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STABLE ELEMENTS IN COHOMOLOGY ALGEBRAS

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1 Introduction

Twenty years have past since the cohomology rings of block ideals of finite group algebras over fields of prime characteristics. Although many properties have been revealed by several authors, there still remain fundamental problems that should be settled. Among them here we would like to revisit the definition of the block cohomology.

M. Linckelmann [6] defined the cohomology of a block ideal to be the subring of the cohomology ring of its defect group consisting of stable elements with respect to the fusion system of subpairs contained in a maximal subpair. He proved that the cohomology of block is embedded into the subring of the Hochshild cohomology ring of the defect group which is *stable* with respect to the source algebra of the block.

Our aim in this report is to show that the converse of this result does hold.

Here we fix notation. Let R be a commutative ring. For G a finite group let δ_G : $H^*(G, R) \to HH^*(RG)$ denote the diagonal embedding, where $HH^*(RG)$ is the Hochshild cohomology ring of the group algebra RG.

2 Stable elemens

First of all we recall the definition of stable elements in Hochshild cohomology rings.

In this section we let R be a symmetric ring and let A and B be finitely generated R-algbras.

Throuout of this section we let X be an (A, B)-bimodule such that as left A-module X is finitely generated and projective and al right B-module X is finitely generated and projective.

Associated with the (A, B)-bimodule X a map $t_X : HH^*(B) \to HH^*(A)$ is defined, which is called the transfer map. The image $\pi_X = t_X(1_B) \in HH^0(A)$, which is isomorphic to $Z(A)$, is called the relatively X-projective element. We should mention that the transfer map and then the relatively X-projective elements depend on the symmetrizing forms of R. See [6] for the detail.

Definition 2.1. A pair $(\zeta, \theta) \in HH^*(A) \oplus HH^*(B)$ is said to be X-stable if the elements $\zeta \otimes \mathrm{Id}_X \in \mathrm{Ext}_{A\otimes B^{\mathrm{op}}}(X, X)$ and $\mathrm{Id}_X \otimes \theta \in \mathrm{Ext}_{A\otimes B^{\mathrm{op}}}(X, X)$ coincide. The element $\zeta \in$

$HH^*(A)$ is said to be X-stable. The set of the X-stable elements in $HH^*(A)$ forms a subring, which is called the X-stable subring and is denoted by $HH^*_X(A)$.

If $(\zeta, \theta) \in HH^*(A) \oplus HH^*(B)$ is X-stable then $(\theta, \zeta) \in HH^*(B) \oplus HH^*(A)$ is X^*-stable.

Linceklmann [6, Corolloary 3.8] says that if relatively projective element $\pi_{X^*} \in Z(B)$ is invertible then an $X \otimes_B X^*$-stable element in $HH^*(A)$ is also X-stable. The following is the converse to this fact.

Proposition 2.1. *Suppose that the relatively projective element $\pi_{X^*} \in Z(B)$ is invertible. Then the followin hold.*

(i) *If $\zeta \in HH^n(A)$ is X stable then the pair $(\zeta, \zeta) \in HH^n(A) \oplus HH^n(A)$ is $X \otimes_B X^*$-stable. In particular we have $HH^n_X(A) \subset HH^n_{X \otimes_B X^*}(A)$.*
(ii) *We have $HH^*_X(A) = HH^*_{X \otimes_B X^*}(A)$; if $\zeta \in HH^*(A)$ is $X \otimes_B X^*$-stable then $(\zeta, \zeta) \in HH^n(A) \oplus HH^n(A)$ is $X \otimes_B X^*$-stable.*

3 Cohomology rings of block ideals

M. Linckelmann defined for a block ideal B of kG the cohomology algebra $H^*(G, B; D_\gamma)$ with respect to a defect pointed group D_γ. It is a subring of the cohomology ring $H^*(D, k)$ of the defect group D consisting of stable elements. Namely

Definition 3.1. Let $i \in \gamma$ be a source idempotent of the block B and let (D, b_D) be a maximal B-Brauer pair associated with i. Then the cohomology ring of the block B is defined as follows.

$$H^*(G, B; D_\gamma)$$
$$= \{ \zeta \in H^*(D, k) \mid \mathrm{res}_Q {}^g\zeta = \mathrm{res}_Q \zeta \quad \forall Q \leqslant D, \ \forall g \in N_G(Q, b_Q), (Q, b_Q) \leq (D, b_D) \}.$$

One of his main theorems is that the diagonal embedding maps the cohomology of the block into the $ikGi$-stable subring of the Hochshild cohomology of the group ring kD.

