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Chaotic Difference Equations in Metric Spaces of Fuzzy Sets

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Abstract. In this article we introduce Kloeden-Li's paper (2006) which is concerning results on the appearance of chaos of difference equations in finite dimensional spaces, Banach spaces and complete metric spaces of fuzzy sets. We discuss the ideas due to Kloeden-Li and illustrate examples of the chaos to difference equations in finite dimensional spacesand complete metric spaces of fuzzy sets.

1.Introduction.

Consider the following difference equation

$$x_{n+1} = f(x_n), \ n = 0,1,2,... \tag{1}$$

where $x_n \in J$ (an interval) and $f: J \to J$ be continuous. For x in J, we denote $f^0(x) = x$ and $f^{n+1}(x) = f(f^n(x))$ for $n = 0,1,2,...$ A point x^* is called a k-periodic point if x^* in J and $x^* = f^k(x^*)$ with $x^* \neq f^p(x^*)$ for $1 \leq p < k$. If $k = 1$, then $x^* = f(x^*)$ is called a fixed point. In Section 2 the Li-Yorke's theorem and Chaos , for which a sufficient condition of a 3-periodic point in the one-dimensional space is mentioned. In Section 3 a generalized Marotto's result in the higher dimensional space is dealt with and our main example of an $\boldsymbol{R}^m$-mapping, where a positive integer m, with a 3-periodic point but no expanding is given. Section 4 introduces a chaos criterion to fuzzy mappings which are due to Kloeden-Li are given.

2.Li-Yorke's Chaos

Li-Yorke's theorem[2] on chaos in the one-dimensional space is as follows:

Theorem 1. Let J be an interval and $f: J \to J$ be continuous. Assume that there is one point $a \in J$, for which the points $b = f(a)$, $c = f(f(a)) = f^2(a)$ and $d = f^3(a)$ satisfy $d \leq a < b < c$ (or, $d \geq a > b > c$). Then the following statements (i) and (ii) hold truly.

(i) For every k= 1,2,..., there is a k-periodic points on J;

(ii) There is an uncountable set $S \subset J$, containing no periodic points, which satisfies the following conditions (a) and (b):

(a) For every no periodic p, q in S with $p \neq q$, it follows that

$$\limsup_{n\to\infty} | f^n(p) - f^n(q) | > 0 \text{ and } \liminf_{n\to\infty} | f^n(p) - f^n(q) | = 0;$$

(b) For every no periodic p in S and periodic q in J, it follows that

$$\limsup_{n\to\infty} | f^n(p) - f^n(q) | > 0.$$

Example 1. The tent map $T(x) = 1 - |1 - 2x|$ for $0 \leq x \leq 1$ is well known as a chaotic function in the sense of Li-Yorke. It has six 3-periodic points {2/9, 4/9, 8/9} and {2/7, 4/7, 6/7}.

3.Generalized Marotto's Theorem for the Li-Yorke's Chaos

In this section we consider an m-dimensional difference equation

$$x_{n+1} = f(x_n), \quad n = 0,1,2,... \tag{2}$$

where $f: \boldsymbol{R}^m \to \boldsymbol{R}^m$ is continuous and differentiable in the neighborhood of the fixed point $x^* = f(x^*)$. Let $||x||$ be the Euclidean norm of x in $\boldsymbol{R}^m$ and denote by $B_r(x)$ the closed ball in $\boldsymbol{R}^m$ of radius r and centered at x. Marotto introduced the following definitions (1) and (2). See [1].

Definition 1

(1) Let f be differentiable on $B_r(x^*)$, where x^* is a fixed point of f. The point x^* is called an expanding fixed point of f on $B_r(x^*)$ if $||Df(x)|| > 1$ for all x in $B_r(x^*)$. Here $Df(x)$ is the Jacobian matrix at x.

(2) Assume that x^* is an expanding fixed point in $B_r(x^*)$ for some $r > 0$. Then x^* is called a snap-back repeller of f if there exists an eventually fixed

point y in $B_r(x^*)$ with $y \neq x$, *i.e.*, $f^M(y) = x^*$ and the determinant $\det(Df^M(y)) \neq 0$ for some positive integer M.

It can be seen that in the one-dimensional space the existence of the snap-back repeller is equivalent to the existence of a 3-periodic point for the map f^p with a positive integer p.

Marotto claimed that Definition1(1) means the following expanding property of f.

