数理解析研究所講究録
第 1682 巻 2010 年 33-40

How to stop near the top in a random walk?

Pieter C. Allaart *

January 27, 2010

Abstract

This note discusses the problem of maximizing the probability of stopping with one of the two highest values in a Bernoulli random walk with arbitrary parameter p and finite time horizon n. The optimal strategy is determined by a non-monotone sequence of constants (critical probabilities) $\{p_n^*\}$. Several properties of this sequence are proved, and additional properties are conjectured.

1 Introduction

Let $\{S_n\}_{n=0,1,2,\ldots}$ be a Bernoulli random walk with parameter $p\in(0,1)$. That is, $S_0\equiv 0$, and for $n\geq 1$, $S_n=X_1+\cdots+X_n$, where $X_1, X_2, \ldots$ are independent, identically distributed random variables with $\mathrm{P}(X_1=1)=p$, and $\mathrm{P}(X_1=-1)=q:=1-p$. Let $M_n:=\max\{S_0, S_1, \ldots, S_n\}$, for $n\in\mathbb{N}$. Suppose that, for some finite time horizon N, we wish to find a stopping time τ (adapted to the process $\{S_n\}$) that will maximize $\mathrm{P}(S_\tau=M_N)$; that is, suppose we wish to maximize the probability of "stopping at the top" of the random walk. What is the optimal τ?

Surprisingly, this simple question was answered in the literature only recently, by Yam et al. [4], though for the case $p=1/2$ it is already implicit in the work of Hlynka and Sheahan [3]. If $p>1/2$, the rule $\tau\equiv N$ is the unique optimal rule; if $p<1/2$, $\tau\equiv 0$ is the unique optimal rule; and if $p=1/2$, any rule τ such that $\mathrm{P}(S_\tau=M_\tau$ or $\tau=N)=1$ is optimal. The proof given by Yam et al. is far from the simplest one; an easier argument can be given by conditioning on the first time the reflected process $Z_n:=M_n-S_n$ returns to 0 and using backward induction.

Suppose now that, more generally, we wish to maximize the expectation of some nonincreasing function f of the distance from the stopped value of the walk to its eventual maximum. That is, we wish to find a stopping time τ that will maximize $\mathrm{E}[f(M_N-S_\tau)]$. Note that if we take $f(0)=1$ and $f(k)=0$ for $k\geq 1$, this reduces to the problem discussed earlier. For the choice $f(k)=d^k$ where $0<d<1$, Yam et al. [4] showed that the optimal rule is exactly as above. This leads one to wonder if there might be some general principle at work. Indeed, the author has shown in [1] that the rule that is optimal for the problem of maximizing the probability of stopping at the top remains optimal for the general problem, as long as f is nonincreasing and *convex*. A similar result holds in continuous time for Brownian motion with drift. In fact, this statement can be generalized well beyond simple random walk and Brownian motion: it applies to any random walk whose steps stochastically

*Address correspondence to P. C. Allaart, Department of Mathematics, University of North Texas, 1155 Union Circle #311430, Denton, TX 76203-5017, USA; E-mail: allaart@unt.edu

dominate their opposites or vice versa. It even applies to many Lévy processes, provided that the "small jumps" of these processes are sufficiently well-behaved. (See [2].)

But what if f is not convex? Then the optimal rule is in general much more complex, even for the seemingly simple case when $f(0) = f(1) = 1$ and $f(k) = 0$ for $k \geq 2$. In that case, the expectation to be maximized is

$$\mathrm{E}[f(M_N - S_\tau)] = \mathrm{P}(M_N - S_\tau \leq 1), \tag{1.1}$$

so we want to maximize the probability of stopping within one unit of the highest point of the walk. The main purpose of the present note is to record what the author knows about the optimal stopping rule for this particular problem, and what the author believes to be true but has been unable to prove. Some of the proofs below are based on lengthy calculations. When that is the case, the conceptual ideas will be emphasized, and many of the algebraic details will be omitted.

