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An Approach to the Construction of Inequivalent Models of Central Limit Theorem for Gaussianization of a Symmetric Probability Measure

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Abstract. For any symmetric probability measure ν on the real line $\mathbb{R}$ with finite moments of all orders such that ν is not finitely supported, we construct a large family of models of central limit theorem related to the 'Gaussianization' of measure ν in the sense of L. Accardi and M. Bożejko. The models are parametrized with infinitely many parameters $q = \{q_n\}_{n=2}^{\infty}$, $q_n \in (-1, 1)$, and constructed so that, for each q, the central limit distribution of the model realizes the same measure ν, but that, for each pair of different values $q \neq q'$, the two limit processes arising from the functional central limit (i.e. Brownian motions) are not stochastically equivalent to each other. Although the models do not explicitly contain the notion of 'independence' (for example, 'independence' as a universal calculation rule in the sense of R. Speicher), our result suggests that, in non-commutative probability theory, the correspondence from 'independence' to 'central limit distribution' is highly 'many to one'. Our result looks similar to the result of T. Cabanal-Duvillard and V. Ionescu, but our approach is different from theirs.

1 Introduction

In non-commutative probability theory, the topic of 'Gaussianization' of probability measures have been studied by several authors [6] [1] [2] [7]. Let $\mathcal{P}_{fm}(\mathbb{R})$ be the set of all probability measures on the real line $\mathbb{R}$ with finite moments of all orders.

A Gaussianization result was first obtained by T. Cabanal-Duvillard and V. Ionescu [6]. They showed that any symmetric probability measure ν in $\mathcal{P}_{fm}(\mathbb{R})$ can be obtained as the central limit distribution of some weakly independent random variables on some non-commutative probability space. Their construction was based on the amalgamated product of algebras with infinitely many states.

In [1] L. Accardi and M. Bożejko have shown that any symmetric probability measure ν in $\mathcal{P}_{fm}(\mathbb{R})$ can be realized as the distribution of the field operator $Q_f = C_f^+ + C_f^-$ on the λ-Fock space $\mathcal{F}_\lambda(\mathcal{H})$ (= one-mode type interacting Fock space), and they call this phenomena Gaussianization of (symmetric) probability measures (see §2 subs.3). Besides they proved that any possibly non-symmetric probability measure μ in $\mathcal{P}_{fm}(\mathbb{R})$ can be realized as the distribution of the operator of the form

$$X_f = C_f^+ + C_f^- + C_{f,t}^\circ$$

(see §2 subs.3). They call also this phenomena Gaussianization of probability measures.

In [2] L. Accardi, V. Crismale and Y. G. Lu have shown that any possibly non-symmetric probability measure μ in $\mathcal{P}_{fm}(\mathbb{R})$ can be obtained as the limit distribution of the scaled sum of some random variables on some interacting Fock space, but in this case random variables are not weakly independent. Also A. D. Krystek and Ł. J. Wojakowski [7] have given another proof of this fact.

In this note we study about the diversity of Gaussianization of symmetric probability measures. We restrict ourselves to the symmetric case because we are interested in the *weakly independent* random variables.

Given a symmetric probability measure $\nu \in P_{fm}(\mathbb{R})$ such that ν is not finitely supported, and given a sequence of possibly non-symmetric probability measures $\{\mu_l\}_{l=1}^\infty$ from $P_{fm}(\mathbb{R})$, then we will construct a family $\{X^{(q)}\}_{q\in Q}$, parametrized by $q = \{q_n\}_{n=2}^\infty \in Q = \prod_{n=2}^\infty(-1,1)$, of sequences $X^{(q)} = \{X_l^{(q)}\}_{l=1}^\infty$of weakly independent random variables $X_l^{(q)}$ on the certain Fock space $\mathcal{F}_q^{(\nu)}(l^2(\mathbb{N}^*))$ (= a new example of interacting Fock space) associated to ν so that the following properties hold: (1) the distribution of $X_n^{(q)}$ realizes μ_n for each n and all q; (2) the distribution of the scaled sum $\frac{1}{\sqrt{n}}\{X_1^{(q)} + \cdots + X_n^{(q)}\}$ converges in moments to the measure ν (not depending on q) whenever the standard conditions on the joint moments for central limit theorem are satisfied; (3) however, for different $q = \{q_n\}_{n=2}^\infty \neq q' = \{q_n'\}_{n=2}^\infty$, the two limit processes (= Brownian motions) $\{B_t^{(q)}\}_{t\geq 0}$ and $\{B_t^{(q')}\}_{t\geq 0}$ arising in the functional central limit are not stochastically equivalent. So this result can be viewed as a construction of a family of inequivalent models of central limit theorem for Gaussianization of a symmetric probability measure ν although our models do not explicitly contain the notion of 'independence' as the universal calculation rule [9].

Our models satisfies the following features.

(a) In the construction any sequence of probability measures $\{\mu_l\}_{l=1}^\infty$ can be used whenever the uniform boundedness condition on the joint moments is satisfied. As a special case, for any possibly non-symmetric probability measure μ with mean 0 variance 1, we can construct some weakly independent identically ditributed random varables with the same distribution $\mu_l = \mu$, $l \in \mathbb{N}^*$, so that in the central limit the prescribed symmetric measure ν can be obtained.

(b) In the construction, for a given symmetric measure ν, we can get a large family of inequivalent models of central limit theorem with the same limit measure ν.

Our result looks similar to the result of Cabanal-Duvillard and Ionescu. But our construction is different from theirs.

The contents of this note is as follows.

In §2 we remind of basic facts on the interacting Fock space (= λ-Fock space) $\mathcal{F}_\lambda(\mathcal{H})$ and Gaussianization of probability measures. In §3 we introduce (λ, q)-Fock space $\mathcal{F}_{\lambda,q}(\mathcal{H})$, a new example of interacting Fock space, which is parametrized with $\lambda = \{\lambda_n\}_{n=1}^\infty$ and $q = \{q_n\}_{n=2}^\infty$, and we construct on the Fock space $\mathcal{F}_{\lambda,q}(\mathcal{H})$ our model $\{X_l^{(q)}\}_{l=1}^\infty$. In §4 we examine central limit theorem and the functional central limit theorem for our model $\{X_l^{(q)}\}_{l=1}^\infty$.

