = \frac{A_k^{(i-1)} + A_{k+1}^{(i-1)}}{2} \quad \text{and} \quad A_n^{(i)} = \frac{A_n^{(i-1)} + A_1^{(i-1)}}{2}.$$

Then

$$\lim_{i\to\infty} A_k^{(i)} = \frac{A_1 + \cdots + A_n}{n} \quad \text{for all } k = 1, 2, \cdots, n$$

in the norm topology.

5. On permutation invariant

We have already obtained that $\mathfrak{L}(A_1, \cdots, A_n)$ satisfies properties (P1)–(P10) except (P3). We hope that $\mathfrak{L}(A_1, \cdots, A_n)$ satisfies (P3), i.e., permutation invariant. But there is a counterexample for the problem as follows:

Theorem 4. *There exist positive matrices A, B, C and D such that*

$$\mathfrak{L}(A, B, C, D), \quad \mathfrak{L}(A, B, D, C) \quad \text{and} \quad \mathfrak{L}(A, C, B, D)$$

are all different from each other.

Proof. Let $U(\theta)$ be a unitary matrix defined by

$$U(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

and let A, B, C and D be positive matrices as follows:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$B = U\left(\frac{\pi}{6}\right)\begin{pmatrix} 1 & 0 \\ 0 & 100 \end{pmatrix} U\left(\frac{\pi}{6}\right)^*,$$

$$C = U\left(\frac{10}{9}\pi\right)\begin{pmatrix} 1 & 0 \\ 0 & 20 \end{pmatrix} U\left(\frac{10}{9}\pi\right)^*,$$

$$D = U\left(\frac{7}{9}\pi\right)\begin{pmatrix} 10 & 0 \\ 0 & 4 \end{pmatrix} U\left(\frac{7}{9}\pi\right)^*.$$

Then concrete computing by MATLAB says that

$$\mathfrak{L}(A,B,C,D) = \begin{pmatrix} 7.830092 & 1.614080 \\ 1.614080 & 2.480581 \end{pmatrix},$$

$$\mathfrak{L}(A,B,D,C) = \begin{pmatrix} 8.201878 & 1.882447 \\ 1.882447 & 2.482545 \end{pmatrix},$$

$$\mathfrak{L}(A,C,B,D) = \begin{pmatrix} 7.773366 & 1.675709 \\ 1.675709 & 2.534766 \end{pmatrix}.$$

Hence the proof is complete. □

Since $\mathfrak{L}(A_1, \cdots, A_n)$ does not satisfy permutation invariant, we obtain $\mathfrak{G}(A_1, \cdots, A_n) \neq \mathfrak{L}(A_1, \cdots, A_n)$ for $n \geq 4$, generally. Moreover we obtain the following fact:

Theorem 5. *There exist positive matrices A, B, C and D such that*

$$\mathfrak{G}(A,B,C,D) = \mathfrak{G}(A\sharp B, B\sharp C, C\sharp D, D\sharp A) \tag{5.1}$$

does not hold.

Proof. If (5.1) holds for all positive operators, since the definition of $\mathfrak{L}(A,B,C,D)$, we have

$$\begin{aligned}
\mathfrak{G}(A,B,C,D) &= \mathfrak{G}(A\sharp B, B\sharp C, C\sharp D, D\sharp A) \\
&= \mathfrak{G}((A\sharp B)\sharp(B\sharp C), (B\sharp C)\sharp(C\sharp D), (C\sharp D)\sharp(D\sharp A), (D\sharp A)\sharp(A\sharp B)) \\
&= \cdots \\
&= \mathfrak{G}(\mathfrak{L}(A,B,C,D), \mathfrak{L}(A,B,C,D), \mathfrak{L}(A,B,C,D), \mathfrak{L}(A,B,C,D)) \\
&= \mathfrak{L}(A,B,C,D).
\end{aligned}$$

Hence $\mathfrak{L}(A,B,C,D)$ satisfies (P3). It is a contradiction to Theorem 4. □

Hence we have

$$\mathfrak{G}(A_1, \cdots, A_n) \neq \mathfrak{G}(A_1\sharp A_2, \cdots, A_n\sharp A_1)$$

for $n \geq 4$, generally.

At the end of the paper, we construct a new geometric mean of 4-operators which satisfies (P1)–(P10).