2 Stopping within one step from the top

From now on we focus on the problem of maximizing (1.1), for a given time horizon N. A useful observation is that, for $0 \leq n \leq N$, we can write

$$M_N - S_n = (M_n - S_n) \vee \max_{n \leq k \leq N} (S_k - S_n) = Z_n \vee M'_{N-n},$$

where $Z_n := M_n - S_n$, and M'_{N-n} is a random variable independent of the walk up to time n, having the same distribution as M_{N-n}. Thus, if we stop at time n and $Z_n = j$, we win with probability $\mathrm{P}(j \vee M_k \leq 1)$, where $k = N - n$. This probability is zero if $j \geq 2$, and simplifies to $\mathrm{P}(M_k \leq 1)$ if $j = 0$ or 1.

The process $(N - n, Z_n)$, $n = 0, 1, \ldots, N$ is a bivariate Markov chain. We can now conclude that in state (k, j) of this process, it is optimal to continue if $j \geq 2$ and $k \geq 1$. (Note that in state $(0, j)$ we must stop regardless of j.) In fact, in state $(k, 0)$ with $k \geq 1$ it is optimal to continue as well. For if we take one more step and then stop, we win with probability $\mathrm{P}(M_{k-1} \leq 1)$, which is at least as large as $\mathrm{P}(M_k \leq 1)$, the win probability if we stop immediately. Thus, the only non-trivial states are those of the form $(k, 1)$, where $k \geq 1$.

Proposition 2.1. *For each $n \geq 1$, there exists a number p_n^* in $[0, 1]$ such that, in state $(n, 1)$, it is optimal to stop if and only if $p \leq p_n^*$.*

Let

$$\begin{aligned} V_n &:= \text{optimal win probability from state } (n,1), \\ W_n &:= \text{optimal win probability from state } (n,1) \\ & \text{ if we take at least one step,} \\ U_n &:= \text{win probability from state } (n,1) \text{ if we stop,} \end{aligned}$$

so that $U_n = \mathrm{P}(M_n \leq 1)$, and $V_n = \max\{W_n, U_n\}$. Also define the hitting times

$$\tau_j := \inf\{n \geq 0 : S_n = j\}, \qquad j \in \mathbb{N}.$$

Proof of Proposition 2.1. In state $(1,1)$ is is optimal to stop regardless of p, so the statement is true for $n = 1$, and $p_1^* = 1$. Let $n \geq 2$, and define the stopping time

$$\sigma := \inf\{j \geq 1 : 1 \vee M_j - S_j = 1\}.$$

We show that $V_n - U_n$ is nondecreasing in p on $0 \leq p \leq 1/2$. Since the proof of part (ii) of the next lemma will show that $W_n \geq U_n$ for all $n \geq 2$ when $p \geq 1/2$, the proposition will follow.

First, observe that we can write

$$V_n = \max\left\{ U_n, \sum_{j=2}^{n} \mathrm{P}(\sigma = j)V_{n-j} + p^n \right\},$$

because if in state $(n,1)$ we continue, we can win only if the walk either comes back to one unit below its running maximum at some future time, or records a string of n straight up-steps. Furthermore, for $n \geq 2$,

$$\begin{aligned} U_n &= pqU_{n-2} + q\sum_{j=2}^{n} \mathrm{P}(\tau_1 = j-1)U_{n-j} + q\,\mathrm{P}(\tau_1 > n-1) \\ &= \sum_{j=2}^{n} \mathrm{P}(\sigma = j)U_{n-j} - \sum_{j=3}^{n} p^{j-1}qU_{n-j} + q\,\mathrm{P}(\tau_1 > n-1) \\ &= \sum_{j=2}^{n} \mathrm{P}(\sigma = j)U_{n-j} - \sum_{j=3}^{n} p^{j-1}\,\mathrm{P}(M_{n-j+1} = 0) + q\,\mathrm{P}(\tau_1 > n-1). \end{aligned}$$

