数理解析研究所講究録
第 1679 巻 2010 年 61-70

On fusion systems and isometries of characters

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1. Introduction

Let G be a finite group, p a prime. The structure of p-local objects of G is important when investigating structure or representations of G. In particular, a fusion system over a Sylow p-subgroup P is crucial. Sometimes G and the normalizer $N_G(P)$ of P in G have the same saturated fusion systems over P. For example, it is always so if P abelian. From a modular representation theoretic point of view, it is interesting to know whether, in general, the principal blocks of two groups having common Sylow p-subgroups P and giving the same saturated fusion systems over P have similar structure. In the case where P is abelian, Broué conjectured that between the principal blocks of G and $N_G(P)$ there is a perfect isometry, [3]. If P is not abelian, we can not expect the existence of such a nice isometry. However, if two groups with a common Sylow p-subgroup P give the same saturated fusion systems over P, then their principal blocks have quite often the same numbers of irreducible characters. In each of these cases, there exists at least a bijection between the sets of ordinary irreducible characters, but, usually there is not a perfect isometry between their principal blocks. However, in [10], we define a new type of isometry, which is a complete generalization of the perfect isometry defined by Broué, and prove that for $p = 3$ or 5 there exists the new isometry between the principal blocks of two groups having extra special p-groups of order p^3 and exponent p as their Sylow p-subgroups over which they give the same saturated fusion systems. But, we do not know the relationship between this new isometry and the structural relationship of their module categories. We hope that in the future such interesting phenomena will be regarded as shadows of some equivalences, like perfect isometries are considered as those of derived equivalences.

In this note we give a brief exposition of our paper [10]. For terminologies and notions of modular representation theory of finite groups, we refer to [8].

2. Fusion systems

In this section, we review fusion systems following [12].

Definition 2.1. *Let P be a finite p-group. A fusion system $\mathcal{F}$ over P is a category whose objects are the subgroups of P, and the morphism set $\mathrm{Hom}_{\mathcal{F}}(Q_1, Q_2)$ for subgroups Q_1 and Q_2 of P satisfies the following.*

(i) *Elements in the morphism set $\mathrm{Hom}_{\mathcal{F}}(Q_1, Q_2)$ are injective group homomorphisms and all the homomorphisms from Q_1 to Q_2 given by the conjugation by the elements of P lie in $\mathrm{Hom}_{\mathcal{F}}(Q_1, Q_2)$.*

(ii) *Every element f in $\mathrm{Hom}_{\mathcal{F}}(Q_1, Q_2)$ can be written as the composition of the isomorphism $f : Q_1 \to f(Q_1)$ and the inclusion $f(Q_1) \subseteq Q_2$, and the both are morphisms of $\mathcal{F}$.*

Let Q be a subgroup of P. Then the set $\mathrm{Aut}_Q(Q)$ of Q-conjugations on Q is a normal subgroup of $\mathrm{Hom}_{\mathcal{F}}(Q, Q)$. We use $\mathrm{Out}_{\mathcal{F}}(Q)$ or simply $\mathrm{Out}(Q)$ to denote $\mathrm{Hom}_{\mathcal{F}}(Q, Q)/\mathrm{Aut}_Q(Q)$. If there is an isomorphism in $\mathrm{Hom}_{\mathcal{F}}(Q_1, Q_2)$, then we say that Q_1 and Q_2 are $\mathcal{F}$-conjugate. The centralizer of Q in P is denoted by $C_P(Q)$.

Definition 2.2. *Let $\mathcal{F}$ be a fusion system over P.*

(i) *A subgroup Q of P is said to be fully centralized in $\mathcal{F}$, if $|C_P(Q)| \geq |C_P(Q_1)|$ for all those $Q_1 \leq P$ that are $\mathcal{F}$-conjugate to Q.*

(ii) *A subgroup Q of P is said to be fully normalized in $\mathcal{F}$, if $|N_P(Q)| \geq |N_P(Q_1)|$ for all those $Q_1 \leq P$ that are $\mathcal{F}$-conjugate to Q.*

(iii) *We say that $\mathcal{F}$ is a saturated fusion system if the following are satisfied.*

(a) *Every fully normalized subgroup Q of P is fully centralized and $\mathrm{Aut}_P(Q)$ is a Sylow p-subgroup of $\mathrm{Hom}_{\mathcal{F}}(Q, Q)$.*

