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Cohomology of the cyclic group $\mathbb{Z}/p$

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1 Introduction

Let p be an odd prime and let $\mathbb{Z}/p$ be the cyclic group of order p. Let V_p be a vector space with the basis $\{v_0, \ldots, v_{p-1}\}$ and suppose that the cyclic group $\mathbb{Z}/p$ acts on V_p by permuting this basis. Then, the cohomology $H^*(\mathbb{Z}/p, \mathbb{Z}/p[V_p])$ is well-known as the cohomology of wreath products. See, for example, [1], Proposition 4.2.8 in [11] for H^0 which is the rings of invariants and [12]. In this paper, we consider similar but slightly different situation.

We fix a generator of $\mathbb{Z}/p$ and denote it by g. Let $A_p = \mathbb{Z}/p[x_0, \ldots, x_{p-1}]$ be a polynomial algebra in p variables $x_0, \ldots, x_{p-1}$. There exists a derivation ∂ on A_p such that $\partial(x_0) = 0$, $\partial(x_i) = x_{i-1}$ for $i = 1, \ldots, p-1$. Using this derivation, we may consider the action of the cyclic group $\mathbb{Z}/p$ on A_p given by $g(x) = x - \partial(x)$.

We compute the cohomology $H^*(\mathbb{Z}/p, A_p)$ and discuss its application to the cohomology of classifying spaces of compact connected Lie groups.

Our computational result is as follows:

Theorem 1.1 With the notation as above, we have

$$H^i(\mathbb{Z}/p, A_p) = \mathbb{Z}/p[x_{p-1}^p]$$

for $i > 0$.

Theorem 1.2 Suppose that p is an odd prime. Let $A_{p-1} = \mathbb{Z}/p[x_0, \dots, x_{p-2}] \subset A_p$. Then, we have

$$H^{2i}(\mathbb{Z}/p, A_{p-1}) = \mathbb{Z}/p[x_{p-2}^p]\{1, x_0\}$$

and

$$H^{2i-1}(\mathbb{Z}/p, A_{p-1}) = \mathbb{Z}/p[x_{p-2}^p]\{1, x_{p-2}\}$$

for $i > 0$.

After proving these theorems, we give their applications to the computation of cohomology of classifying spaces of compact connected Lie groups, in particular, simply-connected exceptional Lie groups.

2 Preliminaries on A_k and H_k^ε

For $k = 1, \dots, p$, let A_k be the polynomial algebra

$$A_k = \mathbb{Z}/p[x_0, \dots, x_{k-1}] \subset \mathbb{Z}/p[x_0, \dots, x_{p-1}] = A_p$$

together with the derivation ∂ given by $\partial(x_0) = 0$, $\partial(x_i) = x_{i-1}$ for $i = 1, \dots, p-1$ and $\partial(x \cdot y) = x \cdot \partial(y) + \partial(x) \cdot y$ for $x, y \in A_p$.

We also consider the length and the weight of monomial x as follows: For a monomial $x = x_0^{i_0} \cdots x_{p-1}^{i_{p-1}}$, let us define $\ell(x)$, $w(x)$ by

$$\ell(x) = i_0 + \cdots + i_{p-1},$$
$$w(x) = 0 \cdot i_0 + 1 \cdot i_1 + \cdots + (p-1) \cdot i_{p-1}.$$

Let $A_k^{\ell,w}$ be the subspace spanned by monomials x in A_k whose length is ℓ and whose weight is w.

Now, we recall the definition of Poincaré series of bigraded $\mathbb{Z}/p$-modules. For a bigraded $\mathbb{Z}/p$-module M, say

$$M = \bigoplus_{i,j \geq 0} M^{i,j},$$

we define the Poincaré series $PS(M, s, t)$ in $\mathbb{Z}[[s, t]]$ by

$$PS(M, s, t) = \sum_{i,j \geq 0} (\dim M^{i,j})\, s^i t^j.$$

For instance, we have

$$PS(A_k, s, t) = \frac{1}{(1-s)(1-st)\cdots(1-st^{k-1})}.$$

The derivation ∂ maps $A_k^{\ell,w}$ to $A_k^{\ell,w-1}$, so we may think of A_k as bigraded vector space over $\mathbb{Z}/p$ where the degree is given by $\ell(x)$ and $w(x)$ and ∂ is a homomorphism of graded vector spaces whose degree is $(0,-1)$.