Expanding Property. There exist $s > 1$ and $r > 0$ such that

$$||f(x) - f(y)|| > s||x - y||$$

for all x, y in $B_r(x^*)$.

The following example shows that the mapping f has a 3-perodic point but it is not expanding.

Example 2. Consider the following $\boldsymbol{R}^2$-valued function.

$$f(x_1, x_2) = \begin{pmatrix} 7\cos\dfrac{2\pi x_1}{7} \\ \sin\dfrac{2\pi x_2}{7} \end{pmatrix} \quad \text{with } \|(x_1, x_2)\| = \sqrt{|x_1|^2 + |x_2|^2}$$

It has three fixed points $(fp_1,0)$, $(fp_2,0)$ and $(7,0)$, where fp_1, fp_2 are about 1.75, 6.65, respectively, and has six 3-peridoc points. See Fig.1.

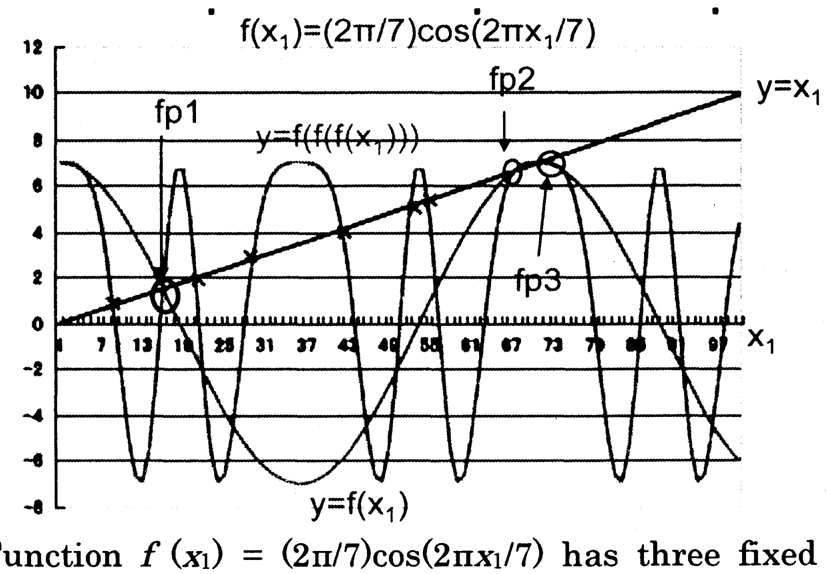

Fig.1. Function $f(x_1) = (2\pi/7)\cos(2\pi x_1/7)$ has three fixed points and six

3-periodic points.

Function f has the Jacobian matrix such as

$$Df(x)=\frac{\partial f}{\partial x}(x_1,x_2)=\begin{pmatrix}-(2\pi)\sin\dfrac{2\pi x_1}{7} & 0\\ 0 & \dfrac{2\pi}{7}\cos\dfrac{2\pi x_2}{7}\end{pmatrix}\quad (x=(x_1,x_2)).$$