(b) *For $Q \leq P$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q, P)$ with $\varphi(Q)$ is fully centralized, let $N = \{g \in N_P(Q) \mid \varphi c_g \varphi^{-1} \in \mathrm{Aut}_P(\varphi(Q))\}$, where c_g is the conjugation by g. Then, there is $\varphi' \in \mathrm{Hom}_{\mathcal{F}}(N, P)$ such that the restriction of φ' to Q is equal to φ.*

Suppose that P is a Sylow p-subgroup of a finite group G. We indicate this situation by $P \in \mathrm{Syl}_p(G)$. Then G gives rise to a saturated fusion system $\mathcal{F}_P(G)$ over P (Proposition 1.3 of [2]). This is given by defining $\mathrm{Hom}_{\mathcal{F}_P(G)}(Q_1, Q_2)$ for subgroups Q_1 and Q_2 of P as the set of conjugation maps from Q_1 to Q_2 given by the elements of G.

Example 2.3. *Suppose that P is abelian. Then saturated fusion systems over P are in one-to-one correspondence with representatives of conjugacy classes of p'-subgroups of $\mathrm{Out}(P)$. Moreover, if $P \in \mathrm{Syl}_p(G)$, then $\mathcal{F}_P(G) = \mathcal{F}_P(N_G(P))$.*

Example 2.4. *For an odd prime p, let*

$$p_+^{1+2} = \left\{ \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \in GL(3,p) \mid a,b,c \in \mathbb{F}_p \right\}.$$

This p_+^{1+2} is called an extra special p-group of order p^3 and exponent p. Saturated fusion systems over p_+^{1+2} are classified by [12].

In particular, for $p = 3$, there are fifteen saturated fusion systems over 3_+^{1+2}. Finite groups with a Sylow p-subgroup p_+^{1+2} and without non-trivial normal p'-subgroups can be classified by using the classification theorem of finite simple groups. For example the following holds.

Let J_4 be the largest sporadic simple group of Janko. We know that $P = 3_+^{1+2} \in \mathrm{Syl}_3(J_4)$. Suppose that a finite group G satisfies $3_+^{1+2} \in \mathrm{Syl}_3(G)$, $\mathcal{F}_P(J_4) = \mathcal{F}_P(G)$ and that G does not have a non-trivial normal $3'$-subgroup. Then G satisfies one of the following. Here Ru is the sporadic simple group of Rudvalis and ${}^2F_4(q^2)$ is the twisted Chevalley group of type F_4.

$$G = Ru, \quad {}^2F_4(q^2) \le G \le \mathrm{Aut}({}^2F_4(q^2)) \ (q^2 \equiv 2 \text{ or } 5 \mod 9), \quad G = J_4.$$

3. Principal blocks

In this section, we explain blocks of finite groups. However, since we treat only principal blocks and characters, instead of giving a definition of general blocks, we give only that of the principal blocks by using values of irreducible characters.

Let G be a finite group and χ an irreducible (complex) character of G. Let p be a prime.

Definition 3.1. *We say that χ belongs to the principal block of G if*

$$\frac{\chi(g)|G|}{\chi(1)|C_G(g)|} \equiv \frac{|G|}{|C_G(g)|} \mod p \quad \forall g \in G$$

The above definition needs some explanations. First of all, the left hand side of the congruence is known to be an algebraic integer, whereas the right hand side is of course a rational integer. Thus " mod p" means modulo a certain prime ideal, lying over $p\mathbb{Z}$, of the ring of integers in a finite extension field of $\mathbb{Q}$. Moreover, the above condition of congruence does not depend on the choice of such a prime ideal. We write $\chi \in B_0(G)$, if χ belongs to the principal block of G.

Example 3.2. *The following give the character tables of the symmetric groups $\mathfrak{S}_3$, $\mathfrak{S}_4$ and $\mathfrak{S}_5$. As is well known, irreducible characters of $\mathfrak{S}_n$ are labeled by partitions of n. For a partition λ of n, the corresponding character is denoted by χ_λ.*

$\mathfrak{S}_3$	e	(12)	(123)
$\chi_{(3)}$	1	1	1
$\chi_{(1,1,1)}$	1	-1	1
$\chi_{(2,1)}$	2	0	-1

$\mathfrak{S}_4$	e	(12)	$(12)(34)$	(123)	(1234)
$\chi_{(4)}$	1	1	1	1	1
$\chi_{(1,1,1,1)}$	1	-1	1	1	-1
$\chi_{(2,2)}$	2	0	2	-1	0
$\chi_{(3,1)}$	3	1	-1	0	-1
$\chi_{(2,1,1)}$	3	-1	-1	0	1

