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Support Varieties for Modules over Stacked Monomial Algebras[1]

東京理科大学 理学部数学科 古谷貴彦 (Takahiko Furuya)
Department of Mathematics,
Tokyo University of Science
e-mail: furuya@ma.kagu.tus.ac.jp

Introduction

Throughout let K be an algebraically closed field, and let $\Lambda = KQ/I$ be an indecomposable finite-dimensional algebra of infinite global dimension, where Q is a finite quiver and I is an admissible ideal. Let Λ^e be the enveloping algebra $\Lambda \otimes_K \Lambda^{\mathrm{op}}$. Then the Hochschild cohomology ring of Λ is defined to be $\mathrm{HH}^*(\Lambda) = \mathrm{Ext}^*_{\Lambda^e}(\Lambda, \Lambda) = \bigoplus_{n\geq 0} \mathrm{Ext}^n_{\Lambda^e}(\Lambda, \Lambda)$ with the Yoneda product. We denote by $\mathcal{N}$ the ideal in $\mathrm{HH}^*(\Lambda)$ which is generated by the homogeneous nilpotent elements, and by $\mathfrak{r}$ the radical of Λ.

In this note, we study the support varieties for modules over (D, A)-stacked monomial algebras. (D, A)-stacked monomial algebras was introduced by Green and Snashall in [4], where the structure of their Hochschild cohomology rings was completely described. They arose from the study of the Ext algebra $\mathrm{Ext}^*_\Lambda(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r})$, and in [5, 6] their interesting and important properties are described. An algebra Λ of infinite global dimension is called a *(D, A)-stacked monomial algebra* if (i) Λ is a monomial algebra; and (ii) all projective modules in a minimal projective resolution of $\Lambda/\mathfrak{r}$ over Λ is generated in a single degree, or equivalently, the Ext algebra is finitely generated as a K-algebra (where $D \geq 2$ and $A \geq 1$ are positive integers which are uniquely determined by minimal generators of I; see, for the detail, [4]). Note that, in the original definition [4, Definition 3.1], (D, A)-stacked monomial algebras are defined in terms of the notion of overlaps of paths and are not assumed to be of infinite global dimension. It is known that the class of (D, A)-stacked monomial algebras contains Koszul monomial algebras and D-Koszul monomial algebras of [1].

Support varieties for modules over any finite-dimensional algebra Λ were introduced by Snashall and Solberg in [7] using the Hochschild cohomology ring. Recall that the *support variety* $V(M)$ of a finitely generated Λ-module M is defined as

$$V(M) = \{\mathfrak{m} \in \operatorname{MaxSpec} \mathrm{HH}^*(\Lambda)/\mathcal{N} \mid \operatorname{Ann}_{\mathrm{HH}^*(\Lambda)} \mathrm{Ext}^*_\Lambda(M, M) \subseteq \mathfrak{m}'\}$$

where $\mathfrak{m}'$ denotes the inverse image of $\mathfrak{m}$ in $\mathrm{HH}^*(\Lambda)$. Then we necessarily have a unique maximal graded ideal $\mathfrak{m}_{\mathrm{gr}}$ in $\mathrm{HH}^*(\Lambda)/\mathcal{N}$, and $\{\mathfrak{m}_{\mathrm{gr}}\} \subseteq V(M)$ for all non-zero finitely generated Λ-modules M ([7, Proposition 3.4]). The variety of M is then said to be trivial if $V(M) = \{\mathfrak{m}_{\mathrm{gr}}\}$.

In this note we are interested in the support varieties of simple modules over (D, A)-stacked monomial algebra. We give necessary and sufficient conditions for a

[1]This note is a survey article of a joint work with Nicole Snashall. See [3] for the detail.

simple Λ-module to have trivial variety. We also provide new structural information on the algebra Λ, namely, we show that if every simple Λ-module has nontrivial variety then $A = 1$ and so Λ is a D-Koszul algebra.

1. Background

In this section, we recall from [3, 4] the notations and definitions which we need in this note.

