Remarks on source algebras of blocks with cyclic defect groups

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1. Introduction and a kind of history

This is a part of joint work with Naoko Kunugi [7].

In representation theory of finite groups, particularly, in modular representation theory, studying structure of p-blocks (block algebras) of finite groups G, where p is a prime number, is one of the most important and exciting things.

Let us look at, for instance, a case where a p-block algebra A of a finite group G has a *cyclic* defect group P. A celebrated work in such a case done by E.C.Dade [3] is one of the high points in representation theory of finite groups. Of course, there had been wonderful works due to R.Brauer, which we can not forget, see [2]. Anyway, after Dade's work, there are several important results, which are, independently, done by G.J.Janusz [5] and H.Kupisch [8], where they describes all finitely generated indecomposable kG-modules that belong to A, where k is an algebraically closed field of characteristic $p > 0$ and A is the block (algebra) of the group algebra kG, as above. Speaking of cyclic defect groups, we should recall that F.Kasch, M.Kneser and H.Kupisch had proven already that, for a finite group G and k mentioned above, the group algebra kG has only finitely many (finitely generated) non-isomorphic indecomposable kG-modules if and only if a Sylow p-subgroup of G is cyclic, in their paper [6] which has only two

pages. Of course, this was a motivation to get a theorem such that the p-block algebra A of kG has only finitely many non-isomorphic indecomposable kG-modules belonging to A if and only if the defect group P of A is cyclic. Essentially and implicitly, depending on these results due to Dade, Janusz-Kupisch, and so on, many important conjectures in representation theory of finite groups have successfully been solved by E.C.Dade [4], J.Rickard [14], M.Linckelmann ([9], [11]) and so on. Here by the conjectures we mean such as Alperin's Weight Conjecture, Dade's Conjecture and Broué's Abelian Defect Group Conjecture.

Then, what else to do in cyclic defect groups case? Is there still anything interesting and important to do in such cyclic defect groups case? Yet, there are still, we would say, many important and interesting problems and questions in block theory of finite groups even where the blocks have *cyclic* defect groups, as far as we understand.

Now, here comes the thing. Namely, in this short note, we shall present kind of interesting theorems in a cyclic defect groups case, which are actually quite useful and helpful to obtain main results in our joint work by Kunugi and the author [7, Theorems 1.2 and 1.6; Corollaries 1.3, 1.4 and 1.8]. We shall, in fact, not mention these main results in [7], but we shall present several theorems which are due to other people, essentially, such as L.Puig, B.Külshammer and M.Linckelmann.

2. Main ingredients

Notation 2.1. Throughout this note we use the following notation and terminology. We denote by G a finite group always, and let p be a prime. Then, a triple $(\mathcal{K}, \mathcal{O}, k)$ is so-called a p-modular system, which is big enough for all finitely many finite groups which we are looking at, including G. Namely, $\mathcal{O}$ is a complete descrete valuation ring, $\mathcal{K}$ is the quotient field of $\mathcal{O}$, $\mathcal{K}$ and $\mathcal{O}$ have characteristic zero, and k is the residue field $\mathcal{O}/\mathrm{rad}(\mathcal{O})$ of $\mathcal{O}$ such that k has characteristic p. We mean by "big enough" above that $\mathcal{K}$ and k are both splitting fields for the finite groups mentioned above.

Let A be a block of $\mathcal{O}G$ (and sometimes of kG) with a defect group P. We always assume that P is cyclic and $P \neq 1$. Then, we write P_1, N_1 and N for the unique subgroup of P of order p, $N_G(P_1)$ and $N_G(P)$, respectively. Since P is cyclic, we know $N \subseteq N_1$. Hence we have other two block algebras B_1 and B of $\mathcal{O}N_1$ and $\mathcal{O}N$, respectively, such that all these blocks A, B_1 and B correspond each other via the Brauer correspondence with respect to P. We denote by E the inertial quotient for A with respect to P, namely, $E := N_G(P,e)/P \cdot C_G(P)$ where e is a block idempotent of $kG_G(P)$ such that $A = (kC_G(P)e)^G$ (block induction), and $N_G(P,e) := \{g \in N_G(P) \mid g^{-1}eg = e\}$, see [15, p.346].

In general, let A be a block algebra of $\mathcal{O}G$ with a defect group P. Then, we say that i is a *source idempotent* of A with respect to P and that the algebra iAi is a *source algebra* of A with respect to P, if i is a primitive idempotent of $A^P := \{a \in A \mid u^{-1}au = a, \ \forall u \in P\}$ and i satisfies that $\mathrm{Br}_P^A(i) \neq 0$, where Br_P^A is the Brauer homomorphism in A with respect to P, see [15, p.321]. For other notation and terminology, see the books of [1], [12] and [15].

The following three results are quite remarkable.