We denote

$$H^{even}(\mathbb{Z}/p, A_k)^{\ell,w} = (\mathrm{Ker}\partial/\mathrm{Im}\partial^{p-1})^{\ell,w}$$

by $H_k^{even,\ell,w}$. Also we denote

$$H^{odd}(\mathbb{Z}/p, A_k)^{\ell,w} = (\mathrm{Ker}\partial^{p-1}/\mathrm{Im}\partial)^{\ell,w}$$

by $H_k^{odd,\ell,w}$. Thus, we have

$$H_k^{even} = H^{even}(\mathbb{Z}/p, A_k) = \bigoplus_{\ell,w} H_k^{even,\ell,w}$$

and

$$H^{odd} = H^{odd}(\mathbb{Z}/p, A_k) = \bigoplus_{\ell,w} H_k^{odd,\ell,w}.$$

Proposition 2.1 For $k = 1, \ldots, p$, there holds

$$PS(H_k^{even}, s, 1) = PS(H_k^{odd}, s, 1).$$

Proof Since

$$H_k^{even} = \mathrm{Ker}\partial/\mathrm{Im}\partial^{p-1},$$

we have

$$PS(H_k^{even}, s, 1) = PS(\mathrm{Ker}\,\partial, s, 1) - PS(\mathrm{Im}\,\partial^{p-1}, s, 1)$$

and

$$PS(\mathrm{Im}\,\partial^{p-1}, s, 1) = PS(A_k, s, 1) - PS(\mathrm{Ker}\,\partial^{p-1}, s, 1),$$

Hence, we have

$$PS(H_k^{even}, s, 1) = PS(\mathrm{Ker}\,\partial, s, 1) + PS(\mathrm{Ker}\,\partial^{p-1}, s, 1) - PS(A_k, s, 1).$$

Similarly, we have

$$PS(H_k^{odd}, s, 1) \quad = \quad PS(\mathrm{Ker}\,\partial, s, 1) + PS(\mathrm{Ker}\,\partial^{p-1}, s, 1) - PS(A_k, s, 1).$$

Therefore, we have $PS(H_k^{odd} s, 1) = PS(H_k^{even}, s, 1)$. □

Let us consider the following short exact sequence of $\mathbb{Z}/p$-modules:

$$0 \to A_k^{\ell,w} \xrightarrow{\cdot x_0} A_k^{\ell+1,w} \to (A_k/(x_0))^{\ell+1,w} \to 0.$$

There is an isomorphism

$$(A_k/(x_0))^{\ell+1,w} \to A_{k-1}^{\ell+1,w-\ell-1}$$

sending x_i to x_{i-1} for $i = 1, \dots, k-1$ and x_0 to 0. We denote by

$$\phi : H_p^{even,\ell,w} \to H_p^{even,\ell+1,w}, \ \phi : H^{odd,\ell,w} \to H^{odd,\ell+1,w}$$

the induced homomorphisms induced by the multiplication by x_0. We also denote by

$$\psi : H_p^{even,\ell,w} \to H_{p-1}^{even,\ell,w-\ell},$$

the homomorphism induced by the composition of the projection

$$A_p \to A_p/(x_0)$$

and the isomorphism

$$A_p/(x_0) \to A_{p-1}.$$

This short exact sequence induces a long exact sequence

$$\cdots \to H_k^{even,\ell,w} \xrightarrow{\phi} H_k^{even,\ell+1,w} \xrightarrow{\psi} H_{k-1}^{even,\ell+1,w-\ell-1} \xrightarrow{\delta} H_k^{odd,\ell,w-1} \to H_k^{odd,\ell+1,w-1} \to \cdots$$

Proposition 2.2 For $\varepsilon = even, odd$, the multiplication by x_0 induces the zero homomorphism

$$\phi : H_p^{\varepsilon,\ell,w} \to H_p^{\varepsilon,\ell+1,w}.$$

Proof If $\partial f = 0$, we have

$$x_0 f = \partial^{p-1}(x_{p-1} f).$$

If $\partial^{p-1} f = 0$, we have

$$x_0 f = \partial(x_1 f - x_2 \partial(f) + \cdots + x_{p-1}\partial^{p-1}(f)). \qquad \square$$

Remark 2.3 This proposition does not hold for $k < p$.

With this proposition, the above long exact sequence splits in the short exact sequences for $k = p$ and we have the following exact sequence

$$0 \to H_p^{even,\ell,w} \xrightarrow{\psi} H_{p-1}^{even,\ell,w-\ell} \xrightarrow{\delta} H_p^{odd,\ell-1,w-1} \to 0.$$

In particular, we have the following proposition.

Proposition 2.4 There holds

$$\dim H_p^{even,\ell,w+1} = \dim H_p^{odd,\ell-1,w} - \dim H_{p-1}^{even,\ell,w+1-\ell}.$$

From now on, we assume k is p or $p-1$.

3 Lower bound for the Poincaré series

In this section, we give lower bounds for

$$H_k^{\varepsilon,\ell,w}$$

where $\epsilon = even,\ odd$ and $k = p-1,\ p$.