Let $\mathcal{Q}$ be a finite quiver. A *closed path C* in $\mathcal{Q}$ at the vertex v is a non-trivial path C in the path algebra $K\mathcal{Q}$ such that $C = vCv$ for some vertex v. For a closed path C at the vertex v, v is said to be *not internal* to C if $C = v\sigma_1 v\sigma_2 v$ for paths σ_1, σ_2 implies that $\sigma_1 = v$ or $\sigma_2 = v$.

For $A \geq 1$, a *closed A-trail* T in $\mathcal{Q}$ is a non-trivial closed path $T = \alpha_0\alpha_1 \cdots \alpha_{m-1}$ in $K\mathcal{Q}$ such that $\alpha_0, \ldots, \alpha_{m-1}$ are all distinct paths of length A. Put $T_0 = T$ and

$$\begin{aligned} T_1 &= \alpha_1 \cdots \alpha_{m-1}\alpha_0 \\ T_2 &= \alpha_2 \cdots \alpha_0\alpha_1 \\ &\vdots \\ T_{m-1} &= \alpha_{m-1}\alpha_0 \cdots \alpha_{m-2}, \end{aligned}$$

then we call the set $\{T_0, T_1, \ldots, T_{m-1}\}$ a *complete set of closed A-trails* on the A-trail $T = \alpha_0\alpha_1 \cdots \alpha_{m-1}$.

Let $d \geq 2$ and set $d = Nm + l$ where $0 \leq l \leq m-1$ and $N \geq 0$. For $t \in \mathbb{N}$, let $[t] \in \{0, 1, \ldots, m-1\}$ denote the residue of t modulo m. Let $W = T_0^N \alpha_0\alpha_1 \cdots \alpha_{l-1}$ with the conventions that if $N = 0$ then $T_0^N = \mathfrak{o}(\alpha_0)$ and if $l = 0$ then $W = T_0^N$. More generally, for $k = 0, 1, \ldots, m-1$, define $\sigma^k(W) = T_k^N \alpha_k\alpha_{k+1} \cdots \alpha_{k+l-1}$ with the conventions that

(i) if $t \geq m$ then $\alpha_t = \alpha_{[t]}$,

(ii) if $N = 0$ then $T_k^N = e_k$, and

(iii) if $l = 0$ then $\sigma^k(W) = T_k^N$.

We define $\rho_T := \{W, \sigma(W), \ldots, \sigma^{m-1}(W)\}$ and call it the *set of paths of length dA that are associated to the A-trail T*. Note that $\{W, \sigma(W), \ldots, \sigma^{m-1}(W)\}$ is also the set of paths of length dA that is associated to each A-trail T_k for $k = 0, \ldots, m-1$.

Let Λ be a (D, A)-stacked monomial algebra. Then, by [4, Proposition 3.3], we have $D = dA$ for some $d \geq 2$.

Let $C_1, \ldots, C_u$ be all the closed paths in the quiver $\mathcal{Q}$ at the vertices $v_1, \ldots, v_u$ respectively, such that for each C_i with $1 \leq i \leq u$, we have $C_i \neq p_i^{r_i}$ for any path p_i with $r_i \geq 2$, $C_i^d \in \rho$, and there are no overlaps of C_i^d with any relation in $\rho \setminus \{C_i^d\}$. (Note that it follows that $\ell(C_i) = A$.)

Let $T_{u+1}, \ldots, T_r$ be all the distinct closed A-trails in the quiver $\mathcal{Q}$ such that for each T_i with $u+1 \leq i \leq r$, the set ρ_{T_i} of paths of length $D = dA$ which are associated to the trail T_i is contained in ρ but, if $T_i = \alpha_{i0}\alpha_{i1}\cdots\alpha_{im_i-1}$, then each path α_{ij} of length A has no overlaps with any relation in $\rho \setminus \rho_{T_i}$. (We assume that there is no repetition amongst these closed paths and closed A-trails, that is, $\{C_1, \dots, C_u\} \cap \{T_{u+1}, \dots, T_r\} = \emptyset$.)

Then, [4, Theorem 3.4] says that there is a ring isomorphism

$$\mathrm{HH}^*(\Lambda)/\mathcal{N} \cong K[x_1, \dots, x_r]/\langle x_a x_b \mid a \neq b \rangle. \tag{1}$$

(See [4, Theorem 3.4] and [3] for the detail of the maps corresponding to $x_1, \dots, x_r$.)