$\mathfrak{S}_5$	e	(12)	$(12)(34)$	(123)	(1234)	(12345)	$(12)(345)$
$\chi_{(5)}$	1	1	1	1	1	1	1
$\chi_{(2,2,1)}$	5	-1	1	-1	1	0	-1
$\chi_{(2,1,1,1)}$	4	-2	0	1	0	-1	1
$\chi_{(1,1,1,1,1)}$	1	-1	1	1	-1	1	-1
$\chi_{(3,2)}$	5	1	1	-1	-1	0	1
$\chi_{(4,1)}$	4	2	0	1	0	-1	-1
$\chi_{(3,1,1)}$	6	0	-2	0	0	1	0

Assume that $p = 3$. Then the characters in the principal blocks are those above the horizontal lines in the individual character tables. In particular, in each case, three irreducible characters belong to the principal block. Moreover, it is known that in each case, exactly two equivalence classes of irreducible representations over an algebraically closed field of characteristic 3 belong to the principal block.

On the other hand, we know that $\mathfrak{S}_3$, $\mathfrak{S}_4$ and $\mathfrak{S}_5$ have the same Sylow 3-subgroup P, that is, a cyclic group C_3 of order three, and moreover,

$$\mathcal{F}_P(\mathfrak{S}_3) = \mathcal{F}_P(\mathfrak{S}_4) = \mathcal{F}_P(\mathfrak{S}_5).$$

The above may suggest that saturated fusion systems and certain invariants of principal blocks are related.

4. Conjectures

The phenomenon described in the previous section can be observed in many cases. In fact, there is a conjecture which is due originally to McKay and Alperin, and later extended by Isaacs and Navarro. Before mentioning it, we need some notations.

Let p be a prime. For an irreducible character χ of a finite group G, we decompose the rational integer $\frac{|G|}{\chi(1)}$ as

$$\frac{|G|}{\chi(1)} = p^{d(\chi)} r(\chi),$$

where $d(\chi)$ is a non-negative integer and $r(\chi)$ is a positive integer relatively prime to p.

Conjecture 1. (McKay–Alperin–Isaacs–Navarro [7], [1], [6], for principal blocks) *Let $P \in \mathrm{Syl}_p(G)$. Let r be an integer with $1 \leq r \leq \frac{p}{2}$. Then*

$$\begin{aligned} &\sharp\{ \chi \in B_0(G) \mid p^{d(\chi)} = |P|, \ \textit{and} \quad r(\chi) \equiv \pm r \mod p\} \\ = &\sharp\{ \theta \in B_0(N_G(P)) \mid p^{d(\theta)} = |P|, \ \textit{and} \ \ r(\theta) \equiv \pm r \mod p\}? \end{aligned}$$

For $\mathfrak{S}_3$, $\mathfrak{S}_4$ and $\mathfrak{S}_5$ and $p = 3$, the above numbers are three as their character tables in the previous section show. Note that the condition on $r(\chi)$ is automatically satisfied if $p = 2$ or 3.

It may be preferable if there exists a natural bijection between the above two sets. A conjecture in this nature is raised by Broué. In order to state Broué's conjecture, we need the following notion.

Definition 4.1. *Let G and H be finite groups. If a generalized character, i.e. a $\mathbb{Z}$-linear combination of irreducible characters, μ of $G \times H$ satisfies the following, then we say that μ is perfect.*

(P1) *If $\mu(g,h) \neq 0$, then either both g and h are p-regular or both are p-singular.*

(P2) *$\mu(g,h)/|C_G(g)|$ and $\mu(g,h)/|C_H(h)|$ are p-local integers.*

Here an element g is called p-regular if its order is relatively prime to p and it is p-singular otherwise.

Let G and H be finite groups. For a bijection $I : B_0(G) \to B_0(H)$ and a map $\varepsilon : B_0(G) \to \{\pm 1\}$, we can define a generalized character μ of $G \times H$ by

$$\mu(g,h) = \sum_{\chi \in B_0(G)} \varepsilon(\chi)\chi(g)I(\chi)(h), \quad (g,h) \in G \times H.$$

Definition 4.2. *Let G and H be finite groups. If there are a bijection $I : B_0(G) \longrightarrow B_0(H)$ and a map $\varepsilon : B_0(G) \longrightarrow \{\pm 1\}$ such that μ defined above is perfect, then we say that μ gives a perfect isometry between $B_0(G)$ and $B_0(H)$.*

Broué's conjecture for principal blocks is stated as follows.