Let us define

$$\sum_{\ell,w} \varphi_p^{even,\ell,w} s^\ell t^w = \sum_{\ell,w} \varphi_p^{odd,\ell,w} s^\ell t^w = \frac{1}{1 - s^p t^{p(p-1)}}$$

$$\sum_{\ell,w} \varphi_{p-1}^{even,\ell,w} s^\ell t^w = \frac{1+s}{1 - s^p t^{p(p-2)}}$$

$$\sum_{\ell,w} \varphi_{p-1}^{odd,\ell,w} s^\ell t^w = \frac{1+st^{p-2}}{1 - s^p t^{p(p-2)}}$$

These Poincaré series are Poincaré series of

$$\mathbb{Z}/p[x_{p-1}^p],\ \ \mathbb{Z}/p[x_{p-2}^p]\{1,x_0\},\ \ \mathbb{Z}/p[x_{p-2}^p]\{1,x_{p-2}\},$$

respectively. Considering the weight of x_{p-2}^{mp}, $x_0x_{p-2}^{mp}$, it is clear that x_{p-2}^{mp}, $x_0x_{p-2}^{mp}$ are not in the image of ∂^{p-1}. Thus, it is clear that $\dim H_{p-1}^{even,\ell,w} \geq \varphi_{p-1}^{even,\ell,w}$. It is also easy to see that $\dim H_k^{\varepsilon,\ell,w} \geq \varphi_k^{\varepsilon,\ell,w}$ for $k = p-1, p$, $\varepsilon = even, odd$. Moreover, we have the following proposition.

Proposition 3.1 There holds

$$\varphi_p^{even,\ell,w+1} = \varphi_p^{odd,\ell-1,w} - \varphi_{p-1}^{even,\ell,w+1-\ell}.$$

Proof Consider the short exact sequence

$$0 \to \mathbb{Z}/p[x_{p-1}^p] \xrightarrow{\psi} \mathbb{Z}[x_{p-2}^p]\{1,x_0\} \xrightarrow{\delta} \mathbb{Z}/p[x_{p-1}^p] \to 0,$$

where $\psi(x_{p-1}^{mp}) = x_{p-2}^{mp}$, $\delta(x_{p-2}^{mp}) = 0$, $\delta(x_0x_{p-2}^{mp}) = x_{p-2}^{mp}$. □

For $\varepsilon = even,\ odd$ and for $k = p,\ p-1$, we say the condition $\Phi_k^{\varepsilon,\ell,w}$ holds if and only if

$$\dim H_k^{\varepsilon,\ell',w'} = \varphi_k^{\varepsilon,\ell',w'}$$

for $\ell' < \ell$ and for $\ell' = \ell$ and $w' \leq w$. In terms of Poincaré series, the condition $\Phi_k^{\varepsilon,\ell,w}$ is equivalent to say that

$$PS(H_k^\varepsilon, s, t) - \sum_{\ell,w} \varphi_k^{\varepsilon,\ell,w} s^\ell t^w$$

is divisible by $s^{\ell-1}$ and the coefficient of s^ℓ in $\mathbb{Z}[t]$ is divisible by t^{w+1}. In particular, we have that $\Phi_k^{\varepsilon,\ell,*}$ is equivalent to say that

$$PS(H_k^\varepsilon, s, t) - \sum_{\ell,w} \varphi_k^{\varepsilon,\ell,w} s^\ell t^w$$

is divisible by s^ℓ.

Since the coefficient of $s^\ell t^w$ in

$$PS(H_k^\varepsilon, s, t) - \sum_{\ell,w} \varphi_k^{\varepsilon,\ell,w} s^\ell t^w$$

is non-negative, the conditions $\Phi_k^{\varepsilon,\ell-1,*}$ ($\Phi_k^{\varepsilon,\ell,w}$ holds for all w) is equivalent to

$$PS(H_k^\varepsilon, s, 1) - \sum_{\ell,w} \varphi_k^{\varepsilon,\ell,w} s^\ell$$

is divisible by s^ℓ. Therefore, we have the following proposition.

Proposition 3.2 If $\Phi_p^{even,\ell,w}$ holds, then $\Phi_p^{odd,\ell-1,*}$ hold.

Proof The condition $\Phi_p^{even,\ell,w}$, by definition, implies the condition $\Phi_p^{even,\ell-1,*}$. By Proposition **??**, we have the condition $\Phi_p^{odd,\ell-1,*}$. □

In terms of above conditions, our main theorem is given as follows:

Theorem 3.3 The condition $\Phi_p^{even,\ell,w}$ holds for all $\ell \geq 0$, $w \geq 0$.

4 Proof of Theorem 3.3

First, we prove two lemmas.

Lemma 4.1 For $\beta \not\equiv 0, 1 \bmod p$, the leading monomial of $\partial^{p-1}(x_{p-2}^\beta)$ is $x_0 x_{p-3}$.