Throughout this note $\mathbb{N}$ denotes the set of all positive integers $\mathbb{Z}_{\geq 0}$, and $\mathbb{N}^*$ denotes the set of all strict positive integers $\mathbb{Z}_{>0}$. The scalar product $\langle\cdot|\cdot\rangle$ is always supposed to be $\mathbb{C}$-linear in the right variable. Also we use the short notation $\langle\cdot\rangle$ to mean the expectation w.r.t the vacuum state $\langle\cdot\rangle := \langle\Omega|\cdot\Omega\rangle$. A non-commutative probability space means a pair of $(\mathcal{A}, \varphi)$ consisting of a unital $*$-algebra $\mathcal{A}$ and a state φ of $\mathcal{A}$. We use the term 'random variable' to mean a non-commutative random variavle, i.e. an element $a \in \mathcal{A}$ from a non-commutative probability space $(\mathcal{A}, \varphi)$. The distribution of self-adjoint random variable $a = a^* \in \mathcal{A}$ is a linear functional $\mu_a : \mathbb{C}[X] \to \mathbb{C}$ over the polynomial algebra $\mathbb{C}[X]$ with $X^* = X$ defined by

$$\mu_a(P) = \varphi(P(a))$$

for all $P \in \mathbb{C}[X]$.

2 λ-Fock space and Gaussinaiztion of probability measures

In this section, let us remind of the basic facts on the interacting Fock spaces and Gaussianization of probability measures (see [1]).

2.1 λ-Fock space

Let $\lambda = \{\lambda_n\}_{n=1}^\infty$ be a sequence of real numbers $\lambda_n \geq 0$ satisfying the condition that $\lambda_n = 0$ implies $\lambda_m = 0$ for all $m \geq n$. Given a Hilbert space $\mathcal{H}$ ($\neq \{0\}$) and an integer $n \geq 1$, we define a new scalar product $\langle\cdot|\cdot\rangle_{\lambda_n}$ on the tensor product Hilbert space $\mathcal{H}^{\otimes n}$ by $\langle u|v\rangle_{\lambda_n} := \lambda_n\langle u|v\rangle$ whenever $\lambda_n > 0$. We denote by $\mathcal{H}_{\lambda_n}^{\otimes n}$ the Hilbert space $\mathcal{H}^{\otimes n}$ with the scalar product $\langle\cdot|\cdot\rangle_{\lambda_n}$. Then the λ-*Fock space (= one-mode type interacting Fock space)* $\mathcal{F}_\lambda(\mathcal{H})$ is defined as the Hilbert space direct sum

$$\mathcal{F}_\lambda(\mathcal{H}) := \mathbb{C}\Omega \oplus \bigoplus_{n\in N^*} \mathcal{H}_{\lambda_n}^{\otimes n},$$

where Ω is the vacuum vector with $\langle\Omega|\Omega\rangle_\lambda \equiv 1$ and $N^* = \{n \in \mathbb{N}^* \mid \lambda_n > 0\}$. Here we denoted by $\langle\cdot|\cdot\rangle_\lambda$ the scalar product of λ-Fock space $\mathcal{F}_\lambda(\mathcal{H})$.

We also denote by $\mathcal{H}^{(n)}$ the algebraic n^{th} tensor product of $\mathcal{H}$ (without completion), i.e. the linear span of vectors of the form $f_1 \otimes \cdots \otimes f_n$ with $f_1, \cdots, f_n \in \mathcal{H}$, and by $F_\lambda(\mathcal{H})$ the corresponding algebraic Fock space over $\mathcal{H}$, i.e. the algebraic direct sum of $\mathbb{C}\Omega$ and $\mathcal{H}^{(n)}$ over $n \in N^*$.

On the λ-Fock space $\mathcal{F}_\lambda(\mathcal{H})$, we have three types of linear operators $C_f^+, C_f^-, C_{f,t}^\circ$, $f \in \mathcal{H}$, $f \neq 0$. For simplicity, the domain $\mathcal{D}$ of these operators is understood as $\mathcal{D} = F_\lambda(\mathcal{H})$. The creation operator C_f^+ is defined by

$$C_f^+(f_1 \otimes \cdots \otimes f_n) \;:=\; f \otimes f_1 \otimes \cdots \otimes f_n$$

for $n \geq 1$ s.t. $n+1 \in N^*$, $C_f^+(f_1 \otimes \cdots \otimes f_n) := 0$ for $n \in N^*$ s.t. $n+1 \notin N^*$, and $C_f^+\Omega := f$. The annihilation operator C_f^- is defined as $C_f^- := (C_f^+)^{*_\lambda}$. Here $*_\lambda$ means the adjoint w.r.t. the scalar product $\langle\cdot|\cdot\rangle_\lambda$. The action of C_f^- on the n-particle vectors is given by

$$C_f^-(f_1 \otimes \cdots \otimes f_n) \;=\; \frac{\lambda_n}{\lambda_{n-1}} \langle f|f_1\rangle\, f_2 \otimes \cdots \otimes f_n$$

and $C_f^-\Omega = 0$. Also, with a sequence of real numbers $t = \{t_n\}_{n=0}^{\infty}$, we define the preservation operator $C_{f,t}^\circ$ by

$$C_{f,t}^\circ(\overbrace{f \otimes \cdots \otimes f}^{n}) \;:=\; t_n\, \overbrace{f \otimes \cdots \otimes f}^{n}$$

for the tensor power. For $u \in \mathcal{H}^{(n)}$ s.t. $\langle u|f^{\otimes n}\rangle_\lambda = 0$, we put $C_{f,t}^\circ u := 0$. Besides we put $C_{f,t}^\circ \Omega = t_0\,\Omega$.