Conjecture 2. (Broué's perfect isometry conjecture for principal blocks) *Suppose that a finite group G has an abelian Sylow p-subgroup P. Then does there exist a perfect isometry between $B_0(G)$ and $B_0(N_G(P))$?*

Remarks. (i) *In the case where a Sylow p-subgroup is not abelian, a perfect isometry does not exist in general. The principal block for $p = 2$ of Suzuki group $G = {}^2B_2(2^{2n+1})$ gives an example. A Sylow 2-subgroup P of G is not abelian and there is no perfect isometry between $B_0(G)$ and $B_0(N_G(P))$. However, we have $\mathcal{F}_P(G) = \mathcal{F}_P(N_G(P))$.*

(ii) *Broué conjectured that if P is abelian, then the derived categories of the module categories of the principal blocks of G and $N_G(P)$ (as algebars) over the ring of p-local integers are equivalent. He also showed that, if this is the case, then there exists a perfect isometry between $B_0(G)$ and $B_0(N_G(P))$.*

We now consider the group p_+^{1+2} for an odd prime p. By using the classification theorems of finite simple groups and of saturated fusion systems over p_+^{1+2}, we can determine the numbers of characters in the principal blocks.

For a non-negative integer d and an integer r with $1 \leq r \leq \frac{p}{2}$, we set

$$k_{d,\pm r}(G) = \sharp\{\ \chi \in B_0(G) \mid d(\chi) = d, \text{ and } r(\chi) \equiv \pm r \mod p\}.$$

Moreover, we denote by $l(G)$ the number of equivalence classes of irreducible representations over an algebraically closed field of characteristic p, which belong to the principal block of G. Namely, irreducible representations which are obtained as irreducible constituents when reduced modulo p of irreducible representations over a field of characteristic 0 belonging to the principal block.

Let $p = 3$ and $P = 3_+^{1+2}$. For G with $P \in \mathrm{Syl}_3(G)$, we can obtain $k_{d,\pm r}(G)$ and $l(G)$ as follows. Recall that there exist fifteen saturated fusion systems over P. In the following table, the column indicated as $|\mathcal{F}^e|$ gives the number of those subgroups Q of P which have order 3^2 and satisfy $SL_2(3) \leq N_G(Q)/C_G(Q)$. In fact, it is known that in general saturated fusion systems over p_+^{1+2} are determined only by $\mathrm{Out}_{\mathcal{F}_P(G)}(P)$ and $|\mathcal{F}^e|$, [2]. The symbol $(*)$ means the case where $Z(P) \leq C_G(N_G(P))$.

Case	$\mathrm{Out}_{\mathcal{F}_P(G)}(P)$	$\lvert\mathcal{F}^e\rvert$	$k_{3,\pm1}(G)$	$k_{2,\pm1}(G)$	$l(G)$
(1)	1 (*)	0	9	2	1
(2)	C_2 (*)	0	6	4	2
(3)	C_2	0	9	1	2
(4)	C_2	1	9	1	3
(5)	C_4 (*)	0	6	8	4
(6)	$C_2 \times C_2$	0	9	2	4
(7)	$C_2 \times C_2$	1	9	2	6
(8)	$C_2 \times C_2$	2	9	2	8
(9)	C_8	0	9	4	8
(10)	Q_8 (*)	0	6	10	5
(11)	D_8	0	9	4	5
(12)	D_8	2	9	4	7
(13)	D_8	4	9	4	9
(14)	SD_{16}	0	9	5	7
(15)	SD_{16}	4	9	5	9

Here, D_8, Q_8 and SD_{16} mean a dihedral group of order eight, a quaternion group of order eight and a semidihedral group of order sixteen, respectively. If d is not 2 nor 3, we have $k_{d,\pm1}(G) = 0$.

A similar thing can be seen for a general p. Namely, if $P = p_+^{1+2} \in \mathrm{Syl}_p(G) \cap \mathrm{Syl}_p(H)$ and $\mathcal{F}_P(G) = \mathcal{F}_P(H)$, then we have for all d and r,

$$k_{d,\pm r}(G) = k_{d,\pm r}(H) \quad \text{and} \quad l(G) = l(H).$$

5. Some invariants and a new isometry

To define a new type of isometry, we introduce some invariants for normal subgroups of p-groups and conjugacy classes of finite groups. For a finite group G, we denote by $\mathbb{Z}\mathrm{Irr}(G)$ the set of generalized characters of G.