Thus

$$\begin{aligned} V_n - U_n = \max\Bigg\{ 0, &\sum_{j=2}^{n} \mathrm{P}(\sigma = j)\left(V_{n-j} - U_{n-j}\right) + p^n \\ &+ \sum_{j=3}^{n} p^{j-1}\,\mathrm{P}(M_{n-j+1} = 0) - q\,\mathrm{P}(\tau_1 > n-1) \Bigg\}. \end{aligned} \tag{2.1}$$

Now each j-step path in the event $\{\sigma = j\}$ must have at least as many up-steps as down-steps, and so $\mathrm{P}(\sigma = j)$ is increasing in p on $p \leq 1/2$, for all j. Thus, by the induction hypothesis, the first summation in (2.1) is nondecreasing on $p \leq 1/2$. We now proceed to prove the same for the remaining terms.

Since $\mathrm{P}(M_j = 0) = 1 - \mathrm{P}(\tau_1 \leq j)$, we obtain after some arithmetic,

$$p^n + \sum_{j=3}^{n} p^{j-1}\,\mathrm{P}(M_{n-j+1} = 0) - q\,\mathrm{P}(\tau_1 > n-1) = \sum_{i=1}^{n-1} a_{n-i}\,\mathrm{P}(\tau_1 = i) - a_n,$$

where

$$a_j := 1 - \frac{p}{q}(1 - p^j), \qquad j \in \mathbb{N}.$$

Now it is easy to see that $p \leq 1/2$ implies $a_j > 0$ for all j. Furthermore,

$$\frac{da_j}{dp} = \frac{1}{q^2}\left[(jq+1)p^j - 1\right] \geq -\frac{1}{q^2}.$$

Also, $d\,\mathrm{P}(\tau_1 = i)/dp \geq 0$ for $p \leq 1/2$, since $\mathrm{P}(\tau_1 = 2j) = 0$, and

$$\mathrm{P}(\tau_1 = 2j+1) = \frac{1}{j+1}\binom{2j}{j} p^{j+1} q^j$$

which has a higher power of p than of $q = 1 - p$. It therefore follows that

$$\begin{aligned}
&\frac{d}{dp}\left[\sum_{i=1}^{n-1} a_{n-i}\,\mathrm{P}(\tau_1 = i) - a_n\right] \\
&\geq \frac{da_{n-1}}{dp}\,\mathrm{P}(\tau_1 = 1) + \sum_{i=2}^{n-1}\left(-\frac{1}{q^2}\right)\mathrm{P}(\tau_1 = i) - \frac{da_n}{dp} \\
&= \frac{1}{q^2}\left[\{((n-1)q+1)p^{n-1} - 1\}\,p - \{(nq+1)p^n - 1\} - \sum_{i=2}^{n-1}\mathrm{P}(\tau_1 = i)\right] \\
&= \frac{1}{q^2}\left[\mathrm{P}(\tau_1 \geq n) - p^n q\right] \geq \frac{1}{q^2}(q^{n-1} - p^n q) \geq 0
\end{aligned}$$

for $p \leq 1/2$, as required. □

Lemma 2.1. *We have*
(i) $p_1^* = 1$ *and* $p_2^* = p_3^* = 1/2$*;*
(ii) $p_n^* < 1/2$ *for all* $n \geq 4$*; and*
(iii) $\limsup_{n\to\infty} p_n^* = 1/2$.

Proof. Statement (i) follows in a straightforward manner by using backward induction. For (ii), consider the stopping time

$$\tau := \begin{cases} n-1, & \text{if } 1 \vee M_{n-1} - S_{n-1} \leq 1 \\ n, & \text{otherwise.} \end{cases}$$

Then in state $(n, 1)$,

$$\begin{aligned}
W_n &\geq \mathrm{P}(\text{win using } \tau) \\
&= \mathrm{P}(1 \vee M_{n-1} - S_{n-1} \leq 1)\cdot 1 + \mathrm{P}(1 \vee M_{n-1} - S_{n-1} = 2)\cdot p,
\end{aligned}$$

while for $p \geq 1/2$,

$$\begin{aligned}
U_n &\leq \mathrm{P}(M_n^q \leq 1) = \mathrm{P}(M_n - S_n \leq 1) \\
&= p\,\mathrm{P}(M_{n-1} - S_{n-1} \leq 1) + q\,\mathrm{P}(1 \vee M_{n-1} - S_{n-1} \leq 1).
\end{aligned}$$