Theorem 3.1. *It follows that*

$$\delta_D(H^*(G, B; D_\gamma)) \subset HH^*_{ikGi}(kD),$$

where $HH^*_{ikGi}(kD)$ is the subring of the Hochchild cohomology ring $HH^*(kD)$ consisting of the $ikGi$-stable elements.

Note that $X = kGi$ is a source module of the block B and $ikGi = X^* \otimes_B X$ is the source algebra of B.

Let us review the theory of the stable elements in cohomology rings of finite groups. Let $H \leqslant G$. An element $\zeta \in H^*(H, k)$ is said to be G-stable if

$$\mathrm{res}_{H \cap {}^gH}\, \zeta = \mathrm{res}_{H \cap {}^gH}\, {}^g\zeta \quad \forall g \in G.$$

Let us consider the stablity condition above through the diagonal embedding $\delta_H : H^*(H, k) \to HH^*(kH)$.

We see by Linckelmann [6, Lemmas 5.3 and 3.3] and Proposition 2.1 for $\zeta \in H^*(H,k)$ and $g \in G$ that

$$\mathrm{res}_{H\cap {}^gH}\,\zeta = \mathrm{res}_{H\cap {}^gH}\,{}^g\zeta \iff \delta_H\zeta \in HH^*(kH) \text{ is } k[HgH]\text{-stable}$$

Therefore we see that

Lemma 3.2. *For $\zeta \in H^*(H,k)$*

$$\zeta \textit{ is } G\textit{-stable} \iff \delta_H\zeta \in HH^*(kH) \textit{ is } {}_{kH}kG_{kH}\textit{-stable.}$$

In particular, if P is a Sylow p-subgroup of G, then we have for an element $\zeta \in H^*(P,k)$ that

$$\zeta \in \mathrm{Im}\,\mathrm{res}_P \simeq H^*(G,k) \iff \delta_P\zeta \in HH^*(kP) \text{ is } {}_{kP}kG_{kP}\text{-stable.}$$

Comparing Theorem 3.1 and the observation above, we expect that the converse of the theorem would hold; the answer is yes.

Theorem 3.3. *An element $\zeta \in H^*(D,k)$ belongs to the cohomology $H^*(G,B;D_\gamma)$ if and only if the embedding $\delta_D\zeta \in HH^*(kD)$ is $ikGi$-stable.*

Proof. Suppose for $\zeta \in H^*(D,k)$ that $\delta_D\zeta \in HH^*_{ikGi}(kD)$. Then we see from Proposition 2.1 that $(\delta_D\zeta, \delta_D\zeta) \in HH^*(kD) \oplus HH^*(kD)$ is $ikGi$-stable. Thus for an arbitrary direct summand $Y \simeq k[DxD]$, as (kD,kD)-bimodule, of $ikGi$ the pair $(\delta_D\zeta, \delta_D\zeta) \in HH^*(kD) \oplus HH^*(kD)$ is $k[DxD]$-stable. Therefore we have by Linckelmann [6, Lemma 5.3] that

$$\mathrm{res}_{D\cap {}^xD}\,{}^x\zeta = \mathrm{res}_{D\cap {}^xD}\,\zeta.$$

We would like to show that $\zeta \in H^*(G,B;D_\gamma)$. It suffices to show that the stablity condition in Definition 3.1 holds for subpairs (Q,b_Q) belonging to a conjugation family $\mathscr{F} \subseteq \{(Q,b_Q) \mid (Q,b_Q) \leq (D,b_D)\}$.