Proof We have

$$\begin{aligned}\partial^{p-1}(x_{p-2}^{\beta}) &= \sum_{(\alpha_1,\ldots,\alpha_\beta)} \frac{(p-1)!}{\alpha_1!\ldots\alpha_\beta!}\partial^{\alpha_1}(x_{p-2})\cdots\partial^{\alpha_\beta}(x_{p-2}) \\ &= \beta(\beta-1)(p-1)x_0x_{p-3}x_{p-2}^{\beta-2} + \text{lower terms.}\end{aligned}$$

□

Lemma 4.2 For $0 \leq \gamma \leq p-2$, $\partial^{p-1}(x_{p-2}x_{p-1}^{\gamma}) = 0$. For $\gamma = p-1$, we have

$$\partial^{p-1}(x_{p-2}x_{p-1}^{\gamma}) = -x_{p-2}^{p}.$$

Proof Let us consider a derivation $\widehat{\partial}$ on $\mathbb{Z}[x_0,\ldots,x_{p-1}]$ defined by

$$\begin{aligned}\widehat{\partial}x_i &= x_{i-1} && \text{for } i = 1,\ldots,p-1, \\ \widehat{\partial}x_0 &= 0 && \text{and} \\ \widehat{\partial}(x\cdot y) &= \partial(x)\cdot y + x\cdot\partial(y).\end{aligned}$$

The derivation ∂ is the mod p reduction of $\widehat{\partial}$. Then, we have

$$\begin{aligned}\widehat{\partial}^{p-1}(x_{p-2}x_{p-1}^{\gamma}) &= \frac{1}{\gamma+1}\widehat{\partial}^{p}(x_{p-1}^{\gamma+1}) \\ &= \frac{1}{\gamma+1}\sum_{(\alpha_1,\cdots,\alpha_{\gamma+1})}\frac{p!}{\alpha_1!\cdots\alpha_{\gamma+1}!}\widehat{\partial}^{\alpha_1}(x_{p-1})\cdots\widehat{\partial}^{\alpha_{\gamma+1}}(x_{p-1})\end{aligned}$$

where $(\alpha_1,\cdots,\alpha_{\gamma+1})$ ranges over the $(\gamma+1)$-partitions of p, so that $\alpha_1+\cdots+\alpha_{\gamma+1} = p$, $\alpha_j \geq 0$ for $j = 1,\ldots,\gamma+1$. If $\gamma+1 < p$, then we have $\partial^{p-1}(x_{p-2}x_{p-1}^{\gamma}) = 0$. Suppose that $\gamma+1 = p$. The symmetric group of p-letters acts on the set of $(\gamma+1)$-partitions of p and the number of elements in each orbit is divisible by p except for the case $(\alpha_1,\cdots,\alpha_{\gamma+1}) = (1,\cdots,1)$. Hence, we have

$$\begin{aligned}\partial^{p-1}(x_{p-2}x_{p-1}^{\gamma}) &= (p-1)!\partial x_{p-1}\cdots\partial x_{p-1} \\ &= -x_{p-2}^{p}.\end{aligned}$$

Thus, we have the required equality. □

We prove Theorem 3.3 by induction on ℓ and w. It is clear that for $\ell = 0$, the theorem holds. It is also clear that for each ℓ, if $\Phi_p^{even,\ell-1,*}$ holds, $\Phi_p^{even,\ell,0}$ holds.

Proposition 4.3 The condition $\Phi_p^{even,\ell,w}$ implies the condition $\Phi_{p-1}^{even,\ell,w}$.

Proof Let us consider an element $[x]$ in $H_{p-1}^{even,\ell,w}$ represented by $x \in A_{p-1}^{\ell,w}$. So, we assume $\partial(x) = 0$ in $A_{p-1}^{\ell,w-1}$.

First, we show that there exists $y \in A_p^{\ell,w+p-1}$ such that $x = \partial^{p-1}(y)$. If $\ell = mp$ and $w = mp(p-2)$ for some $m \geq 0$, then $A_{p-1}^{\ell,w} = \{0\}$. Therefore, we may put $y = 0$. Since $\Phi_p^{even,\ell,w}$ holds, if ℓ is not divisible by p, or if $\ell = mp$ and $w \neq mp(p-2)$ for some $m \geq 0$, then $H_p^{even,\ell,w} = \{0\}$. Hence, there exists $y \in A_p^{\ell,w+p-1}$ such that $\partial^{p-1}(y) = x$. Suppose that $y = y_n x_{p-1}^n + y_{n-1} x_{p-1}^{n-1} + \cdots + y_1 x_{p-1} + y_0$, where $y_n, \ldots, y_0$ are in A_{p-1}.

Now, we prove by induction on n that $[x]$ is represented by $x_0^\epsilon x_{p-2}^\beta$ for some $\epsilon \in \{0, 1\}$, $\beta \geq 0$ divisible by p. In the case $n = 0$, it is trivial. Suppose that $n \geq 1$. Then

$$\partial^{p-1}(y) = \partial^{p-1}(y_n) x_{p-1}^n + \text{terms lower than } x_{p-1}^n.$$

Therefore, we have $\partial^{p-1}(y_n) = 0$ since x is in A_{p-1}.