Here M_n^q denotes the maximum after n steps of a Bernoulli random walk with parameter q, and we have used the well-known fact that $M_n^q \stackrel{d}{=} M_n - S_n$. It follows that for $p \geq 1/2$,

$$W_n - U_n \geq p\left[\mathrm{P}(1 \vee M_{n-1} - S_{n-1} \leq 2) - \mathrm{P}(M_{n-1} - S_{n-1} \leq 1)\right].$$

Now $\{M_{n-1} - S_{n-1} \leq 1\} \subseteq \{1 \vee M_{n-1} - S_{n-1} \leq 2\}$, the inclusion being proper if $n \geq 4$. So $W_n \geq U_n$ for $n \geq 2$, with strict inequality for $n \geq 4$ and all $p \geq 1/2$. Since $W_n - U_n$ is the maximum of several polynomials in p, it is continuous in p. Hence $p_n^* < 1/2$ for $n \geq 4$.

(iii) Suppose, by way of contradiction, that $\limsup p_n^* < 1/2$. By part (ii), there exists p with $\sup_{n\geq 4} p_n^* < p < 1/2$. For this p, the optimal rule in state $(n, 1)$ (where $n \geq 4$) is to wait until the walk reaches one of the states $(3, 1)$, $(2, 1)$, $(1, 1)$ or $(0, j)$ and then stop. But then

$$W_n \leq \mathrm{P}(1 \vee M_{n-3} - S_{n-3} \leq 4) \leq \mathrm{P}(S_{n-3} \geq -3) \to 0$$

as $n \to \infty$, since the walk has a negative drift. On the other hand,

$$\lim_{n\to\infty} \mathrm{P}(M_n \leq 1) = \mathrm{P}(M_n \leq 1 \ \forall n) = \mathrm{P}(\tau_2 = \infty) = 1 - \left(\frac{p}{q}\right)^2 > 0.$$

Thus, for large enough n, $W_n < U_n$ and so $p < p_n^*$, a contradiction. □

Conjecture 2.1. $\lim_{n\to\infty} p_n^* = 1/2$.

Theorem 2.1. *For every $m \geq 4$, we have $p_{2m+1}^* \geq p_{2m-1}^* \geq p_{2m}^*$.*

Conjecture 2.2. *In addition, $p_{2m}^* \leq p_{2m+2}^*$ for all $m \geq 2$.*

The theorem and conjecture are illustrated by Table 1 at the end of this paper. To prove the theorem, we use the formula

$$\mathrm{P}(M_n = k, S_n = l) = a_{n,k,l}\, p^{(n+l)/2} q^{(n-l)/2}, \qquad 0 \leq l \leq k, \tag{2.2}$$

where

$$a_{n,k,l} := \binom{n}{\frac{1}{2}(n + 2k - l)} - \binom{n}{\frac{1}{2}(n + 2k + 2 - l)}.$$

Let

$$t_m := \frac{1}{m}\binom{2m-2}{m-1}, \qquad m \in \mathbb{N}.$$

It is not difficult to derive, for each $m \in \mathbb{N}$, that

$$U_{2m} = U_{2m+1} = 1 - \sum_{j=1}^{m} t_{j+1} p^{j+1} q^{j-1}. \tag{2.3}$$

Proof of Theorem 2.1. For convenience, we re-index and write the statement as

$$p_{2n+5}^* \geq p_{2n+3}^* \geq p_{2n+4}^*, \qquad \text{for all } n \geq 2.$$

That the statement holds for $n = 2$ follows easily by a direct calculation of $p_4^*, \ldots, p_9^*$. Let $m \geq 3$, and assume the statement holds for all values of n up to $m - 1$. Note that this implies $p_{2m+3}^* \geq p_j^*$ for all $4 \leq j \leq 2m + 2$. Let $1/2 > p \geq p_{2m+3}^*$. Then, if in state $(2m + 3, 1)$ or $(2m + 4, 1)$ we continue, the optimal strategy is to wait until there are 3 steps left, and play optimally from then on. Thus, we condition on the state of the process at the time when there are 3 steps remaining. We first show that