2.2 Jacobi coefficients

Let $\mathcal{P}_{fm}(\mathbb{R})$ be the set of all probability measures on the real line $\mathbb{R}$ with finite moments of all orders, i.e. $\displaystyle\int_{\mathbb{R}} |x|^p \mu(dx) < \infty$ for all $p \in \mathbb{N}^*$.

Let μ be any probability measure in $\mathcal{P}_{fm}(\mathbb{R})$, and let $\{P_n(x)\}_{n\in N}$ be the monic orthogonal polynomials associated to μ obtained from the Gram-Schmidt orthogonalization procedure. Here the index set N is taken to be $N := \mathbb{N}$ when the support of μ is an infinite set, and to be $N := \{0,1,2,\cdots,n_0-1\}$ when the support of μ is a finite set of cardinality n_0.

Then, from the theory of orthogonal polynomials, there exists a unique pair of sequences of real numbers $\{\omega_n\}_{n\in N\setminus\{0\}}$ and $\{\alpha_n\}_{n\in N}$ with $\omega_n > 0$ such that the following relation holds:

$$(x-\alpha_n)\,P_n(x) \;=\; P_{n+1}(x) \;+\omega_n\,P_{n-1}(x)$$

for all $n \in N$, with the convention that $\omega_0 \equiv 1$, $P_0(x) \equiv 1$, $P_{-1}(x) \equiv 0$. These sequences $\{\omega_n\}_{n\in N\setminus\{0\}}$ and $\{\alpha_n\}_{n\in N}$ are called the *Jacobi coefficients* associated to the measure μ.

For the Jacobi coefficients, the following properties are well-known. We have

$$\int_{\mathbb{R}} P_n(x)P_m(x)\,\mu(dx) \;=\; \delta_{n,m}\,\omega_1\omega_2\cdots\omega_n$$

for all $m,n \in N$. If the measure μ is symmetric then $\alpha_n = 0$ for all $n \in N$.

2.3 Gaussianization of probability measures

Let μ be a probability measure in $\mathcal{P}_{fm}(\mathbb{R})$, and let a be a self-adjoint random varable in some non-commutative probability space $(\mathcal{A},\varphi)$. We say that μ is realized as the distribution μ_a of a if we have

$$\int_{\mathbb{R}} t^p d\mu(t) \;=\; \varphi(a^p) \quad (\;=\; \mu_a(X^p)\;)$$

for all $p \in \mathbb{N}^*$. It can happen that two different measure $\mu_1 \neq \mu_2$ from $\mathcal{P}_{fm}(\mathbb{R})$ are realized as the distribution μ_a of the same random variable a in $(\mathcal{A}, \varphi)$.

In [1] Accardi and Bożejko showed that, under the vacuum state $\langle \Omega | \cdot \Omega \rangle_\lambda$, the distribution of the field operator $Q_f = C_f^+ + C_f^-$, $\|f\| = 1$, on the interacting Fock space $\mathcal{F}_\lambda(\mathcal{H})$ is given by the symmetric probability measure $\mu \in \mathcal{P}_{fm}(\mathbb{R})$ such that its asscociated Jacobi coefficients $\{\omega_n\}_{n \in N \setminus \{0\}}$ and $\{\alpha_n\}_{n \in N}$ satisfy the relations

$$\begin{cases} \lambda_n = \omega_1 \omega_2 \cdots \omega_n & (n \in N \setminus \{0\}), \\ \alpha_n = 0 & (n \in N), \end{cases}$$

where N is given by $N = \{0\} \cup \{n \in \mathbb{N}^* \mid \lambda_n > 0\}$.

This result means that any symmetric probability measure μ from $\mathcal{P}_{fm}(\mathbb{R})$ can be realized as the distribution of the field operator Q_f on the λ-Fock space $\mathcal{F}_\lambda(\mathcal{H})$ with an appropriate choice of $\lambda = \{\lambda_n\}_{n=1}^\infty$, and hence that the moments of μ can be described by the combinatorics of pair partitions. They call this phenomena *Gaussianization* of (symmetric) probability measures.

Furthermore in [1] they also showed that the distribution of the operator $C_f^+ + C_f^- + C_{f,t}^\circ$, $\|f\| = 1$, is given by the (psossibly) non-symmetric probability measure $\mu \in \mathcal{P}_{fm}(\mathbb{R})$ such that its asscociated Jacobi coefficients $\{\omega_n\}_{n \in N \setminus \{0\}}$ and $\{\alpha_n\}_{n \in N}$ satisfy

$$\begin{cases} \lambda_n = \omega_1 \omega_2 \cdots \omega_n & (n \in N \setminus \{0\}), \\ t_n = \alpha_n & (n \in N) \end{cases}$$

with $N = \{0\} \cup \{n \in \mathbb{N}^* \mid \lambda_n > 0\}$. So any possibly non-symmetric probability measure $\mu \in \mathcal{P}_{fm}(\mathbb{R})$ can be realized as the distribution of $C_f^+ + C_f^- + C_{f,t}^\circ$.

They call also this phenomena Gaussianization of probability measures although (1) the measure μ is (possibly) non-symmetric, (2) a new operator C° is involved, and (3) the combinatorics of moments is not given by the pair partitions but by the partitions consisting of pair or singleton blocks.

2.4 Another realization of probability measures by operators on free Fock space

Any probability measure on $\mathbb{R}$ with finite moments of all orders can be realized also as the distribution of some operators on the *free Fock space* $\mathcal{F}(\mathcal{H})$ (i.e., the λ-Fock space $\mathcal{F}_\lambda(\mathcal{H})$ with $\lambda_n \equiv 1$ for all $n \in \mathbb{N}^*$) as follows.

Let us define, with a sequence $s = \{s_n\}_{n=1}^\infty$ of positive real numbers $s_n \geq 0$, the deformation $C_{f,s}^+$ of creation operator C_f^+ on the free Fock space $\mathcal{F}(\mathcal{H})$ by

$$C_{f,s}^+ (f_1 \otimes \cdots \otimes f_n) := s_{n+1} \, f \otimes f_1 \otimes \cdots \otimes f_n$$

and $C^+_{f,s}\Omega := s_1 f$. The deformation $C^-_{f,s}$ of annihilation operator C^-_f is defined by $C^-_{f,s} := (C^+_{f,s})^{*_0}$. Here $*_0$ means the adjoint w.r.t. the scalar product $\langle\cdot|\cdot\rangle_0$ of the free Fock space $\mathcal{F}(\mathcal{H})$.