Let P be a p-group, and let Q be a normal subgroup of P. We set

$$X(P;Q) = \{ \ \theta \in \mathbb{Z}\mathrm{Irr}(P) \mid \theta(g) = 0 \ \text{ for all } \ g \in P \setminus Q \ \},$$

and

$$V(P;Q) = \{ \sum_{\varphi \in \mathrm{Irr}(Q)} a_\varphi \varphi \uparrow^P \ \mid a_\varphi \in \mathbb{Z} \ \}.$$

Namely, $V(P;Q)$ is the image of the induction map from $\mathbb{Z}\mathrm{Irr}(Q)$ to $\mathbb{Z}\mathrm{Irr}(P)$. Then, $X(P;Q)$ and $V(P;Q)$ are $\mathbb{Z}$-submodules of $\mathbb{Z}\mathrm{Irr}(P)$. Moreover, we have $V(P;Q) \subseteq X(P;Q)$. Furthermore, Lemma 3.3 (ii) in [11] shows that there exists a non-negative integer c such that $p^c X(P;Q) \subseteq V(P;Q)$. Now we define $c(P;Q)$ as follows.

Definition 5.1. *Let P be a p-group and Q a normal subgroup of P. We denote by $c(P;Q)$ the non-negative integer c smallest among those c which satisfy $p^c X(P;Q) \subseteq V(P;Q)$.*

Remarks. (i) *We have $c(P;\{1\}) = 0$, since $X(P;\{1\})$ and $V(P;\{1\})$ are both generated by the regular character of P over $\mathbb{Z}$. In particular, if P is abelian, then we have $c(P;[P,P]) = 0$.*

(ii) *Suppose that $P = p_+^{1+2}$. Then it is easy to see that $c(P;[P,P]) = 1$. In fact, we can describe $X(P;[P,P])$ and $V(P;[P,P])$ as follows. Let $\psi = 1_{[P,P]}\uparrow^P$, and let $\chi_1, \chi_2, \cdots, \chi_{p-1}$ be distinct irreducible characters of P with degree p. Then, $X(P;[P,P])$ is generated by $\psi, \chi_1, \chi_2, \cdots, \chi_{p-1}$ over $\mathbb{Z}$, whereas $V(P;[P,P])$ is generated by $\psi, p\chi_1, p\chi_2, \cdots, p\chi_{p-1}$ over $\mathbb{Z}$.*

Next, we let G be a finite group and fix a p-subgroup Q of G. Let

$$\mathrm{Tr}_Q^G : (\mathbb{Z}G)^Q \to (\mathbb{Z}G)^G = Z(\mathbb{Z}G)$$

be the trace map, where $(\mathbb{Z}G)^Q$ is the set of Q-invariant elements in $\mathbb{Z}G$ under the conjugate action. Thus $(\mathbb{Z}G)^G$ is the center $Z(\mathbb{Z}G)$ of $\mathbb{Z}G$. Let $g \in G$ and C the G-conjugacy class of g. Then the sum $\hat{C}$ of all the elements in C lies in $Z(\mathbb{Z}G)$.

Definition 5.2. *With the above notation, we denote by $s(g) = s_Q(g)$ the non-negative integer s smallest among those s such that there exists a positive integer m relatively prime to p with $mp^s\hat{C} \in \mathrm{Im}\mathrm{Tr}_Q^G$.*

Remarks. (i) *If $g' \in G$ is G-conjugate to g, then $s_Q(g) = s_Q(g')$.*

(ii) *Since $\mathrm{Tr}_{\{1\}}^G(g) = |C_G(g)|\hat{C}$, it follows that $p^{s_{\{1\}}(g)}$ is the order of a Sylow p-subgroup of $C_G(g)$.*

(iii) *The invariant $s_Q(g)$ is related to relatively projective modules. Green showed that, if μ is the character of a Q-projective G-module, then the value $\mu(g)/p^{s_Q(g)}$ is a p-local integer. See IV. Theorem 2.3 of* [4].

We apply the above notion to the direct product $G \times H$ of two finite groups G and H having a common Sylow p-subgroup P. Let Q be a normal subgroup of P. Consider the subgroup $(Q \times Q)\Delta(P)$ of $P \times P$, where $\Delta(P) = \{(u, u^{-1}) \mid u \in P\}$. For $(g,h) \in G \times H$, we denote $s_{(Q\times Q)\Delta(P)}((g,h))$ simply by $s_Q(g,h)$.