Since, by Proposition 3.2, the condition $\Phi_{p-1}^{odd,\ell-1,*}$ holds, there exist z in $A_{p-1}^{\ell-n,*}$ and α in $\mathbb{Z}/p$ such that $y_n = \alpha x_{p-2}^{\ell-n} + \partial(z)$. Replacing y by $y + \partial(z x_{p-1}^n)$, we have

$$x = \partial^{p-1}(\alpha x_{p-2}^{\ell-n} x_{p-1}^n + \text{terms lower than } x_{p-1}^n).$$

If $\alpha = 0$, by inductive hypothesis, $[x]$ is represented by a linear combination of $x_0^\epsilon x_{p-2}^\beta$. Suppose that $\alpha \neq 0$. Then, we have

$$w(y) = (\ell-n)(p-2) + n(p-1) > (\ell-n+k)(p-2) + (n-k)(p-1) = \max w(y_{n-k} x_{p-1}^{n-k}).$$

Thus, $y = \alpha x_{p-2}^{\ell-n} x_{p-1}^n$.

If $\ell - n \not\equiv 0, 1 \bmod p$, then, by Lemma 4.1, the leading monomial of x is $x_0 x_{p-3} x_{p-2}^{\ell-n-2} x_{p-1}^n$. So, if x is in A_{p-1}, then $n = 0$ and so y is also in A_{p-1}.

If $\ell - n \equiv 0 \bmod p$ and if n is divisible by p, then $x = 0$. If $\ell - n \equiv 0 \bmod p$ and if n is not divisible by p, then the leading monomial of x is $x_0 x_{p-2}^{\ell-n} x_{p-1}^{n-1}$. Since x is in A_{p-1}, $n = 1$. So, $[x]$ is represented by a scalar multiple of $x_0 x_{p-2}^{\ell-1}$ and $\ell - 1$ is divisible by p.

If $\ell - n \equiv 1 \bmod p$, then, by Lemma 4.2, we have $x = 0$ or x is a scalar multiple of x_{p-2}^ℓ, where $n = p - 1$. So, ℓ is divisible by p and $[x]$ is represented by a scalar multiple of x_{p-2}^ℓ. □

Proposition 4.4 For $\ell \geq 1$, the condition $\Phi_p^{even,\ell,w}$ implies the condition $\Phi_p^{even,\ell,w+1}$.

Proof By Proposition 4.3, we have the condition $\Phi_{p-1}^{even,\ell,w+1-\ell}$. In particular, we have

$$\dim H_{p-1}^{even,\ell,w+1-\ell} = \varphi_{p-1}^{even,\ell,w+1-\ell}.$$

By Proposition ??, we have the condition $\Phi_p^{odd,\ell-1,*}$. In particular, we have

$$\dim H_{p-1}^{odd,\ell-1,w} = \varphi_{p-1}^{odd,\ell-1,w}.$$

Hence, we have

$$\begin{aligned}
\dim H_p^{even,\ell,w+1} &= \dim H_{p-1}^{even,\ell,w+1-\ell} - \dim H_p^{odd,\ell-1,w} \\
&= \varphi_{p-1}^{even,\ell,w+1-\ell} - \varphi_p^{odd,\ell-1,w} \\
&= \varphi_p^{even,\ell,w+1}.
\end{aligned}$$

□

As we already mentioned, it is clear that for $\ell = 0$, the theorem holds. It is also clear that for each ℓ, if $\Phi_p^{even,\ell-1,*}$ holds, $\Phi_p^{even,\ell,0}$ holds. So, the above propositions complete the proof of Theorem 3.3.

Remark 4.5 Let us consider the tensor product of m-copies of A_{p-1} and n-copies of A_p, say $A_{p-1}^m \otimes A_p^n$. One may compute $H^\varepsilon(\mathbb{Z}/p, A_{p-1}^m \otimes A_p^n)$ using the theorem

$$H^\varepsilon(\mathbb{Z}/p, M \otimes A_p) = H^\varepsilon(\mathbb{Z}/p, M) \otimes \mathbb{Z}/p[x_{p-1}^p]$$

and cohomology long exact sequence associated with

$$0 \longrightarrow A_p \xrightarrow{\times x_0} A_p \longrightarrow A_{p-1} \longrightarrow 0.$$

