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Multiple Sums-the-Odds Theorem[1]

穴太 克則 (Katsunori Ano)
東京工業大学・情報理工学研究科 柿沼 英夫 (Hideo Kakinuma)
Graduate School of Information Science and Engineering
Tokyo Institute of Technology
東京工業大学・情報理工学研究科 三好 直人 (Naoto Miyoshi)
Graduate School of Information Science and Engineering
Tokyo Institute of Technology

Abstract

We extends so-called *Sum-the-Odds Theorem* in optimal stopping to the multiple stopping case. The optimal multiple stopping rule is shown to be the form of Multiple Sums-the-Odds. We give the recursive equation of the maximum probability of win (that is, obtaining the last success) with multiple stopping chances. Further, the asymptotic maximum probability of win with double stopping chances is studied.

1 Introduction

For a positive integer N, let $X_1, X_2, \cdots, X_N$ be independent Bernoulli random variables on the probability space $(\Omega, \mathcal{F}, P)$. Let $p_i = P(X_i = 1) = 1 - q_i = 1 - P(X_i = 0)$. The $X_i's$ are observed sequentially. We call " success" if $X_i = 1$. The problem is to find a stopping rule τ to maximize the probability of stopping at the last success with exactly one stopping chance. Bruss [4] shows with the elegant simpleness that the optimal stopping rule $\tau_*^{(1)}$ stops when the sum of the odds of future successes is less than one;

$$\tau_*^{(1)} = \min \left\{ i \in [1, N] : X_i = 1 \ \& \ \sum_{j=i+1}^{N} r_i \leq 1 \right\}, \tag{1.1}$$

and obtains the maximum probability of "win" (that is, obtaining the last success), $P^{(1)}(\text{win})$, as follows.

$$P^{(1)}(\text{win}) = \prod_{k=i_*^{(1)}}^{N} q_k \sum_{k=i_*^{(1)}}^{N} r_k, \tag{1.2}$$

where $r_i = p_i/(1-p_i)$ is the odds, if $p_i = 1$, r_i is taken to be $+\infty$ and $i_*^{(1)} = \min\{i \in [1, N] : \sum_{i=k+1}^{N} r_i \leq 1\}$. This problem and the result are referred to as the odds problem and the Sum-the-Odds Theorem. Hill and Krengel [10] and Bruss [5] remarkably find that the lower bounds of the maximum probability of obtaining the last success is e^{-1} whatever be the values of the p_i. This value is known as the asymptotic probability of win for the Classical Secretary Problem (CSP) having the specific $p_i = 1/i$ for $i = 1, \cdots, N$. Ferguson [8] extends it in several ways.

[1]This paper is an abbreviated version of Ano, Kakinuma and Miyoshi [1].

An infinite number of Bernoulli trial is allowed or the Bernoulli variables are allowed to be dependent. Here, we extend the single stopping chance of the odds problem to the multiple stopping chances. First main result is that when we have m ($m > 1$) stopping chances, the optimal stopping time for each $k = 1, 2, \cdots, m$ is also shown to be the multiple sums-the-odds form.

As the second main result, we shows that the asymptotic maximum probability of win for the odds problem with the double stopping chances are shown to be $e^{-1} + e^{-3/2}$ under some appropriate conditions. It is nice to see that this asymptotic probability of win coincides with the asymptotic probability of win for the the CSP with double stopping chances.

This paper is organized as follows. In Section 2, we derive the optimal multiple stopping rule. To find it, our approach is essentially based on the method of Ano and Ando [2], in which they study the condition for the one-step look-ahead stopping rule to be optimal in the monotone multiple stopping problem. For the monotone stopping problem, see Chow, Robbins and Siegmund [6] or Ferguson [7]. In Section 3, we give the recursive formula of $P^{(m)}(\text{win})$. Using this formula and the method in Bruss [4], the asymptotic probability of win with double stopping chances are discussed.