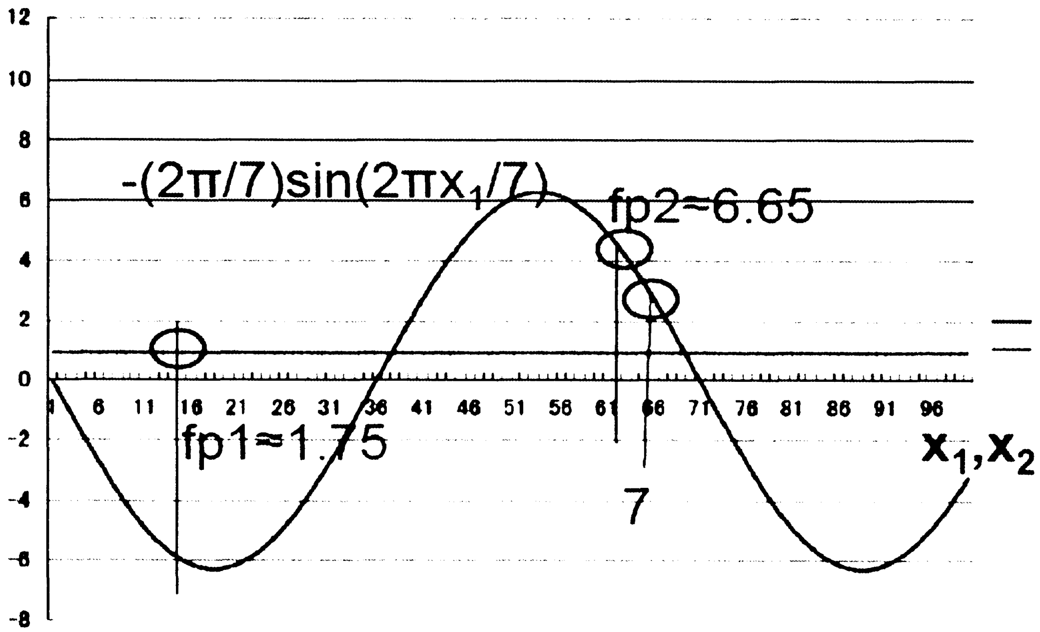

Fig.2. The Euclidean norms of the Jacobian matrix are larger than 1 at x = fp2 and 7.

It follows that the values of the Euclidean norm to the Jacobian matrix are larger than 1 at x = fp2 and 7. See Fig.2. Then Definition1(1) are satisfies with f.

If suppose that $\|f(x)-f(y)\| > s\|x-y\|$ with $s>1$, then at $x={}^t(7,0)$ and $y={}^t(7,\varepsilon)$ it follows that, $f(7,0)={}^t(7,0)$ with $s\varepsilon > 2\pi/7$,

$$\|f(7,\varepsilon)-f(7,0)\| > s\|{}^t(7,\varepsilon)-{}^t(7,0)\|,$$

so that

$$\|{}^t(0,(2\pi/7)\cos(2\pi c/7))\| > s\|{}^t(0,\varepsilon)\|$$

for $0<c<\varepsilon$ in the mean value theorem, which means

$$2\pi/7 > s\varepsilon > 2\pi/7$$

with a contradiction.

The existence of snap-back repellers show that function f of (2) has homoclinic orbits under that f satisfies Definition1(2) and that (R) there exists an eventually fixed point $x_0 = f^n(z)$ for a fixed point z and positive integer n, provided that
$\det(Df^j(x_0)) \neq 0$ for $j=1,2,..,n$. See Fig.3.

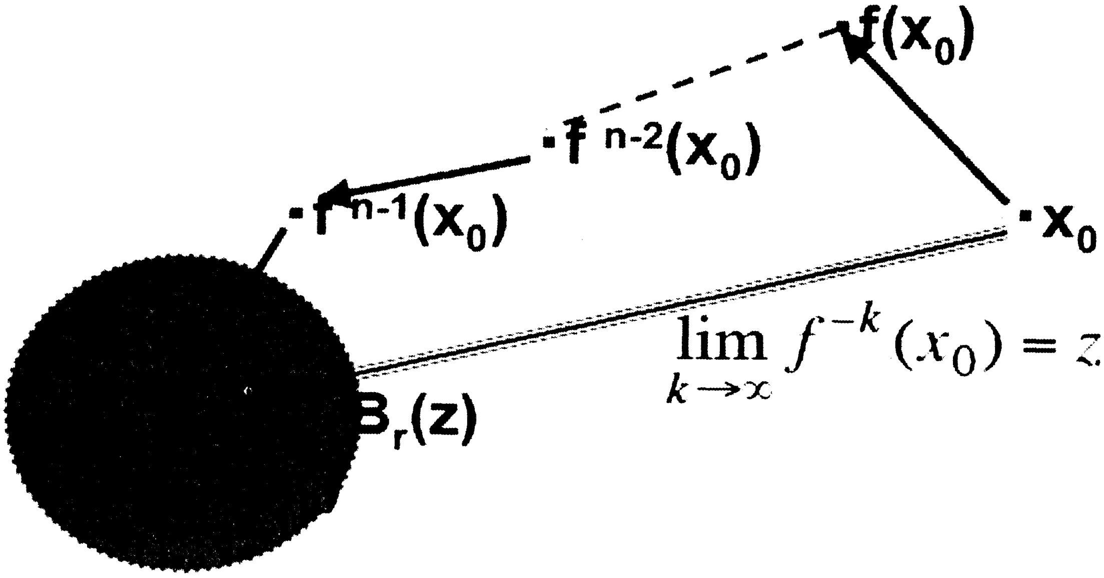

Fig.3. Function f has a homoclinic orbit.