Furthermore, the family

$$\mathscr{F} = \{(Q,b_Q) \mid (Q,b_Q) \leq (D,b_D) \text{ is extremal}\}$$

is a conjugation family; if $(Q,b_Q) \leq (D,b_D)$ is extremal, then $C_D(Q)$ is a defect group of the block b_Q of $kC_G(Q)$. (Alperin–Broué [1])

Linckelmann [5, Lemma 3.3 (v)] says for a subpair (Q,b_Q) that if $C_D(Q)$ is a defect group of b_Q, then for $g \in N_G(Q,b_Q)$, as a (kQ,kQ)-bimodule, $k[gQ]$ is a direct summand of $ikGi$. Now as a (kD,kD)-bimodule we can write $ikGi \simeq \bigoplus_{x\in I} k[DxD]$ as a direct sum of indecomposables and let us assume as (kQ,kQ)-bimodules that

$$k[gQ] \mid k[DxD].$$

As a $k[Q\times Q^{\mathrm{op}}]$-module $k[gQ]$ has a trivial source and $\mathrm{vtx}\,k[gQ] = {}^{(g,1)}\Delta Q$:

$$k[gQ] = k[Q\times Q^{\mathrm{op}}] \otimes_{k[{}^{(g,1)}\Delta Q]} k.$$

As a $k[D \times D^{\mathrm{op}}]$-module, $k[DxD]$ has a trivial source and vtx $k[DxD] = {}^{(x,1)}\Delta\left({}^{x^{-1}}D \cap D\right)$:

$$k[DxD] = k[D \times D^{\mathrm{op}}] \otimes_{k\left[{}^{(x,1)}\Delta({}^{x^{-1}}D \cap D)\right]} k.$$

Therefore we see that

$$k[Q \times Q^{\mathrm{op}}] \otimes_{k[{}^{(g,1)}\Delta Q]} k \;\Big|\; {}_{k[Q \times Q^{\mathrm{op}}]}k[D \times D^{\mathrm{op}}] \otimes_{k\left[{}^{(x,1)}\Delta({}^{x^{-1}}D \cap D)\right]} k.$$

Applying Mackey decomposition to the right hand side we have for an element $(a, b^{-1}) \in D \times D^{\mathrm{op}}$ that

$$k[Q \times Q^{\mathrm{op}}] \otimes_{k[{}^{(g,1)}\Delta Q]} k \;\Big|\; k[Q \times Q^{\mathrm{op}}] \otimes_{k\left[Q \times Q^{\mathrm{op}} \cap {}^{(a,b^{-1})(x,1)}\Delta({}^{x^{-1}}D \cap D)\right]} k.$$

Thus we may assume by Green's indecomposablity theorem that

$${}^{(g,1)}\Delta Q = Q \times Q^{\mathrm{op}} \cap {}^{(a,b^{-1})(x,1)}\Delta({}^{x^{-1}}D \cap D).$$

From this equation we see for some element $y \in C_G(Q)$ that $g = axby$. Notice moreover that ${}^{b}Q \leq {}^{x^{-1}}D \cap D$.

Recall that

$$\mathrm{res}_{D \cap {}^{x}D}\, \zeta = \mathrm{res}_{D \cap {}^{x}D}\, {}^{x}\zeta = {}^{x}\mathrm{res}_{{}^{x^{-1}}D \cap D}\zeta.$$

Using the decription that $g = axby$ and the equation above, we can verify that the stablity condition does hold:

$$\mathrm{res}_Q\, {}^{g}\zeta = \mathrm{res}_Q\, \zeta.$$

□

4 Characteristic biset

Let P be a p-group and let $\mathscr{F}$ be a fusion system on P. Then the cohomology ring $H^*(\mathscr{F})$ of $\mathscr{F}$ is defined in a similar way to that of the cohomology ring of block ideals. Broto–Levi–Oliver [2] says that there exists a (P, P)-biset X, which induces a map

$$t_X : H^*(P, k) \to H^*(P, k)$$

with nice properties. In particular

$$t_X(H^*(P, k)) = H^*(\mathscr{F}).$$

Such a biset X is called a characteristic biset.