2 Multiple Sums-the-Odds Theorem

Let $V_i^{(m)}$ be the maximum probability of win when we have at most m stopping chances hereafter and we stop at $X_i = 1$. Let $W_i^{(m)}$ be the maximum probability of win when we have at most m stopping chances hereafter and we continue at $X_i = 1$. Then $V_i^{(m)}$ and $W_i^{(m)}$ are given as follows.

$$\begin{aligned} V_i^{(m)} &= P(X_{i+1} = 0, X_{i+2} = 0, \cdots, X_N = 0 | X_i = 1) + W_i^{(m-1)} \\ &= \prod_{j=i+1}^{N} P(X_j = 0) + W_i^{(m-1)} \\ &= \prod_{j=i+1}^{N} q_j + W_i^{(m-1)}, \end{aligned} \tag{2.1}$$

where, $V_i^{(0)} = 0$. The second equality follows from the independence of $X_i^{\prime}$s.

$$\begin{aligned} W_i^{(m)} &= \sum_{j=i+1}^{N} P(X_{i+1} = X_{i+2} = \cdots = X_{j-1} = 0, X_j = 1) M_j^{(m)} \\ &= \sum_{j=i+1}^{N} \Big[\prod_{k=i+1}^{j-1} P(X_k = 0) \Big] P(X_j = 1) M_j^{(m)} \\ &= \sum_{j=i+1}^{N} \Big[\prod_{k=i+1}^{j-1} q_k \Big] p_j M_j^{(m)}. \end{aligned} \tag{2.2}$$

where, $\prod_{k=i+1}^{i} \cdot = 1$ and $W_i^{(0)} = 0$ for each i. Hence, the optimality equation is as follows. For each $m = 1, 2, \cdots, N$,

$$M_i^{(m)} = \max\{V_i^{(m)}, W_i^{(m)}\}, \quad i = 1, 2, \cdots, N-1. \tag{2.3}$$

When we face $X_N = 1$ and we have m more stopping chances, we then win with probability 1. So that $M_N^{(m)} = V_N^{(m)} = 1$. When we continue at $X_N = 1$, we lose with probability 1. Hence, $W_N^{(m)} = 0$.

2.1 Double stopping odds problem

As a preparation to the double stopping odds problem, we give another proof of the Sum-the-Odds Theorem by the one-step look-ahead stopping rule. The one-step look-ahead stopping region for the single stopping odds problem is given by $B^{(1)} = \{i : G_i^{(1)} \geq 0\}$, where

$$G_i^{(1)} := V_i^{(1)} - \sum_{j=i+1}^{N} \Big[\prod_{k=i+1}^{j-1} q_k \Big] p_j V_j^{(1)}. \tag{2.4}$$

$B^{(1)}$ is the region that the probability of win by the immediately stopping at $X_i = 1$ is not less than the probability of win when we continue at $X_i = 1$ and then stop at the first success arriving after X_i. Substituting $V_i^{(1)} = \prod_{j=i+1}^{N} q_j$ into (2.4),

$$\begin{aligned} G_i^{(1)} &= \prod_{j=i+1}^{N} q_j - \sum_{j=i+1}^{N} \Big[\prod_{k=i+1}^{j-1} q_k \Big] p_j \Big[\prod_{k=j+1}^{N} q_k \Big] \\ &= \prod_{j=i+1}^{N} q_j \left(1 - \sum_{j=i+1}^{N} r_j \right). \end{aligned} \tag{2.5}$$

Hence $B^{(1)}$ is written as $B^{(1)} = \left\{ i : \sum_{j=i+1}^{N} r_j \leq 1 \right\}$. Since $i \mapsto \sum_{j=i+1}^{N} r_j$ is decreasing, $B^{(1)}$ is "closed" in the sense of monotone problem of Chow et al [6]. Therefore, $\tau_*^{(1)} = \min\{i \in [1, N] : i \in B^{(1)}\} = \min\{i \geq i_*^{(1)} : X_i = 1\}$, where $i_*^{(1)} = \min\{i \in [1, N] : \sum_{i=k+1}^{N} r_i \leq 1\}$, is optimal. This coincides the optimal stopping rule in (1.1).