Then it is known that the distribution of the operator $C^+_{f,s} + C^-_{f,s} + C^\circ_{f,t}$ is given by the probability measure μ with the Jacobi coefficients $\{\omega_n\}_{n\in N\setminus\{0\}}$ and $\{\alpha_n\}_{n\in N}$ satisfying

$$\begin{cases} \omega_n = s_n^2 & (n \in N \setminus \{0\}), \\ \alpha_n = t_n & (n \in N), \end{cases}$$

where N is given as the largest interval such that $\{0\} \subset N \subset \{0\} \cup \{n \in \mathbb{N}^* \mid s_n > 0\}$.

This means that any probability measure $\mu \in \mathcal{P}_{fm}(\mathbb{R})$ can be realized also on the free Fock space $\mathcal{F}(\mathcal{H})$ by *deformation of operators* (rather than *deformation of scalar product*) with an appropriate choice of $\{s_n\}_{n=1}^\infty$ and $\{t_n\}_{n=0}^\infty$.

In §3, we will jointly use both methods of realization of probability measures on Fock space (deformation of scalar product and of operators) to construct a family of sequences of weakly independent (non-commutative) random variables with prescribed probabilty distributions.

3 (λ, q)-Fock space and construction of the model

In this section, we explain about the (λ, q)-Fock space $\mathcal{F}_{\lambda,q}(\mathcal{H})$ introduced in [8], a new example of interacting Fock space, which is a deformation of λ-Fock space $\mathcal{F}_\lambda(\mathcal{H})$ by infinitely many parameters $q = \{q_n\}_{n=2}^\infty$. It is also a generalization of generalized q-deformed Fock space of H. Yoshida [10]. Although we do not construct explicitly the notion of 'independence' in this note, we can say intuitively that we obtain a variety of (some weak notions of) 'independence' which is controlled by parameters $\lambda = \{\lambda_n\}_{n=1}^\infty$ and $q = \{q_n\}_{n=2}^\infty$. Besides we will deform Fock space operators on $\mathcal{F}_{\lambda,q}(\mathcal{H})$ so that we obtain a variety of ditributions for each random variables (= operators) so that we get a family $\{X^{(q)}\}_{q\in Q}$ of sequences $X^{(q)} = \{X_l^{(q)}\}_{l=1}^\infty$ of weakly independent random variables with prescribed distributions $\{\mu_l\}_{l=1}^\infty$. The family $\{X^{(q)}\}_{q\in Q}$ will be used in §4 as inequivalent models of central limit theorem for Gaussianization of a symmetric measure ν.

3.1 q-Scalar product on the n-particle space

Given a Hilbert space $\mathcal{H}$ ($\neq \{0\}$), an integer $n \geq 1$ and a real number $q \in (-1, 1)$, we define a new scalar product $\langle\cdot|\cdot\rangle_q^{(n)}$ on the algebraic tensor product $\mathcal{H}^{(n)}$ of $\mathcal{H}$ by

$$\langle f_1 \otimes \cdots \otimes f_n | g_1 \otimes \cdots \otimes g_n \rangle_q^{(n)} := \sum_{\sigma \in S(n)} q^{i(\sigma)} \langle f_1 | g_{\sigma(1)}\rangle \cdots \langle f_n | g_{\sigma(n)}\rangle$$

where $S(n)$ denotes the symmetric group of $\{1, 2, \cdots, n\}$ and $i(\sigma)$ denotes the number of inversions in a permutation σ. Then it is known in the theory of q-Fock space of Bożejko and

Speicher [4] that the sesquilinear form $\langle\cdot|\cdot\rangle_q^{(n)}$ is positive definite and can be represented by the positive operator $T_q^{(n)}$ on $\mathcal{H}^{(n)}$ as $\langle u|v\rangle_q^{(n)} = \langle u\,|T_q^{(n)}v\,\rangle_0$ where $\langle\cdot|\cdot\rangle_0$ is the natural scalar product of $\mathcal{H}^{(n)}$. Of course we have $T_q^{(n)}\mathcal{H}^{(n)} \subset \mathcal{H}^{(n)}$ and $(T_q^{(n)})^{-1}\mathcal{H}^{(n)} \subset \mathcal{H}^{(n)}$.

3.2 (λ, q)-Fock space

Let $\lambda = \{\lambda_n\}_{n=1}^{\infty}$ be a sequence of real numbers $\lambda_n \geq 0$ satisfying the conditions that $\lambda_n = 0$ implies $\lambda_m = 0$ for all $m \geq n$, and let $q = \{q_n\}_{n=2}^{\infty}$ be a sequence of real numbers $q_n \in (-1, 1)$.

Denote by $\mathcal{H}_{\lambda_n,q}^{\otimes n}$ the completion of the pre-Hilbert space $\mathcal{H}^{(n)}$ with respect to the scalar product $\langle\cdot|\cdot\rangle_{\lambda_n,q}^{(n)} := \lambda_n \langle\cdot|\cdot\rangle_q^{(n)}$ whenever $\lambda_n > 0$. Then the (λ, q)-*Fock space* $\mathcal{F}_{\lambda,q}(\mathcal{H})$ is defined as the Hilbert space direct sum

$$\mathcal{F}_{\lambda,q}(\mathcal{H}) \;:=\; \mathbb{C}\Omega \oplus \bigoplus_{n\in N^*} \mathcal{H}_{\lambda_n,q_n}^{\otimes n},$$

where $N^* = \{n \in \mathbb{N}^* \mid \lambda_n > 0\}$. The scalar product of $\mathcal{F}_{\lambda,q}(\mathcal{H})$ is denoted by $\langle\cdot|\cdot\rangle_{\lambda,q}$. Denote by $F_{\lambda,q}(\mathcal{H})$ the corresponding algebraic (λ, q)-Fock space defined in the same way as for $F_\lambda(\mathcal{H})$.