Theorem 3. ([1]) Let z be a fixed point of f. Assume that Function f is continuously differentiable and absolute values of all eigenvalues to $Df(x)$ at x in a neighborhood of z are larger than 1 under the above condition(R). Then there exists a positive integer N such that for each positive integer $p \geq N$, f has a p-periodic point. Moreover there exists an uncountable set S such that $S \supset f(S)$ and that statements (ii)(a-b) of Theorem 1 hold truly.

4.Chaos Criterion to Fuzzy Mappings

Let E^m be the set of all functions, called fuzzy sets, $u : \boldsymbol{R}^m \to [0,1]$ for which u is normal, fuzzy convex, upper semi-continuous and has the compact support. Let d be the Hausdorff metric and $D(u,v) = \sup_{0<\alpha\leq 1} d([u]^\alpha,[v]^\alpha)$. Here

$$[u]^\alpha = \{x \in R^m : u(x) \geq \alpha\}.$$

Then the metric space (E^m, D) is complete.

Kloeden-Li[1] gives criteria on the Li-Yorke's chaos.

Theorem 4. Let $f: E^m \to E^m$ be continuous and suppose that there exist non-empty compact subsets A and B of E^m and integers $p, q \geq 1$ such that

(i) A is homeomorphic to a convex subset of E^m ;

(ii) $A \subset f(A)$;

(iii) there exists $s > 1$ such that $D(f(u), f(v)) > sD(u,v)$ for all u, v in A;

(iv) $B \subset A$;

(v) $f^p(B) \cap A = \varnothing$;

(vi) $A \subset f^{p+q}(B)$;

(vii) f^{p+q} is one-to-one on B.

Then the mapping f satisfies the conclusions of Theorem 1.

Denote

$a(\alpha) = \inf[u]^\alpha$, $b(\alpha) = \sup[u]^\alpha$, $E_0^1 = \{u \in E^1 : a(0) = 0\}$,

$I_0^1 = \{u \in E_0^1 : a(\alpha) = \frac{\alpha}{2}(b(0) - L)$ and $b(\alpha) = b(0) - \frac{\alpha}{2}(b(0) - L)\}$ for $0 \leq L \leq b(0)$, $\Delta_0^1 = \{u \in I_0^1 : L = 0\}$.

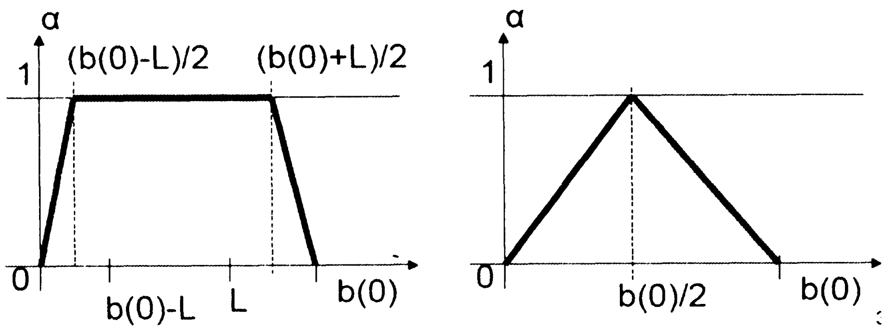

Fig.4. membership functions of I^1_0(left) and Δ^1_0(right).

Consider a fuzzy mapping $f: E^1 \to E^1$ by $f(u) = f_3(f_2(f_1(u)))$, which is continuous with D and maps Δ^1_0 into itself. Here

$f_1 : E^1 \to E_0^1$ by $[f_1(u)]^\alpha = [a(\alpha) - a(0), b(\alpha) - a(0)]$;

$f_2 : E_0^1 \to I_0^1$ by $[f_2(u)]^\alpha = [\alpha M, b(0) - \alpha M]$, where $M = \frac{1}{2}b(0) - \frac{1}{8}(b(1) - a(1)) > 0$;

$f_3 : I_0^1 \to I_0^1$ by $[f_3(u)]^\alpha = g(b(0))[u]^\alpha$, where $g(x) = T(x)/x$.

$T(x)$ is the tent map with $T(x) = 0$ for $x \leq 0$, $x \geq 1$. See Fig. 5-6.