Theorem 2.1 *When we have at most double stopping chances, the optimal first and second stopping times are given by* $\tau_*^{(2)} = \min\{i \geq i_*^{(2)} : X_i = 1\}$ *and* $\tau_*^{(1)} = \inf\{i \geq i_*^{(1)} : X_i = 1\}$, *respectively, where*

$$i_*^{(2)} = \min \left\{ i \in [1, N] : \sum_{j=i+1}^{i_*^{(1)}-1} r_j + \sum_{j_1 = i+1 \vee i_*^{(1)}}^{N} r_{j_1} \sum_{j_2 = j_1 + 1}^{N} r_{j_2} \leq 1 \right\}, \tag{2.6}$$

$$i_*^{(1)} = \min \left\{ i \in [1, N] : \sum_{j=i+1}^{N} r_j \leq 1 \right\}. \tag{2.7}$$

Proof. When we have at most double stopping chances, the one-step look-ahead stopping region $B^{(2)}$ is given by $B^{(2)} \left\{ i : \; G_i^{(2)} \geq 0 \right\}$, where

$$G_i^{(2)} := V_i^{(2)} - \sum_{j=i+1}^{N} \Big[\prod_{k=i+1}^{j-1} q_k \Big] p_j V_j^{(2)}. \tag{2.8}$$

We show that $B^{(2)}$ is closed by two steps as follows.

(Step 1) First, we shall show that the following equation holds.

$$G_i^{(2)} = G_i^{(1)} + \sum_{j=i+1}^{N} \Big[\prod_{k=i+1}^{j-1} q_k \Big] p_j G_j^{(1)} I_{\{j \geq i_*^{(1)}\}}. \tag{2.9}$$

From (2.1) and (2.2), it follows that

$$\begin{aligned} G_i^{(2)} &= V_i^{(1)} + W_i^{(1)} - \sum_{j=i+1}^{N} \Big[\prod_{k=i+1}^{j-1} q_k \Big] p_j (V_j^{(1)} + W_j^{(1)}) \\ &= \left(V_i^{(1)} - \sum_{j=i+1}^{N} \Big[\prod_{k=i+1}^{j-1} q_k \Big] p_j V_j^{(1)} \right) + W_i^{(1)} - \sum_{j=i+1}^{N} \Big[\prod_{k=i+1}^{j-1} q_k \Big] p_j W_j^{(1)} \\ &= G_i^{(1)} + \sum_{j=i+1}^{N} \Big[\prod_{k=i+1}^{j-1} q_k \Big] p_j (M_j^{(1)} - W_j^{(1)}). \end{aligned} \tag{2.10}$$

On the other hand, observe that if $j \geq i_*^{(1)}$, then $M_j^{(1)} = V_j^{(1)}$ and if $j < i_*^{(1)}$, then $M_j^{(1)} = W_j^{(1)}$. From these it follows that

$$M_j^{(1)} - W_j^{(1)} = (V_j^{(1)} - W_j^{(1)}) I_{\{j \geq i_*^{(1)}\}}, \tag{2.11}$$

where I_A is the indicator function on A. Further, we have

$$W_j^{(1)} = \sum_{\ell=j+1}^{N} \Big[\prod_{k=j+1}^{\ell-1} q_k \Big] p_\ell M_\ell^{(1)} = \sum_{\ell=j+1}^{N} \Big[\prod_{k=j+1}^{\ell-1} q_k \Big] p_\ell V_\ell^{(1)}.$$

Substituting the equation above into (2.11), we have

$$M_j^{(1)} - W_j^{(1)} = \left(V_j^{(1)} - \sum_{\ell=j+1}^{N} \Big[\prod_{k=j+1}^{\ell-1} q_k \Big] p_\ell V_\ell^{(1)} \right) I_{\{j \geq i_*^{(1)}\}} = G_j^{(1)} I_{\{j \geq i_*^{(1)}\}}.$$

Hence, substituting this equation into (2.12), we have (2.9).