5 Exceptional Lie groups

Let p be an odd prime and let G be a compact connected Lie group. If the integral homology of G has no p-torsion, then the cohomology of BG is a polynomial algebra generated by even degree elements. If G is a simply-connected simple Lie group, then by classification theory, G is one of classical groups $SU(n)$, $Sp(n)$, $\mathrm{Spin}(n)$ or one of exceptional Lie groups G_2, F_4, E_6, E_7, E_8. Among these simple Lie groups, it is known that $H_*(G;\mathbb{Z})$ has p-torsion if and only if (G,p) is one of $(F_4,3)$, $(E_6,3)$, $(E_7,3)$, $(E_8,3)$, $(E_8,5)$. So, the computation of the mod p cohomology of classifying spaces of simply-connected simple Lie groups is a finite number of computational problems (5 problems, to be exact), so that we can compute them one by one by ad hoc computation. It seems to me that it is the strategy of Mimura and Sambe in their work

[5], [4], [9], [10], [8], on the computation of the cotorsion products $\mathrm{Cotor}_A(\mathbb{Z}/p,\mathbb{Z}/p)$ of $A = H^*(G;\mathbb{Z}/p)$ for these (G,p)'s. There exists the Rothenberg-Steenrod spectral sequence

$$\mathrm{Cotor}_A(\mathbb{Z}/p,\mathbb{Z}/p) \Rightarrow grH^*(BG;\mathbb{Z}/p).$$

The spectral sequence collapses at the E_2-level for $(G,p) = (F_4,3)$, $(E_6,3)$, $(E_7,3)$, $(E_8,5)$. So, the computation of the cotorsion product is nothing but the computation of $H^*(BG;\mathbb{Z}/p)$ at least as a graded $\mathbb{Z}/p$-module. However, the computation of Mimura and Sambe seems to be too complicated and I think a comprehensive approach for the cotorsion products is desired. We believe our approach is somewhat more comprehensive than the computation of Mimura and Sambe.

In the case $(G,p) = (F_4,3)$, $A = \mathbb{Z}/3[x_8]/(x_8^3) \otimes \Lambda(x_3,x_{11},x_7,x_{15})$, the reduced coproduct is given by

$$\begin{aligned} \bar{\phi}(x_{11}) &= x_8 \otimes x_3, \\ \bar{\phi}(x_{15}) &= x_8 \otimes x_7, \end{aligned}$$

and $\bar{\phi}(x_k) = 0$ for $k = 3,7,8$. Associated with the extension of Hopf algebras

$$\mathbb{Z}/3[x_8]/(x_8^3) \longrightarrow A \longrightarrow \Lambda(x_3,x_{11},x_7,x_{15}),$$

we have the change-of-rings spectral sequence

$$\mathrm{Cotor}_\Gamma(\mathbb{Z}/3,\mathrm{Cotor}_A(\Gamma,\mathbb{Z}/3)) \Rightarrow gr\mathrm{Cotor}_A(\mathbb{Z}/3,\mathbb{Z}/3),$$

where $\Gamma = \mathbb{Z}/3[x_8]/(x_8^3)$. The E_2-term of this spectral sequence could be given by the cohomology of cyclic group $H^i(\mathbb{Z}/3,A_2\otimes A_2)$ and in the case $(G,p) = (F_4,3)$, all spectral sequence turn out to collapse at the E_2-level. So, we have the following theorem for $(G,p) = (F_4,3)$, $(E_6,3)$, $(E_7,3)$, $(E_8,5)$.

Theorem 5.1 After giving suitable degrees for generators of each copy of A_{p-1}, A_p's, respectively, we have following isomorphisms of graded $\mathbb{Z}/p$-modules. For $p = 3$, we have

$$\begin{aligned} H^*(BF_4;\mathbb{Z}/3) &= \bigoplus_{i,\varepsilon} H^{2i+\varepsilon}(\mathbb{Z}/3,A_2\otimes A_2)\{a_9^\varepsilon x_{26}^i\}, \\ H^*(BE_6;\mathbb{Z}/3) &= \bigoplus_{i,\varepsilon} H^{2i+\varepsilon}(\mathbb{Z}/3,A_2\otimes A_2\otimes A_2)\{a_9^\varepsilon x_{26}^i\}, \\ H^*(BE_7;\mathbb{Z}/3) &= \bigoplus_{i,\varepsilon} H^{2i+\varepsilon}(\mathbb{Z}/3,A_2\otimes A_2\otimes A_3)\{a_9^\varepsilon x_{26}^i\}. \end{aligned}$$

For $p = 5$, we have

$$H^*(BE_8;\mathbb{Z}/5) = \bigoplus_{i,\varepsilon} H^{2i+\varepsilon}(\mathbb{Z}/5,A_4\otimes A_4)\{a_{13}^\varepsilon x_{62}^i\}.$$

In the case $p = 3$, $G = F_4$, we put $A_2 = \mathbb{Z}/3[y_4, y_{12}]$, $A_2 = \mathbb{Z}/3[y_8, y_{16}]$ where the index indicates the degree. Then, we have the Poincaré series

$$PS(H^*(BF_4;\mathbb{Z}/3),t) = \sum_{i\geq 0} \dim H^i(BF_4;\mathbb{Z}/3)t^i$$

is equal to

$$\frac{1}{(1-t^4)(1-t^{12})(1-t^{16})(1-t^{24})} + \frac{t^8+t^9+t^{20}+t^{21}+t^{25}+t^{26}+t^{29}+t^{30}}{(1-t^{36})(1-t^{48})(1-t^{26})}.$$

The Poincaré series of $H^*(BG;\mathbb{Z}/p)$ for $(G,p) = (E_6,3)$, $(E_7,3)$, $(E_8,5)$ can be computed from the above theorem easily.