Theorem 2.2 (Dade-Janusz-Kupisch). *The Brauer trees of B_1 and B are both stars with exceptional vertex in the center. In other words, the k-algebras B_1 and B are both Nakayama (uniserial) algebras.*

Theorem 2.3 (Puig [13]). *Let j be a source idempotent of B with respect to P. Then, the source algebra jBj of B with respect to P has the following structure:*

$$jBj \cong \mathcal{O}[P \rtimes E]$$

as interior P-algebras, where E is the inertial quotient for A (and hence for B) with respect to P.

Theorem 2.4 (Linckelmann [9], [11]). *Let j_1 be a source idempotent of B_1 with respect to P. Then, the source algebra $j_1B_1j_1$ of B_1 with*

respect to P has the following structure:

$$j_1 B_1 j_1 \ \cong \ \mathrm{End}_{\mathcal{O}}(V) \otimes_{\mathcal{O}} \mathcal{O}[P \rtimes E]$$

as interior P-algebras, where E is the inertial quotient for A (and hence for B and B_1) with respect to P, and V is an indecomposable **endo-permutation** *right $\mathcal{O}P$-lattice with vertex P and with $P_1 \subseteq \ker(V)$.*

By looking at Theorems 2.3 and 2.4, the following question then comes up quite naturally, we believe. That is to say,

Question 2.5. On the other hand, if we assume that our original biggest block algebra A has a Brauer tree which is a star with the exceptional vertex in the center, then what can we say about the structure of a source algebra, say, iAi of A with respet to P? Of course, here i is a source idempotent of A with respect to P. Actually, the structure of iAi has been essentially determined by M.Linckelmann [11], but, we would say, implicitly. Thus, it should be worthwhile and meaningful to state it explicitly in here. Namely, we get the following:

Theorem 2.6 (see M.Linckelmann [11], see also [7]). *Let A be a block algebra of $\mathcal{O}G$ with a cyclic defect group P with $P \neq 1$. Let i be a source idempotent of A with respect to P. Then, the following three conditions are equivalent:*

(1) *The block algebra $A \otimes_{\mathcal{O}} k$ over k is a Nakayama (uniserial) algera.*

(2) *The Brauer tree of A is a star with the exceptional vertex in the center.*

(3) *The source algebra iAi has the following structure:*

$$iAi \ \cong \ \mathrm{End}_{\mathcal{O}}(V) \otimes_{\mathcal{O}} \mathcal{O}[P \rtimes E] \quad \text{or} \quad \mathrm{End}_{\mathcal{O}}(\Omega V) \otimes_{\mathcal{O}} \mathcal{O}[P \rtimes E]$$

as interior P-algebras, where V is the same as in Theorem 2.4.

Proof. See [7, Proposition 2.11]. ∎

Acknowledgment. The author is grateful to Professor Hiroki Sasaki for organizing such a wonderful meeting held in Shinshu University as a RIMS meeting during 31 August – 4 September, 2009.

REFERENCES

[1] J.L. Alperin, Local Representation Theory, Cambridge Univ. Press, 1986.

[2] J.L. Alperin, MathReview MR1235022 (94i:20012).

[3] E.C. Dade, Blocks with cyclic defect groups. Ann. of Math. **84** (1966), 20–48.

[4] E.C. Dade, Counting characters in blocks with cyclic defect groups I, J.Algebra **186** (1996), 934–969.

[5] G.J. Janusz, Indecomposable modules for finite groups, Ann. of Math. **89** (1969), 209–241.

[6] F. Kasch, M. Kneser and H. Kupisch, Unzerlegbare modulare Darstellungen endlicher Gruppen mit zyklischer p-Sylow-Gruppe, Arch. Math. **8** (1957), 320–321.

[7] S. Koshitani and N. Kunugi, Trivial source modules in blocks with cyclic defect groups, To appear in Math. Z. DOI 10.1007/s00209-009-0508-9 (2009).

[8] H. Kupisch, Unzerlegbare Moduln endlicher Gruppen mit zyklischer p-Sylow-Gruppe, Math. Z. **108** (1969), 77–104.

[9] M. Linckelmann, Variations sur let blocs à groupes de défaut cycliques. Séminaire sur les Groupes Finis, Tome IV, Publ. Univ. Paris 7, **tome 34** (1988), 1–83.

[10] M. Linckelmann, The source algebras of blocks with a Klein four defect groups. J. Algebra **167** (1994), 821–854.

[11] M. Linckelmann, The isomorphism problem for cyclic blocks and their source algebras. Invent. Math. **125** (1996), 265–283.

[12] H. Nagao and Y. Tsushima, Representations of Finite Groups, Academic Press, New York, 1988.

[13] L. Puig, Nilpotent blocks and their source algebras. Invent. Math. **93** (1988), 77–116.

[14] J. Rickard, Derived categories and stable equivalences, J.Pure Appl. Algebra **61** (1989), 303–317.

[15] J. Thévenaz, G-Algebras and Modular Representation Theory. Clarendon Press, Oxford, 1995.