(Step 2) Let $H_i^{(1)} := 1 - \sum_{j=i+1}^{N} r_j$. From (2.5), it follows that $G_i^{(1)} = \prod_{j=i+1}^{N} q_j H_i^{(1)}$. Hence, substituting this $G_i^{(1)}$ into (2.9), we have

$$G_i^{(2)} = \prod_{j=i+1}^{N} q_j \left(H_i^{(1)} + \sum_{j=i+1 \vee i_*^{(1)}}^{N} r_j H_j^{(1)} \right). \tag{2.12}$$

Let $H_i^{(2)} := H_i^{(1)} + \sum_{j=i+1 \vee i_*^{(1)}}^{N} r_j H_j^{(1)}$. Substituting $H_i^{(1)} = 1 - \sum_{j=i+1}^{N} r_j$ into $H_i^{(2)}$, we have

$$H_i^{(2)} = 1 - \sum_{j=i+1}^{i_*^{(1)}-1} r_j - \sum_{j=i+1 \vee i_*^{(1)}}^{N} r_j \sum_{\ell=j+1}^{N} r_\ell. \tag{2.13}$$

Therefore, we have

$$B^{(2)} = \{ i : H_i^{(2)} \geq 0 \} = \left\{ i : \sum_{j=i+1}^{i_*^{(1)}-1} r_j + \sum_{j=i+1 \vee i_*^{(1)}}^{N} r_j \sum_{\ell=j+1}^{N} r_\ell \leq 1 \right\}.$$

Since $H_j^{(1)} \geq 0$ for $j \geq i_*^{(1)}$, $\sum_{j=i+1 \vee i_*^{(1)}}^{N} r_j H_j^{(1)}$ is nonnegative and $H_i^{(2)} \geq H_i^{(1)}$. $H_i^{(2)} \geq 0$ for $i \geq i_*^{(1)}$. For $i < i_*^{(1)}$, $i \mapsto H_i^{(1)}$ is increasing. Therefore, for $i < i_*^{(1)}$, $i \mapsto H_i^{(2)}$ is also increasing. Hence, $B^{(2)}$ is "closed" and the optimal stopping region. Hence, the optimal first stopping time is τ_2^*. From Bruss' Theorem, the optimal second stopping time is τ_1^*. □

From (2.6) and (2.7), we immediately have the next Corollary.

Corollary 2.1 $1 \leq i_*^{(2)} \leq i_*^{(1)} \leq N$.

2.2 Odds theorem for multiple stopping problem

When more m $(1 \leq m \leq N)$ stopping chances are allowed, the one-step look-ahead stopping region, $B^{(m)}$, is $B^{(m)} = \{i : G_i^{(m)} \geq 0\}$. where

$$G_i^{(m)} := V_i^{(m)} - \sum_{j=i+1}^{N} \Big[\prod_{k=i+1}^{j-1} q_k \Big] p_j V_j^{(m)}. \tag{2.14}$$

If we set for each $m = 1, 2, \cdots, N$,

$$H_i^{(m)} := H_i^{(1)} + \sum_{j=i+1 \vee i_*^{(m-1)}}^{N} r_j H_j^{(m-1)}, \tag{2.15}$$

where $H_i^{(1)} = 1 - \sum_{j=i+1}^{N} r_j$ and $i_*^{(m)} = \min\{i \geq 1 : H_i^{(m)} \geq 0\}$, then $B^{(m)} = \{i : H_i^{(m)} \leq 1\}$.

Theorem 2.2 *When we have at most m stopping chances, the optimal stopping times* $\tau_*^{(m)}$ *for each* $m = 1, 2, \cdots, N$ *are* $\tau_*^{(m)} = \min\{i \geq i_*^{(m)} : X_i = 1\}$, *where,* $i_*^{(m)} = \min\{i \in [1, N] : h_i^{(m)} \leq 1\}$,