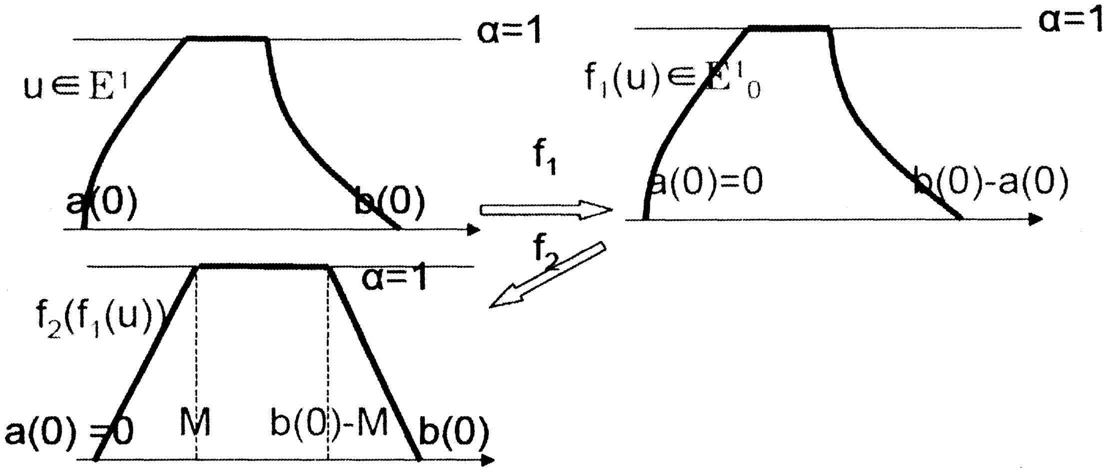

Fig. 5. Fuzzy mappings f_1 and f_2.

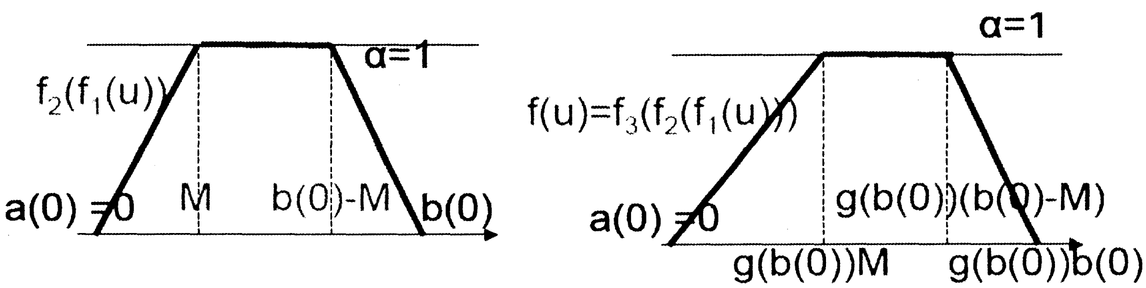

Fig.6. Fuzzy mappings f_3 and f.

Denote $b = b(0)$, then we have $f(u_b) = u_{T(b)}$.

Example 3. In order to apply Theorem 4 we consider the following compact sets A, B and $p = q = 1$ with

$$A = \{u_b \in \Delta_0^1 : \frac{9}{16} \le b \le \frac{7}{8}\},\ \ B = \{u_b \in \Delta_0^1 : \frac{3}{4} \le b \le \frac{7}{8}\}.$$

Then

$$f(A) = \{u_b \in \Delta_0^1 : \frac{1}{4} \le b \le \frac{7}{8}\},\ f(B) = \{u_b \in \Delta_0^1 : \frac{1}{4} \le b \le \frac{1}{2}\}, f^2(B) = \{u_b \in \Delta_0^1 : \frac{1}{2} \le b \le 1\}.$$

Conditions (ii),(v) and (vi) hold truly and it follows that for u_x, u_y in A

$$D(f(u_x), f(u_y)) = 2D(u_x, u_y).$$

and f^2 is one-to-one on B. By Theorem 4 f is chaotic in the sense of Li-Yorke.

5. Concluding Remarks

In this article we introduced Kloeden-Li's paper (2006) which is concerning results on the appearance of chaos of difference equations in complete metric spaces of fuzzy sets. We discussed the ideas due to Kloeden-Li and illustrate examples of the chaos to difference equations in complete metric spaces of fuzzy sets.

References

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2. T. V. Li and J. A. Yorke : Period Three Implies Chaos, Amer. Math. Monthly 82, pp.985-992. 1975.