$$W_{2m+4} - U_{2m+4} \geq W_{2m+3} - U_{2m+3}. \tag{2.4}$$

This inequality implies that, if in state $(2m + 3, 1)$ it is optimal to continue, then it is optimal to continue in state $(2m + 4, 1)$ as well; thus, $p_{2m+4}^* \leq p_{2m+3}^*$.

First, by (2.3),

$$U_{2m+4} - U_{2m+3} = -t_{m+3}p^{m+3}q^{m+1}. \tag{2.5}$$

Let $\pi_{n,k}$ denote the optimal win probability from state (n, k); that is,

$$\pi_{n,k} := \sup_{\tau \le n} \mathrm{P}(k \vee M_\tau - S_\tau \le 1).$$

Let $\Delta\pi_{3,k} := \pi_{3,k} - \pi_{3,k+1}$. Then

$$W_{2m+3+j} = \sum_{k=0}^{4} \mathrm{P}(1 \vee M_{2m+j} - S_{2m+j} \le k)\Delta\pi_{3,k} \qquad \text{for } j = 0, 1,$$

and so

$$W_{2m+4} - W_{2m+3} = \sum_{k=0}^{4} \Delta P_{2m,k}\Delta\pi_{3,k},$$

where

$$\Delta P_{n,k} := \mathrm{P}(1 \vee M_{n+1} - S_{n+1} \le k) - \mathrm{P}(1 \vee M_n - S_n \le k).$$

It is easy to see that

$$\mathrm{P}(1 \vee M_{n+1} - S_{n+1} \le k) = p\,\mathrm{P}(M_n - S_n \le k) + q\,\mathrm{P}(2 \vee M_n - S_n \le k),$$

and since

$$\begin{aligned}
\mathrm{P}(M_n - S_n \le k) - \mathrm{P}(1 \vee M_n - S_n \le k) &= \mathrm{P}(M_n = 0, S_n = -k), \\
\mathrm{P}(1 \vee M_n - S_n \le k) - \mathrm{P}(2 \vee M_n - S_n \le k) &= \mathrm{P}(M_n \le 1, S_n = 1 - k),
\end{aligned}$$

it follows that

$$\Delta P_{n,k} = p\,\mathrm{P}(M_n = 0, S_n = -k) - q\,\mathrm{P}(M_n \le 1, S_n = 1 - k).$$

Now put $n = 2m$ and apply the last identity for $k = 0, 1, \dots, 4$. Also use (2.2) and the notation

$$d_{m,j} := \binom{2m}{m+j} - \binom{2m}{m+j+1}.$$

Then:

$$\begin{aligned}
\Delta P_{2m,0} &= p\,\mathrm{P}(M_{2m} = 0, S_{2m} = 0) = d_{m,0}p^m q^m, \\
\Delta P_{2m,1} &= -q\,\mathrm{P}(M_{2m} \le 1, S_{2m} = 0) = (d_{m,0} + d_{m,1})p^m q^m, \\
\Delta P_{2m,2} &= p\,\mathrm{P}(M_{2m} = 0, S_{2m} = -2) = d_{m,1}p^{m-1}q^{m+1}, \\
\Delta P_{2m,3} &= -q\,\mathrm{P}(M_{2m} \le 1, S_{2m} = -2) = (d_{m,1} + d_{m,2})p^{m-1}q^{m+1}, \\
\Delta P_{2m,4} &= p\,\mathrm{P}(M_{2m} = 0, S_{2m} = -4) = d_{m,2}p^{m-2}q^{m+2}.
\end{aligned}$$