$$h_{j_0}^{(m)} = \sum_{j_1=j_0+1}^{i_*^{(m-1)}} r_{j_1} + \sum_{k=1}^{m-2} \prod_{\ell=1}^{k-1} R(j_{\ell-1}, m) \sum_{j_k=j_{k-1}+1}^{i_*^{(m-k)}} r_{j_k} + \prod_{\ell=1}^{m-1} R(j_{\ell-1}, m) \sum_{j_m=j_{m-1}+1}^{N} r_{j_m} \tag{2.16}$$

and $R(j_{\ell-1}, m) := \sum_{j_\ell=j_{\ell-1}+1 \vee i_*^{(m-\ell)}}^{N} r_{j_\ell}$. *Further, for each* $m = 1, 2, \cdots, N$,

$$1 \leq i_*^{(m)} \leq i_*^{(m-1)} \leq \cdots \leq i_*^{(1)} \leq N. \tag{2.17}$$

Proof. For $m \geq 3$, we prove by induction on m. As induction hypotheses, we assume for fixed $m \geq 3$ that

(i) $B^{(m)} = \{i \in \{1, \cdots, N\} : H_i^{(m)} \geq 0\}$.(2.15) holds and $i \mapsto H_i^{(m)}$ changes the sign at most once from negative to nonnegative. (i.e. $B^{(m)}$ is "closed") .

(ii) $H_i^{(m)} \geq H_i^{(m-1)}$, $\quad i = 1, 2, \cdots, N-1$.

Note that (ii) implies $i_*^{(m)} \le i_*^{(m-1)}$. Then, we have $B^{(m+1)} = \{i \in \{1, \cdots, N\} : G_i^{(m+1)} \ge 0\}$, where

$$G_i^{(m+1)} := V_i^{(m+1)} - \sum_{j=i+1}^{N} \Big[\prod_{k=i+1}^{j-1} q_k \Big] p_j V_j^{(m+1)}. \tag{2.18}$$

From the similar approach to the one in the proof of Theorem 2.1, it follows that

$$\begin{aligned} G_i^{(m+1)} &= G_i^{(1)} + \sum_{j=i+1}^{N} \Big[\prod_{k=i+1}^{j-1} q_k \Big] p_j G_j^{(m)} I_{\{j \ge i_*^{(m)}\}} \\ &= \prod_{j=i+1}^{N} q_j H_i^{(1)} + \sum_{j=i+1 \vee i_*^{(m)}}^{N} \Big[\prod_{k=i+1}^{j-1} q_k \Big] p_j \Big[\prod_{l=j+1}^{N} q_l H_j^{(m)} \Big] \\ &= \prod_{j=i+1}^{N} q_j \left(H_i^{(1)} + \sum_{j=i+1 \vee i_*^{(m)}}^{N} r_j H_j^{(m)} \right). \end{aligned} \tag{2.19}$$

Hence, setting

$$H_i^{(m+1)} := H_i^{(1)} + \sum_{j=i+1 \vee i_*^{(m)}}^{N} r_j H_j^{(m)}, \tag{2.20}$$

and taking the difference between (2.15) and (2.20), we have

$$\begin{aligned} H_i^{(m+1)} - H_i^{(m)} &= \sum_{j=i+1 \vee i_*^{(m)}}^{N} r_j H_j^{(m)} - \sum_{j=i+1 \vee i_*^{(m-1)}}^{N} r_j H_j^{(m-1)} \\ &\ge \sum_{j=i+1 \vee i_*^{(m-1)}}^{N} r_j \{ H_j^{(m)} - H_j^{(m-1)} \} \ge 0 \end{aligned}$$

The first inequality follows from $H_j^{(m)} I_{\{j \ge i_*^{(m)}\}} \ge 0$ and $i_*^{(m)} \le i_*^{(m-1)}$. The second inequality follows form the induction hypothesis (ii). Therefore, (ii) holds for $m+1$. From $H_i^{(m)} I_{\{i \ge i_*^{(m)}\}} \ge 0$ and the induction hypothesis (ii), it follows that $H_i^{(m+1)} \ge 0$. Since the second term of RHS in (2.20) for $i < i_*^{(m)}$ is a constant value, $i \mapsto H_i^{(m+1)}$ is increasing. Hence, $i \mapsto H_i^{(m+1)}$ changes the sign at most once from negative to nonnegative. We then have $B^{(m+1)} = \{i \in \{1, \cdots, N\} : H_i^{(m+1)} \ge 0\}$ and see that it is"closed". Therefore, (i) holds for $m+1$. The proof completes. □