We have three types of linear operators $A_f^+, A_f^-, A_{f,t}^\circ$, $f \in \mathcal{H}$, $f \neq 0$, on $\mathcal{F}_{\lambda,q}(\mathcal{H})$ the domain $\mathcal{D}$ of which is understood as $F_{\lambda,q}(\mathcal{H})$. The creation operator A_f^+ is defined by

$$A_f^+(f_1 \otimes \cdots \otimes f_n) \;=\; f \otimes f_1 \otimes \cdots \otimes f_n$$

and $A_f^+\Omega := f$. The annihilation operator A_f^- is defined as $A_f^- := (A_f^+)^{*_{\lambda,q}}$. Here $*_{\lambda,q}$ denotes the adjoint w.r.t. the scalar product $\langle\cdot|\cdot\rangle_{\lambda,q}$. The action of A_f^- on the n-particle vectors is given by

$$A_f^-(f_1 \otimes \cdots \otimes f_n) \;=\; \frac{\lambda_n}{\lambda_{n-1}}\,(T_{q_{n-1}}^{(n-1)})^{-1}T_{q_n}^{(n-1)} \sum_{i=1}^{n} q_n^{i-1}\langle f|f_i\rangle\, f_1 \otimes \cdots \otimes \check{f}_i \otimes \cdots f_n$$

and $A_f^-\Omega = 0$. Here the notation "$\cdots \otimes \check{f}_i \otimes \cdots$" means "omit the i^{th} factor in the tensor product." Besides we define, with a sequence of real numbers $t = \{t_n\}_{n=0}^{\infty}$, the preservation operator $A_{f,t}^\circ$ by

$$A_{f,t}^\circ(\overbrace{f \otimes \cdots \otimes f}^{n}) \;=\; t_n\, \overbrace{f \otimes \cdots \otimes f}^{n}$$

and $A_{f,t}^\circ u = 0$ for $u \in \mathcal{H}^{(n)}$ with $\langle u|f^{\otimes n}\rangle_\lambda = 0$.

3.3 Construction of the model

Le us first consider the disribution of the field operator $Q_f = A_f^+ + A_f^-$ on the (λ, q)-Fock space $\mathcal{F}_{\lambda,q}(\mathcal{H})$. Then we have the following Gaussianization result.

Theorem 3.1. *For any symmetric probability mesaure ν in $\mathcal{P}_{fm}(\mathbb{R})$ and any $q = \{q_n\}_{n=2}^{\infty}$, there exists $\lambda = \{\lambda_n\}_{n=1}^{\infty}$ such that ν can be realized as the distribution of operator Q_f on $\mathcal{F}_{\lambda,q}(\mathcal{H})$, $\|f\| = 1$, under the vacuum state $\langle\Omega|\cdot\Omega\rangle_{\lambda,q}$.*

Proof. Let $\{\omega'_n\}_{n\in N\setminus\{0\}}$ and $\{\alpha'_n\}_{n\in N}$ (with $\alpha'_n \equiv 0$) be the Jacobi coefficients associated to ν, then there exists λ-Fock space $\mathcal{F}_{\lambda'}(\mathcal{H})$ such that

$$\begin{cases} \lambda'_n = \omega'_1\omega'_2\cdots\omega'_n & (n \in N\setminus\{0\}), \\ \lambda'_m = 0 & (m \in \mathbb{N}^*\setminus N) \end{cases}$$

(see §2 subs.3). Besides let us define $\{\lambda_n\}_{n=1}^{\infty}$ by the relations

$$\lambda_n \sum_{\sigma\in S(n)} q_n^{i(\sigma)} = \lambda'_n \qquad (n \in \mathbb{N}^*),$$

then we have, for the one-dimensional Hilbert space $\mathbb{C}f$, the identification $\mathcal{F}_{\lambda,q}(\mathbb{C}f) = \mathcal{F}_{\lambda'}(\mathbb{C}f)$ as a Hilbert space because of $\langle\cdot|\cdot\rangle_{\lambda,q} = \langle\cdot|\cdot\rangle_{\lambda'}$.

Let a_f^+ (resp. ${c'_f}^+$) be the creation operator on $\mathcal{F}_{\lambda,q}(\mathbb{C}f)$ (resp. $\mathcal{F}_{\lambda'}(\mathbb{C}f)$) defined in §3 subs.2 (resp. §2 subs.1). Then we have $a_f^+ = {c'_f}^+$ under the identification $\mathcal{F}_{\lambda,q}(\mathbb{C}f) = \mathcal{F}_{\lambda'}(\mathbb{C}f)$, and hence $a_f^- = (a_f^+)^{*_{\lambda,q}} = ({c'_f}^+)^{*_{\lambda'}} = {c'_f}^-$. So the p^{th} moment of Q_f is shown to be

$$\begin{aligned}
\langle Q_f^p\rangle_{\mathcal{F}_{\lambda,q}(\mathcal{H})} &= \langle (A_f^+ + A_f^-)^p\rangle_{\mathcal{F}_{\lambda,q}(\mathcal{H})} \\
&= \sum_{(\varepsilon_1\varepsilon_2\cdots\varepsilon_p)\in\{+,-\}^p} \langle A_f^{\varepsilon_1}A_f^{\varepsilon_2}\cdots A_f^{\varepsilon_p}\rangle_{\mathcal{F}_{\lambda,q}(\mathcal{H})} \\
&= \sum_{(\varepsilon_1\varepsilon_2\cdots\varepsilon_p)\in\{+,-\}^p} \langle a_f^{\varepsilon_1}a_f^{\varepsilon_2}\cdots a_f^{\varepsilon_p}\rangle_{\mathcal{F}_{\lambda,q}(\mathbb{C}f)} \\
&= \sum_{(\varepsilon_1\varepsilon_2\cdots\varepsilon_p)\in\{+,-\}^p} \langle {c'_f}^{\varepsilon_1}{c'_f}^{\varepsilon_2}\cdots {c'_f}^{\varepsilon_p}\rangle_{\mathcal{F}_{\lambda'}(\mathbb{C}f)} \\
&= \langle ({c'_f}^+ + {c'_f}^-)^p\rangle_{\mathcal{F}_{\lambda'}(\mathbb{C}f)}
\end{aligned}$$

for all $p \in \mathbb{N}^*$. This means that the distribution of Q_f realizes the symmetric measure ν. ◇

Given a pair of ν and q, let us put $\mathcal{F}_q^{(\nu)}(\mathcal{H}) := \mathcal{F}_{\lambda,q}(\mathcal{H})$ which is the (λ, q)-Fock space dermined from Theorem 3.1.