A direct calculation yields $\Delta\pi_{3,0} = p^3$, $\Delta\pi_{3,1} = -p^2q + 2pq^2 + q^3$, $\Delta\pi_{3,2} = -p^3 + 2p^2q + pq^2$, $\Delta\pi_{3,3} = p^2q$, and $\Delta\pi_{3,4} = p^3$. Thus, we obtain

$$\begin{aligned}
W_{2m+4} - W_{2m+3} = p^m q^m \big[&d_{m,0}p^4 - d_{m,1}p^3q + (d_{m,0} + 3d_{m,1} + d_{m,2})p^2q^2 \\
&- (2d_{m,0} + 2d_{m,1} + d_{m,2})pq^3 - (d_{m,0} + d_{m,1})q^4\big].
\end{aligned}$$

It follows using (2.5) that $W_{2m+4} - U_{2m+4} \geq W_{2m+3} - U_{2m+3}$ if and only if

$$d_{m,0}p^4 + (t_{m+3} - d_{m,1})p^3q + (d_{m,0} + 3d_{m,1} + d_{m,2})p^2q^2 \\ - (2d_{m,0} + 2d_{m,1} + d_{m,2})pq^3 - (d_{m,0} + d_{m,1})q^4 \geq 0.$$

Dividing by $\binom{2m}{m}$, multiplying by $(m+1)(m+2)(m+3)$ and substituting $q = 1 - p$, we can (eventually) write this last inequality as

$$(12m^2 + 18m + 6)p^4 - (40m^2 + 44m + 12)p^3 + (30m^2 + 12m + 6)p^2 \\ + (3m^2 + 33m + 12)p - (4m^2 + 14m + 6) \geq 0. \quad (2.6)$$

Now since $p > p_4^*$, we have $W_4 \geq U_4$. A straightforward calculation gives

$$W_4 - U_4 = p^4 - 2p^3 + p^2 + 2p - 1 \geq 0. \quad (2.7)$$

With some further algebra, it can be seen that (2.7) implies (2.6) when $p \leq 1/2$. Thus, we have (2.4).

Next, we show that

$$W_{2m+5} - U_{2m+5} \leq W_{2m+3} - U_{2m+3}. \quad (2.8)$$

At $p = p_{2m+3}^*$, the right hand side of this inequality is zero, so $W_{2m+5} \leq U_{2m+5}$. Thus, (2.8) implies that $p_{2m+5}^* \geq p_{2m+3}^*$.

Define

$$\Delta^2 P_{n,k} := \mathrm{P}(1 \vee M_{n+2} - S_{n+2} \leq k) - \mathrm{P}(1 \vee M_n - S_n \leq k).$$

In a manner similar to that applied to $\Delta P_{n,k}$ in the first part of the proof, we can show that

$$\Delta^2 P_{n,k} = p^2\,\mathrm{P}(M_n = 0, S_n = -k) - q^2\,\mathrm{P}(M_n \leq 1, S_n = 1 - k) \\ - q^2\,\mathrm{P}(M_n \leq 2, S_n = 2 - k).$$

Applying this again with $n = 2m$ and $k = 0, 1, \ldots, 4$, we obtain

$$\begin{aligned}
\Delta^2 P_{2m,0} &= d_{m,0}p^{m+2}q^m - d_{m,1}p^{m+1}q^{m+1}, \\
\Delta^2 P_{2m,1} &= -(d_{m,0} + d_{m,1})p^m q^{m+2}, \\
\Delta^2 P_{2m,2} &= d_{m,1}p^{m+1}q^{m+1} - (d_{m,0} + d_{m,1} + d_{m,2})p^m q^{m+2}, \\
\Delta^2 P_{2m,3} &= -(d_{m,1} + d_{m,2})p^{m-1}q^{m+3}, \\
\Delta^2 P_{2m,4} &= d_{m,2}p^m q^{m+2} - (d_{m,1} + d_{m,2} + d_{m,3})p^{m-1}q^{m+3}.
\end{aligned}$$