Remark 5.2 Computation of the case $(G,p) = (E_8,3)$ remains to be an open problem. It is known that the Rothenberg-Steenrod spectral sequence does not collapse at the E_2-level, so that $\mathrm{Cotor}_A(\mathbb{Z}/3,\mathbb{Z}/3) \neq H^*(BE_8;\mathbb{Z}/3)$ as graded $\mathbb{Z}/3$-modules. See [2] in detail.

6 Projective unitary groups

The special unitary group $SU(n)$ has the center C_n which is a cyclic group of order n. The projective unitary group $PU()$ is the central quotient $SU(n)/C_n$. In this section, we denote by C_r the cyclic subgroup of order r of the center products of special unitary groups. Little is known for the mod p cohomology of classifying spaces of projective unitary groups $PU(m)$ when p divides m. The case $p = 2$ and m is not divisible by 4 was computed by Kono and Mimura in [6]. As for odd primes, only the mod 3 cohomology of $BPU(3)$ was known in [5]. The mod p cohomology of $BPU(p)$ was computed by Vistoli in [13] and by Kameko and Yagita in [3], recently.

Theorem 6.1 Suppose that p does not divide m. After given suitable degrees for generators of each copy of A_{p-1}, A_p, we have an isomorphism

$$H^*(BPU(pm);\mathbb{Z}/p) = \bigoplus_{\varepsilon,i} H^{2i+\varepsilon}(\mathbb{Z}/p, A_{p-1}\otimes A_p^{m-1})\{a_3^\varepsilon x_{2p+2}^i\}$$

as a graded $\mathbb{Z}/p$-module where A_p^{m-1} is the tensor product of $(m-1)$-copies of A_p.

The result of Kono and Mimura could be stated in the same manner. Moreover, we have the following proposition.

Proposition 6.2 After giving suitable degrees for generators of each copy of A_{p-1}, A_p's, respectively, we have following isomorphisms of graded $\mathbb{Z}/p$-modules.

$$\begin{aligned} H^*(B(SU(p) \times SU(p)/C_p); \mathbb{Z}/p) &= \bigoplus_{i,\varepsilon} H^{2i+\varepsilon}(\mathbb{Z}/3, A_2 \otimes A_2)\{a_3^{\varepsilon} x_{2p+2}^{i}\}, \\ H^*(B(SU(p) \times SU(p) \times SU(p)/C_p); \mathbb{Z}/p) &= \bigoplus_{i,\varepsilon} H^{2i+\varepsilon}(\mathbb{Z}/3, A_2 \otimes A_2 \otimes A_2)\{a_3^{\varepsilon} x_{2p+2}^{i}\}, \\ H^*(B(SU(p) \times SU(2p)/C_p); \mathbb{Z}/p) &= \bigoplus_{i,\varepsilon} H^{2i+\varepsilon}(\mathbb{Z}/3, A_2 \otimes A_2 \otimes A_3)\{a_3^{\varepsilon} x_{2p+2}^{i}\}. \end{aligned}$$

This result corresponds to the computation of the cohomology of classifying spaces of exceptional Lie groups in Theorem 5.1.

Thus, it seem to be interesting to investigate the cohomology of classifying spaces of central quotients of products of unitary groups.

The special unitary group $SU(p^n)$ has a maximal torus T^{p^n-1} whose Weyl group is the symmetric group Σ_{p^n}. It contains a p-Sylow subgroup $\mathbb{Z}/p \int \cdots \int \mathbb{Z}/p$. The diagonal map induces a monomorphism

$$\mathbb{Z}/p \times \cdots \times \mathbb{Z}/p \to \mathbb{Z}/p \int \cdots \int \mathbb{Z}/p.$$

Consider the subgroup of the normalizer of the maximal torus T^{p^n-1}/C_{p^n} in $PU(p^n)$ generated by this elementary abelian p-subgroup and the maximal torus T^{p^n-1}/C_p^n. Let us denote it by

$$N_0 = (\mathbb{Z}/p \times \cdots \times \mathbb{Z}/p) \ltimes (T^{p^n-1}/C_{p^n}).$$

I think this subgroup plays an important role in the study of the cohomology of classifying spaces.

Conjecture 6.3 The induced homomorphism $H^*(BPU(p^n); \mathbb{Z}/p) \to H^*(BN_0; \mathbb{Z}/p)$ is a monomorphism.