Let us fix ν and q, and let us deform operators A_f^ε, $\varepsilon \in \{+,-\}$, on the Fock space $\mathcal{F}_q^{(\nu)}(\mathcal{H})$ ($= \mathcal{F}_{\lambda,q}(\mathcal{H})$). For any sequence $s = \{s_n\}_{n=0}^{\infty}$ of positive real numbers $s_n \geq 0$, let $A_{f,s}^+$ be the deformation of creation operator A_f^+ defined by

$$A_{f,s}^+(f_1\otimes\cdots\otimes f_n) := s_{n+1}\, f\otimes f_1\otimes\cdots\otimes f_n$$

and $A_{f,s}^+\Omega := s_1 f$. Also let $A_{f,s}^-$ be the deformation of annihilation operator A_f^- defined by $A_{f,s}^- := (A_{f,s}^+)^{*_{\lambda,q}}$.

For the notational convenience we put

$$B_f^+ := A_{f,s}^+, \quad B_f^- := A_{f,s}^-, \quad B_f^\circ := A_{f,t}^\circ, \quad X_f := X_{f,s,t} := B_f^+ + B_f^- + B_f^\circ.$$

Let us consider the distribution of the operator X_f under the vacuum state. Then we have

Theorem 3.2. *Let $\nu \in \mathcal{P}_{fm}(\mathbb{R})$ be a symmetric probability measure with infinite support. Then any probability measure μ in $\mathcal{P}_{fm}(\mathbb{R})$ can be realized as the distribution of operator X_f on $\mathcal{F}_q^{(\nu)}(\mathcal{H})$, with an appropriate choice of $\{s_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=0}^{\infty}$, under the vacuum state $\langle\Omega|\cdot\Omega\rangle_q^{(\nu)}$.*

Proof. Let $\lambda' = \{\lambda'_n\}_{n=1}^{\infty}$ and $\lambda = \{\lambda_n\}_{n=1}^{\infty}$ be two sequences given in the proof of Theorem 3.1, let $\{\omega''_n\}_{n\in M\setminus\{0\}}$ and $\{\alpha''_n\}_{n\in M}$ be the Jacobi coefficients associated to μ, and let $\lambda'' = \{\lambda''_n\}_{n=1}^{\infty}$ be the sequence such that the λ-Fock space $\mathcal{F}_{\lambda''}(\mathcal{H})$ realizes the measure μ as the distribution of operator $C_f^+ + C_f^-$, $\|f\| = 1$. Here M is the index set determined from the from the measure μ.

Let us use small letters to mean the operators on the Fock spaces over the one-dimensional space $\mathbb{C}f$ as follws:

$$\begin{cases} b_f^+ = a_{f,s}^+ , & b_f^- = a_{f,s}^- , & b_f^\circ = a_{f,t}^\circ & \text{on } \mathcal{F}_{\lambda,q}(\mathbb{C}f), \\ b'^+_f = c'^+_{f,s} , & b'^-_f = c'^-_{f,s} , & b'^\circ_f = c'^\circ_{f,t} & \text{on } \mathcal{F}_{\lambda'}(\mathbb{C}f), \\ c''^+_f , & c''^-_f , & c''^\circ_f = c''^\circ_{f,t} & \text{on } \mathcal{F}_{\lambda''}(\mathbb{C}f). \end{cases}$$

Here we distinguished the operators on $\mathcal{F}_{\lambda'}(\mathbb{C}f)$ from the operators on $\mathcal{F}_{\lambda''}(\mathbb{C}f)$ by the notation c' and c''.

For the operators b_f^ε, $\varepsilon \in \{+, \circ, -\}$, on $\mathcal{F}_{\lambda,q}(\mathbb{C}f)$, we choose $s = \{s_n\}_{n=1}^{\infty}$ and $t = \{t_n\}_{n=0}^{\infty}$ so that the relations

$$\begin{cases} \omega''_n &= \omega'_n\,(s_n)^2 & (n \in M\setminus\{0\}), \\ \alpha''_n &= t_n & (n \in M) \end{cases}$$

and $s_n = t_n = 0$ $(n \in \mathbb{N}^* \setminus M)$ hold. Note that $N = \mathbb{N}$ since ν is not finitely supported, and hence that $\omega'_n > 0$ for all $n \in \mathbb{N}^*$.