For $m = 2, 3$, (2.16) in Theorem 2.2 are as follows.

$$\begin{aligned} h_i^{(2)} &= \sum_{j=i+1}^{i_*^{(1)}-1} r_j + \sum_{j_1=i+1 \vee i_*^{(1)}}^{N} r_{j_1} \sum_{j_2=j_1+1}^{N} r_{j_2}, \\ h_i^{(3)} &= \sum_{j=i+1}^{i_*^{(2)}} r_j + \sum_{j_1=i+1 \vee i_*^{(2)}}^{N} r_{j_1} \sum_{j_2=j_1+1}^{i_*^{(1)}} r_{j_2} + \sum_{j_1=i+1 \vee i_*^{(2)}}^{N} r_{j_1} \sum_{j_2=j_1+1 \vee i_*^{(1)}}^{N} r_{j_2} \sum_{j_3=j_2+1}^{N} r_{j_3}. \end{aligned}$$

3 Maximum probability of win

Theorem 3.1 *When we have at most $m(m>1)$ stopping chances,*

$$P^{(m)}(\text{win}) = \prod_{k=i_*^{(m)}}^{N} q_k \sum_{j=i_*^{(m)}}^{N} r_j + \sum_{j=i_*^{(m)}}^{N} \Big[\prod_{k=i_*^{(m)}}^{j} q_k \Big] r_j W_j^{(m-1)}. \tag{3.1}$$

Proof. It follows from

$$P^{(m)}(\text{win}) = W^{(m)}_{i_*^{(m)}-1} = \sum_{j=i_*^{(m)}}^{N} \prod_{k=i_*^{(m)}}^{j-1} q_k p_j M_j^{(m)} = \sum_{j=i_*^{(m)}}^{N} \prod_{k=i_*^{(m)}}^{j-1} q_k p_j \left(\prod_{\ell=j+1}^{N} q_\ell + W_j^{(m-1)} \right).$$

□

For example,

$$\begin{aligned} P^{(2)}(\text{win}) &= \sum_{k=i_*^{(2)}}^{N} r_k \prod_{\ell=i_*^{(2)}}^{N} q_\ell \\ &+ \left(\sum_{k=i_*^{(2)}}^{i_*^{(1)}-1} r_k \prod_{\ell=i_*^{(2)}}^{k} q_\ell \right) \left(\sum_{j=i_*^{(1)}}^{N} r_j \prod_{\ell=i_*^{(1)}}^{N} q_\ell \right) + \left(\sum_{k=i_*^{(1)}}^{N} r_k \sum_{j=k+1}^{N} r_j \right) \prod_{\ell=i_*^{(2)}}^{N} q_\ell . \end{aligned} \tag{3.2}$$

Bruss [5] finds that for any p_i, the lower bounds of the probability of win for the single stopping odds problem is e^{-1}. For the double stopping odds problem, we have the following asymptotic probability of win.

Theorem 3.2 *Let $R_1 = \sum_{j=i_*^{(1)}}^{N} r_j$, $R_2 = \sum_{j=i_*^{(2)}}^{N} r_j$, $R_2^{(2)} = \sum_{j=i_*^{(2)}}^{N} r_j^2$, then*

$$P^{(2)}(\text{win}) > R_1 e^{-R_1} + (1 + R_1 - R_1 e^{R_2^{(2)}}) e^{-R_2}. \tag{3.3}$$

Further, if $R_1 \to 1$, $R_2 \to 3/2$, $R_1^{(2)} = \sum_{j=i_*^{(1)}}^{N} r_j^2 \to 0$, $R_2^{(2)} \to 0$, as $N \to \infty$, then

$$P^{(2)}(\text{win}) > e^{-1} + e^{-3/2}. \tag{3.4}$$

Proof. From the result of Bruss [4], it follows that

$$\text{First term of the RHS of (3.2)} > R_2 e^{-R_2}.$$

Since the first blanket of the second term is equivalent to the probability that no success arrive between $i_*^{(2)}$ and $i_*^{(1)} - 1$, it follows from the result of Bruss [4] that