Definition 2. Let A, B, C and D be positive operators. The geometric mean $\mathfrak{GL}(A,B,C,D)$ is defined by

$$\mathfrak{GL}(A,B,C,D) = \mathfrak{L}(\mathfrak{L}(A,B,C,D), \mathfrak{L}(A,B,D,C), \mathfrak{L}(A,C,B,D)).$$

Theorem 6. *Let A, B, C and D be positive operators. The geometric mean $\mathfrak{GL}(A,B,C,D)$ satisfies (P1)–(P10).*

Proof. We have only to prove that $\mathfrak{GL}(A,B,C,D)$ satisfies (P3). By the definition of $\mathfrak{L}(A,B,C,D)$, it invariants under some permutation, exactly, rotation and reflection. So we only consider the case $\mathfrak{L}(A,B,C,D)$, $\mathfrak{L}(A,B,D,C)$ and $\mathfrak{L}(A,C,B,D)$. Since $\mathfrak{L}(X,Y,Z) = \mathfrak{G}(X,Y,Z)$ for each positive operators X, Y, and Z, $\mathfrak{L}(X,Y,Z)$ satisfies (P3). Hence $\mathfrak{GL}(A,B,C,D)$ is so. □

We remark that $\mathfrak{GL}(A,B,C,D)$ is different from $\mathfrak{G}(A,B,C,D)$, for example, let A, B, C and D be defined in the proof of Theorem 4. Then MATLAB says

$$\mathfrak{GL}(A,B,C,D) = \begin{pmatrix} 7.931468 & 1.723281 \\ 1.723281 & 2.494825 \end{pmatrix},$$

$$\mathfrak{G}(A,B,C,D) = \begin{pmatrix} 7.935831 & 1.722989 \\ 1.722989 & 2.493326 \end{pmatrix}.$$

In the number case, geometric mean is only defined by $(a_1 \cdots a_n)^{1/n}$. But since operators are non-commutative, geometric mean can be defined by some forms. So one might think that some geometric means of n-operators are useful in some cases, but some ones also useful in other cases. We can apply geometric mean of n-operators according to the situation. The above geometric mean $\mathfrak{GL}(A,B,C,D)$ is better for computing than the geometric mean by Ando-Li-Mathias.

References

[1] M. Alic, B. Mond, J. Pecaric, and V. Volenec, *Bounds for the Differences of Matrix Means*, SIAM. J. Matrix Anal. Appl. **18** (1997), 119–123.

[2] W.N. Anderson and R.J. Duffin, *Series and parallel addition fo matrices*, J. Math. Anal. Appl., **26** (1969), 576–594.

[3] T. Ando, C.K. Li and R. Mathias, *Geometric means*, Linear Algebra Appl., **385** (2004), 305–334.

[4] G. Corach, H. Porta and L. Recht, *Convexity of the geodesic distance on spaces of positive operators*, Illinois J. Math., **38** (1994), 87–94.

[5] J.I. Fujii, M. Fujii, M. Nakamura, J. Pečarić and Y. Seo, *A reverse inequality for the weighted geometric mean due to Lawson-Lim*, Linear Algebra Appl., **427** (2007), 272–284.

[6] T. Furuta, *$A \geq B \geq 0$ assures $(B^rA^pB^r)^{1/q} \geq B^{(p+2r)/q}$ for $r \geq 0$, $p \geq 0$, $q \geq 1$ with $(1+2r)q \geq p+2r$*, Proc. Amer. Math. Soc., **101** (1987), 85–88.

[7] T. Furuta, J. Mićić, J. Pečarić and Y. Seo, *Mond-Pečarić method in operator inequalities*, Monographs in Inequalities 1, Element, Zagreb, 2005.

[8] K.E. Gustafson and D.K.M. Rao, *Numerical range, the field of values of linear operators and matrices*, Springer-Verlag, New-York, 1997.

[9] G.H. Hardy, J.E. Littlewood and G. Pólya, *Inequalities second edition*, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1988.

[10] F. Kubo and T. Ando, *Means of positive linear operators*, Math. Ann., **246** (1980), 205–224.

[11] J. Lawson and Y. Lim, *Metric convexity of symmetric cones*, Osaka J. Math., **44** (2007), 795–816.

[12] K. Löwner, *Über monotone matrixfunktionen*, Math. Z., **38** (1934), 177–216.

[13] M. Marcus and B.N. Shure, *The numerical range of certain* 0, 1*-matrices*, Linear and Multilinear Algebra, **7** (1979), 111–120.

[14] W. Pusz and S.L. Woronowicz, *Functional calculus for sesquilinear forms and the purification map*, Rep. Math. Phys., **8** (1975), 159–170.

[15] A.C. Thompson, *On certain contraction mappings in a partially ordered vector space*, Proc. Amer. Math. Soc., **14** (1963), 438–443.

[16] G.E. Trapp, *Hermitian semidefinite matrix means and related matrix inequalities–an introduction*, Linear and Multilinear Algebra, **16** (1984), 113–123.

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