Then the p^{th} moment of X_f is shown to be

$$\begin{aligned} \langle X_f^p \rangle_{\mathcal{F}_q^{(\nu)}(\mathcal{H})} &= \langle (B_f^+ + B_f^- + B_f^\circ)^p \rangle_{\mathcal{F}_q^{(\nu)}(\mathcal{H})} \\ &= \sum_{(\varepsilon_1\varepsilon_2\cdots\varepsilon_p)\in\{+,\circ,-\}^p} \langle B_f^{\varepsilon_1} B_f^{\varepsilon_2}\cdots B_f^{\varepsilon_p} \rangle_{\mathcal{F}_{\lambda,q}(\mathcal{H})} \\ &= \sum_{(\varepsilon_1\varepsilon_2\cdots\varepsilon_p)\in\{+,\circ,-\}^p} \langle b_f^{\varepsilon_1} b_f^{\varepsilon_2}\cdots b_f^{\varepsilon_p} \rangle_{\mathcal{F}_{\lambda,q}(\mathbb{C}f)} \\ &= \sum_{(\varepsilon_1\varepsilon_2\cdots\varepsilon_p)\in\{+,\circ,-\}^p} \langle b'^{\varepsilon_1}_f b'^{\varepsilon_2}_f\cdots b'^{\varepsilon_p}_f \rangle_{\mathcal{F}_{\lambda'}(\mathbb{C}f)} \\ &= \sum_{(\varepsilon_1\varepsilon_2\cdots\varepsilon_p)\in\{+,\circ,-\}^p} \langle c''^{\varepsilon_1}_f c''^{\varepsilon_2}_f\cdots c''^{\varepsilon_p}_f \rangle_{\mathcal{F}_{\lambda''}(\mathbb{C}f)} \\ &= \langle (c''^+_f + c''^-_f + c''^\circ_{f,t})^p \rangle_{\mathcal{F}_{\lambda''}(\mathbb{C}f)} \end{aligned}$$

for all $p \in \mathbb{N}^*$. This means that the distribution of X_f realizes the measure μ. ◇

Let us given a sequence of probability measures $\{\mu_l\}_{l=1}^{\infty}$ from $\mathcal{P}_{fm}(\mathbb{R})$. Then we can construct a sequence of operators $\{X_l\}_{l=1}^{\infty}$ on the Fock space $\mathcal{F}_q^{(\nu)}(\mathcal{H})$ with $\mathcal{H} := l^2(\mathbb{N}^*)$, the l^2-space over the natural numbers $\mathbb{N}^*$, as follows. Let $\{e_l\}_{l=1}^{\infty}$ be the natural orthonormal basis of $l^2(\mathbb{N}^*)$. For each $l \in \mathbb{N}^*$, we put

$$X_l := X_{e_l} = X_{e_l, s^{(l)}, t^{(l)}},$$

where two sequences $s^{(l)} = \{s^{(l)}\}_{n=1}^{\infty}$ and $t^{(l)} = \{t^{(l)}\}_{n=0}^{\infty}$ are choosed so that the distribution of X_l coincides with μ_l. This is possible from Theorem 3.2 whenever ν is not finitely supported. We write $X_l^{(q)} = X_l$ when the explicit mention on the dependence on q is needed. The sequence of random variables (= operators) $\{X_l\}_{l=1}^{\infty}$ on $\mathcal{F}_q^{(\nu)}(\mathcal{H})$ can be viewed as 'independent' random variables because of the following.

Theorem 3.3 *(Weak Independence). The factorization*

$$\langle X_{l_1}^{p_1} X_{l_2}^{p_2} \cdots X_{l_k}^{p_k} \rangle \;=\; \langle X_{l_1}^{p_1} \rangle \langle X_{l_2}^{p_2} \rangle \cdots \langle X_{l_k}^{p_k} \rangle$$

holds for all $p_1, \cdots, p_k \in \mathbb{N}^*$ whenever $\#\{l_1, l_2, \cdots, l_k\} = k$.

4 Central limit theorem

In §3, we have constructed on the Fock space $\mathcal{F}_q^{(\nu)}(\mathcal{H})$ a family $\{X^{(q)}\}_{q\in Q}$ of models $X^{(q)} = \{X_l^{(q)}\}_{l=1}^{\infty}$ of weakly independent random variables with prescribed distributions $\{\mu_l\}_{l=1}^{\infty}$, which is parametrized by $q = \{q_n\}_{n=2}^{\infty} \in Q = \prod_{n=2}^{\infty}(-1, 1)$. This construction is possible whenever the symmetric probability measure ν is not finitely supported. For these weakly independent random variables $\{X_l^{(q)}\}_{l=1}^{\infty}$, let us examine central limit theorem and functional central limit theorem.

4.1 Central limit theorem

For the weakly independent random variables $\{X_l^{(q)}\}_{l=1}^{\infty}$, we have the following central limit theorem where the limit measure is shown to be the same measure ν (not dependent on q).

Theorem 4.1 *(central limit theorem). Let $\nu \in \mathcal{P}_{fm}(\mathbb{R})$ be a symmetric probability measure with infinite support, $\{\mu_l\}_{l=1}^{\infty}$ be a sequence from $\mathcal{P}_{fm}(\mathbb{R})$, and $X^{(l)} = \{X_l^{(q)}\}_{l=1}^{\infty}$ be weakly independent random variables on $\mathcal{F}_q^{(\nu)}(\mathcal{H})$ corresponding to $\{\mu_l\}_{l=1}^{\infty}$ as constructed in §3 subs.3. Besides suppose that each μ_l has mean 0 and variance 1, and that the joint moments are uniformly bounded in the sense*

$$\sup_{(i_1 i_2 \cdots i_p)\in(\mathbb{N}^*)^p} |\langle X_{i_1}^{(q)} X_{i_2}^{(q)} \cdots X_{i_p}^{(q)} \rangle| \;<\; \infty$$

for all $p \in \mathbb{N}^*$. Then we have, under the vacuum state,

$$\lim_{N\to\infty} \left\langle \left(\frac{1}{\sqrt{N}} \left\{ X_1^{(q)} + X_2^{(q)} + \cdots + X_N^{(q)} \right\} \right)^p \right\rangle \;=\; \int_{\mathbb{R}} x^p \, d\nu(x)$$

for all $p \in \mathbb{N}^*$.