Now

$$U_{2m+5} - U_{2m+3} = -t_{m+3}p^{m+3}q^{m+1}$$

because $U_{2m+5} = U_{2m+4}$. So we get, putting everything together,

$$\begin{aligned}
&W_{2m+5} - W_{2m+3} - (U_{2m+5} - U_{2m+3}) \\
&= \sum_{k=0}^{4} \Delta^2 P_{2m,k}\Delta\pi_{3,k} + t_{m+3}p^{m+3}q^{m+1} \\
&= p^m q^m \big[(d_{m,0}p^5 - 2d_{m,1}p^4q + (d_{m,0} + 3d_{m,1} + 2d_{m,2})p^3q^2 \qquad (2.9) \\
&\qquad - (d_{m,0} + d_{m,1} + 3d_{m,2} + d_{m,3})p^2q^3 \\
&\qquad - (3d_{m,0} + 4d_{m,1} + 2d_{m,2})pq^4 \\
&\qquad - (d_{m,0} + d_{m,1})q^5 + t_{m+3}p^3q\big].
\end{aligned}$$

n	p_n^*
1	1
2	1/2
3	1/2
4	$.468989\cdots$
5	$.482881\cdots$
6	$.471448\cdots$
7	$.482686\cdots$
8	$.474706\cdots$
9	$.483681\cdots$
10	$.477526\cdots$

n	p_n^*
11	$.484704\cdots$
12	$.479846\cdots$
13	$.485666\cdots$
14	.4818
15	.4863
16	.4834
17	.4870
18	.4846
19	.4877
20	.4856

Table 1: Values of p_n^* for $n \leq 20$. Values to 6 decimal places are exact; others are numerical estimates obtained by backward induction.

Now we can rewrite

$$
\begin{aligned}
&t_{m+3}p^3q - 2d_{m,1}p^4q \\
&\quad = \binom{2m}{m}\frac{(10m^2+14m+12)p^4q + 4(2m+1)(2m+3)p^3q^2}{(m+1)(m+2)(m+3)},
\end{aligned}
$$

which is increasing in p on $p \leq 1/2$. Since p^5 also increases in p, and p^2q^3, pq^4 and q^5 all decrease in p on $p \geq 2/5$, we see from (2.9) that $W_{2m+5}-W_{2m+3}-(U_{2m+5}-U_{2m+3})$ is increasing in p on $p_4^* \leq p \leq 1/2$. (Note that $p_4^* \approx .4690 > 2/5$; see Table 1) So it suffices to consider the value of this difference at $p = 1/2$. Then we can calculate

$$
\begin{aligned}
&W_{2m+5} - W_{2m+3} - (U_{2m+5} - U_{2m+3}) \\
&\quad = -\frac{4(1/2)^5\binom{2m}{m}}{(m+1)(m+2)(m+3)(m+4)}(m-6)(2m^2+3m+1) \\
&\quad \leq 0,
\end{aligned}
$$

for $m \geq 6$. For $m = 3,4,5$ the expression in square brackets in (2.9), call it $f_m(p)$, must be investigated more carefully. Calculating $p_5^*,\dots,p_{13}^*$ using backward induction (see Table 1) and determining the zeroes of $f_m(p)$ for $m = 3,4,5$, it can be seen that for these values of m, $f_m(p) < 0$ if p is sufficiently close to p_{2m+3}^*. This completes the proof. □

References

[1] Allaart, P. C. (2009a). A general "bang-bang" principle for predicting the maximum of a random walk. *arXiv:0910.0545*.

[2] Allaart, P. C. (2009b). A "bang-bang" principle for predicting the supremum of a random walk or Lévy process. *arXiv:0912.0615*.

[3] Hlynka, M. and Sheahan, J. N. (1988). The secretary problem for a random walk. *Stoch. Proc. Appl.* **28** 317–325.

[4] Yam, S. C. P., Yung, S. P. and Zhou, W. (2009). Two rationales behind 'buy-and-hold or sell-at-once'. *J. Appl. Probab.* **46** 651–668.