If P is a Sylow p-subgroup of G and $\mathscr{F}_P(G)$ is the fusion system, then the map $t_X : H^*(P, k) \to H^*(P, k)$ is the following: for $\zeta \in H^*(P, k)$

$$t_X : \zeta \mapsto \mathrm{res}_P\, \mathrm{tr}^G \zeta = \sum_{PgP \in P\backslash G/P} \mathrm{tr}^P\, \mathrm{res}_{P \cap {}^{g}P}\, {}^{g}\zeta.$$

Note that the following commutes:

$$\begin{tikzcd} H^*(P,k) \arrow[r, "\delta_P"] \arrow[d, "t_X"'] \arrow[dr, phantom, "\circlearrowleft"] & HH^*(kP) \arrow[d, "t_{kP}kG_{kP}"] \\ H^*(P,k) \arrow[r, "\delta_P"'] & HH^*(kP) \end{tikzcd}$$

The cohomology of the fusion system $\mathscr{F}_{(D,b_D)}(B)$ is $H^*(G, B; D_\gamma)$.

Let X be a characteristic (D, D)-biset for $\mathscr{F}_{(D,b_D)}(B)$. Using the properties of the map $t_X : H^*(D, k) \to H^*(D, k)$, Linckelmann obtained the stratification theorem for block varieties of modules.

However we would like to get the map t_X more convenient to handle with. The reason is as follows. Let $DC_G(D) \leqslant H \leqslant G$ and let C be a block ideal of kH that corresponds to B under Brauer correspondence and has defect group D. Under some further conditions there should exit maps

$$r : H^*(G, B) \to H^*(H, C),$$
$$t : H^*(H, C) \to H^*(G, B).$$

These maps should have the properties similar to the restriction maps and corestriction maps for cohomology rings of finite groups.

If Y is a characteristic biset for the block C, then we should have the following commutative diagram

$$\begin{tikzcd} H^*(H,C) \arrow[rr, dashed, "t"] & & H^*(G,B) \\ & H^*(D,k) \arrow[ul, "t_Y"] \arrow[ur, "t_X"'] & \end{tikzcd}$$

To define the map $t : H^*(H, C) \to H^*(G, B)$ the maps t_X and t_Y must be easy to understand.

Now let us consider the restriction $t : H^*(D, k) \to H^*(D, k)$ of the transfer map t_{ikGi} : $HH^*(kD) \to HH^*(kD)$ defined by the (kD, kD)-bimodule $ikGi$:

$$\begin{tikzcd} H^*(D,k) \arrow[r, "\delta_D"] \arrow[d, "t"'] \arrow[dr, phantom, "\circlearrowleft"] & HH^*(kD) \arrow[d, "t_{ikGi}"] \\ H^*(D,k) \arrow[r, "\delta_D"'] & HH^*(kD) \end{tikzcd}$$

We hope that

$$H^*(G, B; D_\gamma) = t(H^*(D, k)).$$

We note that

$$H^*(G, B; D_\gamma) = t(H^*(D, k)) \Longleftrightarrow \delta_D t(H^*(D, k)) \subset HH^*_{ikGi}(kD).$$

If the inclusion above holds, then we have by our Theorem 3.3 for $\zeta \in H^*(G, B; D_\gamma)$ that $t(H^*(D, k)) \subset H^*(G, B; D_\gamma)$ and the following diagram commutes:

$$\begin{tikzcd} H^*(G, B; D_\gamma) \arrow[rr, "\cdot\pi_{ikGi}"] \arrow[dr, hook] & & H^*(G, B; D_\gamma) \\ & H^*(D, k) \arrow[ur, "t"'] & \end{tikzcd}$$

Since the relatively projective element $\pi_{ikGi} \in k$ does not vanish, the horizontal map is an isomorphism and thus we have $H^*(G, B; D_\gamma) = t(H^*(D, k))$.

If the defect group is abelian or normal in G then the equality we want holds. However in general cases we have no progress so far.

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