Remark. *In the above situation we can show for any such a Q that*

$$\max(s_Q(g), s_Q(h)) \le s_Q(g,h) \le s_{\{1\}}(g) + s_{\{1\}}(h) = s_{\{1\}}(g,h).$$

In particular, the orders of Sylow p-subgroups of $C_G(g)$ and $C_H(h)$ are smaller than or equal to $p^{s_{\{1\}}(g,h)}$.

Using the invariants given above, we introduce the notion of Q-perfectness. Every element g of a finite group G can be written uniquely as the product $g = g_p g_{p'}$ of two elements g_p and $g_{p'}$ of G with $g_p g_{p'} = g_{p'} g_p$, such that the order of g_p is a power of p and that of $g_{p'}$ is relatively prime to p. Note that g is p-regular if $g_p = 1$ and is p-singular if $g_p \neq 1$. For an element g of G and a subgroup H of G, we write $g \in_G H$ to mean that some G-conjugate of g lies in H.

Definition 5.3. *Assume that $P \in \mathrm{Syl}_p(G) \cap \mathrm{Syl}_p(H)$, and let Q be a normal subgroup of P. If a generalized character μ of $G \times H$ satisfies the following for every $(g, h) \in G \times H$, then we say that μ is $(P; Q)$-perfect or simply Q-perfect.*

(RP1) *If $\mu(g, h) \neq 0$, then $(g_p, h_p) \in_{G\times H} (Q \times Q)\Delta(P)$.*

(RP2) *$p^{c(P;Q)-s_Q(g,h)}\mu(g, h)$ is a p-local integer.*

It is easy to see the following.

Proposition 5.4. *Suppose that μ is $\{1\}$-perfect. Then, it is perfect and is Q-perfect for any normal subgroup Q of P.*

The Q-perfect isometry is defined as follows.

Definition 5.5. *Assume that $P \in \mathrm{Syl}_p(G) \cap \mathrm{Syl}_p(H)$, and let Q be a normal subgroup of P. If there exist a bijection $I : B_0(G) \longrightarrow B_0(H)$ and a map $\varepsilon : B_0(G) \longrightarrow \{\pm 1\}$ such that they give a Q-perfect generalized character μ of $G \times H$, then we say that μ is a Q-perfect isometry between $B_0(G)$ and $B_0(H)$.*

Now, we ask the following question.

Conjecture 3. *Assume that $P \in \mathrm{Syl}_p(G) \cap \mathrm{Syl}_p(H)$ and $\mathcal{F}_P(G) = \mathcal{F}_P(H)$. Then does there exist a normal subgroup Q of P with $Q \leq [P, P]$ such that there exists a Q-perfect isometry between $B_0(G)$ and $B_0(H)$? Moreover, can we take a bijection $I : B_0(G) \longrightarrow B_0(H)$ such that $d(\chi) = d(I(\chi))$ and $r(\chi) \equiv \pm r(I(\chi)) \mod p$ for all $\chi \in B_0(G)$? Furthermore, $l(G) = l(H)$?*

Remarks. (i) *Suppose that $P \in \mathrm{Syl}_p(G) \cap \mathrm{Syl}_p(H)$, and $N_G(P)/PC_G(P)$ and $N_H(P)/PC_H(P)$ are conjugate in $\mathrm{Out}(P)$. Then, in many cases there is a $[P, P]$-perfect isometry between $B_0(G)$ and $B_0(H)$, even if $\mathcal{F}_P(G) \neq \mathcal{F}_P(H)$. However, perhaps it is not true in general and we can ask only whether $k_{d,\pm r}(G) = k_{d,\pm r}(H)$ for d with $p^d = |P|$ and all r. This is nothing but the McKay-Alperin-Isaacs-Navarro conjecture.*

(ii) *Broué gives a notion of isotypic. It can also be generalized by using Q-perfectness. In fact, in the situation of Conjecture 3, we ask in* [10] *whether or not the principal blocks of G and H are $[P, P]$-isotypic.*

In [10] the following is proved. Proof uses the arguments concerning splendid equivalence for the existence of $\{1\}$-perfect isometries, and GAP [13], CHEVIE [5] and MAPLE for the existence of $[P, P]$-perfect isometries.

Theorem 5.6. *Assume that $p = 3$ or 5 and let P be an extra special p-group of order p^3 and exponent p. Then Conjecture 3 is true.*

Remarks. (i) *In* [10], *not only the existence of a $[P, P]$-perfect isometry but also $[P, P]$-isotypic is shown.*

(ii) *For $p \geq 7$, most cases are treated in* [9], *where the existence of $[P, P]$-perfect isometry is proved.*

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