Second term of the RHS of (3.2)

$$= \left(1 - \prod_{\ell=i_*^{(2)}}^{i_*^{(1)}-1} q_\ell \right) \left(\sum_{j=i_*^{(1)}}^{N} r_j \prod_{\ell=i_*^{(1)}}^{N} q_\ell \right) = \sum_{j=i_*^{(1)}}^{N} r_j \left(\prod_{\ell=i_*^{(1)}}^{N} q_\ell - \prod_{\ell=i_*^{(2)}}^{N} q_\ell \right) > R_1 (e^{-R_1} - e^{-R_2 + R_2^{(2)}}).$$

From the definition of $i_*^{(2)}$ and the result of Bruss [4], it follows that

$$\text{Third term of the RHS of (3.2)} \geq \left(1 - \sum_{j=i_*^{(2)}}^{i_*^{(1)}-1} r_j\right) \prod_{\ell=i_*^{(2)}}^{N} q_\ell > (1 - (R_2 - R_1))e^{-R_2}.$$

Hence,

$$P^{(2)}(\text{win}) > R_1 e^{-R_1} + (1 + R_1 - R_1 e^{R_2^{(2)}})e^{-R_2}.$$

Then, under the conditions of Theorem 3.2, $P^{(2)}(\text{win}) > e^{-1} + e^{-3/2}$, as $N \to \infty$. □

The asymptotic probability of win, $e^{-1} + e^{-3/2}$, equals to the asymptotic one for the CSP with double stopping chances (for example, see Ano and Ando [2]). For the multiple stopping odds problem, our conjecture of the lower bounds of probability of win for any p_i is equivalent to the asymptotic probability of win for the CSP with multiple stopping chances as follows;

$$P^{(m)}(\text{win}) > \lim_{N\to\infty} \sum_{j=1}^{m} \frac{i_*^{(j)}}{N}. \tag{3.5}$$

References

[1] Ano, K., Kakinuma, H. and Miyoshi, N. (2009). Multiple Sums-the-Odds Theorem. *The 15th INFORMS Applied Probability Society Conference,* Cornell University.

[2] Ano, K. and Ando, A. (2000). A note on Bruss' stopping problem with random availability, IMS Lecture Note **35**, *Game Theory, Optimal Stopping, Probability and Statistics, Papers in Honor of Thomas S. Ferguson*, F. T. Bruss and L. Le. Cam eds., pp.71–82.

[3] Ano, K. (2000). Mathematics of Timing – Optimal Stopping Problem (in Japanese), Asakura Publ., Tokyo.

[4] Bruss, F. T. (2000). Sum the odds to one and stop, *Annals of Prob.* **28** 1384–1391.

[5] Bruss, F. T. (2003). A note of bounds for the odds of optimal stopping, *Annals of Prob.* **31** 1859–1861.

[6] Chow, Y. S., Robbins, H. and Siegmund, D. (1971). Great Expectations: The Theory of Optimal Stopping, Houghton Mifflin Co., Boston.

[7] Ferguson, T. S. (2006). Optimal Stopping and Applications, Electronic Text at http://www.math.ucla.edu/~tom/Stopping/Contents.html

[8] Ferguson, T. S. (2008). The Sum-the-Odds Theorem with application to a stopping game of Sakaguchi, preprint.

[9] Gilbert, J. P. and Mosteller, F. (1966). Recognizing the maximum of a sequence, J. Amer. Statist. Assoc. **61** 35-73.

[10] Hill, T. P. and Krengel, U. (1992). A prophet inequality related to the secretary problem, in *Strategies for Sequential Search and Selection in Real Time,* Contemporary Mathematics **125**, F. T. Bruss, T. S. Ferguson and S. M. Samuels eds., 209-215.