2. Support varieties for simple modules

Now, we can use (1) to provide the necessary and sufficient condition for the support varieties for simple modules to be nontrivial.

Under the notation in Section 1, let S_j denote the simple module corresponding to the vertex v_j of $\mathcal{Q}$ for $1 \leq j \leq u$. Then we say that S_j is *associated to the closed path* C_j. Similarly, for k with $0 \leq k \leq m_{j-1}$, let S_{jk} be the simple module corresponding to $\mathfrak{o}(T_{j,k})$, where $\mathfrak{o}(T_{j,k})$ denotes the origin of $\mathfrak{o}(T_{j,k})$. Then we say that the simple modules S_{jk}, for $0 \leq k \leq m_{j-1}$, are *associated to the closed A-trail* T_j.

Theorem 2.1. (FS]) Let S be a simple module. Then the variety of S is trivial if and only if S is not associated to one of the closed paths $C_1, \dots, C_u$ or to one of the closed A-trails $T_{u+1}, \dots, T_r$ in the quiver $\mathcal{Q}$.

Example. (a) Let $\Lambda = K\mathcal{Q}/I$ where $\mathcal{Q}$ is the quiver

$$\begin{tikzcd}
2 \arrow[dd, "\beta"'] & & 4 \arrow[dd, "\eta"] \\
& 1 \arrow[ul, "\alpha"'] \arrow[ur, "\zeta"] & \\
3 \arrow[ur, "\gamma"'] & & 5 \arrow[ul, "\theta"]
\end{tikzcd}$$

and $I = \langle \alpha\beta, \beta\gamma, \gamma\alpha, \zeta\eta, \eta\theta, \theta\zeta \rangle$. Then Λ is a Koszul monomial algebra, so that Λ is a $(2,1)$-stacked monomial algebra. By [4, Theorem 3.4] $\mathrm{HH}^*(\Lambda)/\mathcal{N}$ is isomorphic to the subalgebra $K[x,y]/(xy)$ of $\mathrm{HH}^*(\Lambda)$ where $\deg x = \deg y = 6$. Also, all simple modules of Λ are associated to closed A-trails in $\mathcal{Q}$. In fact, the simple module corresponding to the vertex 1 is associated to both the closed 1-trails $\alpha\beta\gamma$ and $\zeta\eta\theta$. Thus, by Theorem 2.1, the varieties of all simple modules of Λ are nontrivial.

(b) Let $\Lambda = K\mathcal{Q}/I$ where $\mathcal{Q}$ is the quiver

$$\begin{tikzcd}
1 \arrow[r, "\alpha"] & 2 \arrow[d, "\beta"] \\
4 \arrow[u, "\delta"] & 3 \arrow[l, "\gamma"]
\end{tikzcd}$$

and $I = \langle \alpha\beta\gamma\delta\alpha\beta, \gamma\delta\alpha\beta\gamma\delta \rangle$. Then, Λ is a $(6,2)$-stacked monomial algebra, and the path $\alpha\beta\gamma\delta$ is a closed 2-trail. It follows by [4, Theorem 3.4] that $\mathrm{HH}(\Lambda)/\mathcal{N} \cong$

$K[x]/(x)$, where $\deg x = 2$. Also, the simple modules S_1, S_3 corresponding to the vertices 1, 3 are associated to the closed 2-trail $\alpha\beta\gamma\delta$. Hence, by Theorem 2.1, the varieties of S_1, S_3 are nontrivial, whereas the varieties of the simple modules corresponding to the vertices 2, 4 are trivial.

Note that, in both examples above, we can directly show that Λ satisfies the finiteness conditions **(Fg1)**, **(Fg2)** found in [2]. Then, we see form [2, Theorem 5.2] that, for a finitely generated module M, the variety of M is trivial if and only if the projective dimension of M is finite. For algebras that do not satisfy the finiteness conditions, see [3].

Finally, we give information on the structure of (D,A)-stacked monomial algebras, where all simple modules have nontrivial variety.

Theorem 2.2 ([FS]) Let $\Lambda = KQ/I$ be a (D, A)-stacked monomial algebra. Suppose that each simple module has nontrivial variety. Then $A = 1$. Thus Λ is a D-Koszul monomial algebra.

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