Conjecture 6.4 There exists filtrations on the cohomology of $BPU(p^n)$ and BN_0 such that associated graded algebra of $H^*(BPU(p^n); \mathbb{Z}/p)$ and the associated graded algebra of $H^*(BN_0; \mathbb{Z}/p)$ are isomorphic to each other as ungraded algebras.

The second conjecture calls for some explanation. We say $H^*(\mathbb{C}P^\infty; \mathbb{Z}/2)$ and $H^*(\mathbb{R}P^\infty; \mathbb{Z}/2)$ are isomorphic as ungraded algebras since both are isomorphic to a polynomial algebra $\mathbb{Z}/2[x]$. Indeed, there is no map which induces an isomorphism between $H^*(\mathbb{C}P^\infty; \mathbb{Z}/2)$and $H^*(\mathbb{R}P^\infty; \mathbb{Z}/2)$. Also there exists a map

$$\mathbb{R}P^\infty \to \mathbb{C}P^\infty$$

such that the induced homomorphism $H^*(\mathbb{C}P^\infty;\mathbb{Z}/2) \to H^*(\mathbb{R}P^\infty;\mathbb{Z}/2)$ is a monomorphism. With this conjecture, we expect the computation of the cohomology of BN_0 is, to some extent, algebraically similar to the computation of the cohomology of $BPU(p^n)$.

For $(G,p) = (F_4,3)$ $(E_8,5)$, we have the following inclusions:

$$\mathbb{Z}/3 \ltimes ((T^2 \times T^2)/C_3) \longrightarrow SU(3) \times SU(3)/C_3 \longrightarrow F_4,$$

$$\mathbb{Z}/5 \ltimes ((T^4 \times T^4)/C_5) \longrightarrow SU(5) \times SU(5)/C_5 \longrightarrow E_8.$$

For $(G,p) = (E_6,3),(E_7,3),(E_8,3)$, we have the following inclusions:

$$\begin{tikzcd}
\mathbb{Z}/3 \ltimes ((T^2 \times T^2 \times T^2)/C_3) \arrow[r] \arrow[d] & SU(3) \times SU(3) \times SU(3)/C_3 \arrow[r] \arrow[d] & E_6 \\
\mathbb{Z}/3 \ltimes ((T^2 \times T^5)/C_3) \arrow[r] \arrow[d] & SU(3) \times SU(6)/C_3 \arrow[r] \arrow[d] & E_7 \\
(\mathbb{Z}/3 \times \mathbb{Z}/3) \ltimes (T^8/C_3) \arrow[r] & SU(9)/C_3 \arrow[r] & E_8.
\end{tikzcd}$$

We consider the left-hand-side groups as N_0 which is a subgroup of the normalizers of maximal tori. Theorem 5.1 and Proposition 6.2 implies that for

$$(G,p) = (F_4,3),(E_6,3),(E_7,3),(E_8,5),$$

the associated graded algebra of the cohomology of the classifying space of the right-hand-side group is isomorphic to the associated graded algebra of the cohomology of the classifying space of the middle group as an ungraded algebra but the obvious induced homomorphism is not an isomorphism. We hope such an isomorphism exists for $(G,p) = (E_8,3)$. We expect the cohomology of BG is controlled by N_0 rather than the normalizer of the maximal torus and the cohomology of BN_0 is easier than the cohomology of the classifying space of the normalizer of the maximal torus itself.

By replacing a maximal torus by elementary abelian p-subgroups, Quillen proved that the induced homomorphism

$$H^*(BG;\mathbb{Z}/p) \to \varprojlim H^*(BA;\mathbb{Z}/p)$$

is an F-isomorphism. It may have a nilpotent kernel. As a matter of fact, for $(G,p) = (\mathrm{Spin}(11),2)$, $(E_7,2)$, this Quillen homomorphism has non-trivial(but nilpotent) kernel. See Kono and Yagita [7]. Still, for odd prime p, Adams and Kono conjectured that the above Quillen homomorphism is a monomorphism. In conjunction with this conjecture, we have the following conjecture. For G such that $H_*(G;\mathbb{Z})$ has no p-torsion, N_0 is nothing but a maximal torus itself.

Conjecture 6.5 Let p be an odd prime. For all simply-connected simple Lie group G, the induced homomorphism $H^*(BG;\mathbb{Z}/p) \to H^*(BN_0;\mathbb{Z}/p)$ is a monomorphism.

Only the case $(G,p) = (E_8,3)$ remains unsettled.

We end this paper with the following conjecture.

Conjecture 6.6 For any prime p and for any connected compact Lie group G there exists a subgroup N_0 of the normalizer of its maximal torus T such that

(1) N_0/T is an elementary abelian p-group and
(2) the induced homomorphism $H^*(BG;\mathbb{Z}/p) \to H^*(BN_0;\mathbb{Z}/p)$ is a monomorphism.

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