Proof. We use the standard method in quantum probability (= moment method). For simplicity we write X_l instead of $X_l^{(q)}$. At first the p^{th} moment of $\frac{1}{\sqrt{N}}\{X_1 + X_2 + \cdots + X_N\}$ is given by

$$\begin{aligned}
&\left\langle \left(\frac{1}{\sqrt{N}} \left\{ X_1 + X_2 + \cdots + X_N \right\} \right)^p \right\rangle \\
&= \left(\frac{1}{\sqrt{N}} \right)^p \sum_{(i_1 \cdots i_p) \in \{1,2,\cdots,N\}^p} \langle X_{i_1} \cdots X_{i_p} \rangle \\
&= \sum_{\mathcal{V} \in \mathcal{P}(p)} \left(\frac{1}{\sqrt{N}} \right)^p \sum_{\substack{(i_1 \cdots i_p) \in \{1,2,\cdots,N\}^p \\ (i_1 \cdots i_p) \eta \mathcal{V}}} \langle X_{i_1} \cdots X_{i_p} \rangle \\
&= \sum_{\mathcal{V} \in \mathcal{P}(p)} \sum_{(\varepsilon_1 \cdots \varepsilon_p) \in \{+, \circ, -\}^p} \left(\frac{1}{\sqrt{N}} \right)^p \sum_{\substack{(i_1 \cdots i_p) \in \{1,2,\cdots,N\}^p \\ (i_1 \cdots i_p) \eta \mathcal{V}}} \langle B_{i_1}^{\varepsilon_1} \cdots B_{i_p}^{\varepsilon_p} \rangle,
\end{aligned}$$

where $\mathcal{P}(p)$ is the set of all partitions on $\{1, 2, \cdots, p\}$. Here we have written $(i_1 \cdots i_p)\eta\mathcal{V}$ when a sequence $(i_1 \cdots i_p)$ satisfies the condition that $i_k = i_l$ if and only if k and l belongs to a same block in the partition $\mathcal{V}$.

We can show that $\langle B_{i_1}^{\varepsilon_1} \cdots B_{i_p}^{\varepsilon_p} \rangle = 0$ for all $(i_1 \cdots i_p)\eta\mathcal{V}$ and all $(\varepsilon_1 \cdots \varepsilon_p)$, whenever $\mathcal{V}$ has some singleton block. Using the uniform boundedness condition for moments, we can show that, in the calculation of the limit of p^{th} moment with $N \to \infty$, only the pair partitions can contribute to the limit. Denote by $\mathcal{P}_2(p)$ the set of all pair partitions of $\{1, 2, \cdots, p\}$. Then we have for large N

$$\begin{aligned}
&\left\langle \left(\frac{1}{\sqrt{N}} \left\{ X_1 + X_2 + \cdots + X_N \right\} \right)^p \right\rangle \\
&= \sum_{\mathcal{V} \in \mathcal{P}(p)} \left(\frac{1}{\sqrt{N}} \right)^p \sum_{\substack{(i_1 \cdots i_p) \in \{1,2,\cdots,N\}^p \\ (i_1 \cdots i_p) \eta \mathcal{V}}} \langle X_{i_1} \cdots X_{i_p} \rangle \\
&\sim \sum_{\mathcal{V} \in \mathcal{P}_2(p)} \left(\frac{1}{\sqrt{N}} \right)^p \sum_{\substack{(i_1 \cdots i_p) \in \{1,2,\cdots,N\}^p \\ (i_1 \cdots i_p) \eta \mathcal{V}}} \langle X_{i_1} \cdots X_{i_p} \rangle \\
&= \sum_{\mathcal{V} \in \mathcal{P}_2(p)} \sum_{(\varepsilon_1 \cdots \varepsilon_p) \in \{+, \circ, -\}^p} \left(\frac{1}{\sqrt{N}} \right)^p \sum_{\substack{(i_1 \cdots i_p) \in \{1,2,\cdots,N\}^p \\ (i_1 \cdots i_p) \eta \mathcal{V}}} \langle B_{i_1}^{\varepsilon_1} \cdots B_{i_p}^{\varepsilon_p} \rangle \\
&= \sum_{\mathcal{V} \in \mathcal{P}_2(p)} \left(\frac{1}{\sqrt{N}} \right)^p \sum_{\substack{(i_1 \cdots i_p) \in \{1,2,\cdots,N\}^p \\ (i_1 \cdots i_p) \eta \mathcal{V}}} \langle B_{i_1}^{\varepsilon_1} \cdots B_{i_p}^{\varepsilon_p} \rangle,
\end{aligned}$$

where, in the last expression, $(\varepsilon_1, \cdots, \varepsilon_p)$ is the unique sequence in $\{-,+\}^p$ associated to the pair partition $\mathcal{V}$, which is defined by

$$\begin{cases} \varepsilon_k := + & \text{if } \ k = r \ \text{ for some pair block } \{l, r\} \in \mathcal{V} \text{ with } l < r, \\ \varepsilon_k := - & \text{if } \ k = l \ \text{ for some pair block } \{l, r\} \in \mathcal{V} \text{ with } l < r. \end{cases}$$

By the way note that, since X_i has mean 0 and variance 1, we have $s_1^{(i)} = 1$, and hence we have, for the above sequence $(\varepsilon_1, \cdots, \varepsilon_p)$ uniquely associated to $\mathcal{V} \in \mathcal{P}_2(p)$,

$$\langle B_{i_1}^{\varepsilon_1} \cdots B_{i_p}^{\varepsilon_p} \rangle = \langle A_{e_{i_1}}^{\varepsilon_1} \cdots A_{e_{i_p}}^{\varepsilon_p} \rangle.$$

Besides, for the sequence $(\varepsilon_1, \cdots, \varepsilon_p)$ uniquely associated to $\mathcal{V} \in \mathcal{P}_2(p)$, we have

$$\begin{aligned} &\langle A_{e_{i_1}}^{\varepsilon_1} \cdots A_{e_{i_p}}^{\varepsilon_p} \rangle \\ &= \sum_{\mathcal{U} \in \mathcal{P}_2(p)} t(\mathcal{U}) \prod_{(l,r) \in \mathcal{U}} \langle e_{i_l} | e_{i_r} \rangle \, Q(\varepsilon_l, \varepsilon_r) \\ &= \sum_{\mathcal{U} \in \mathcal{P}_2(p)} t(\mathcal{U}) \Big(\delta_{\mathcal{U},\mathcal{V}} \prod_{(l,r) \in \mathcal{U}} \langle e_{i_l} | e_{i_r} \rangle \, Q(\varepsilon_l, \varepsilon_r) \Big) \\ &= t(\mathcal{V}) \prod_{(l,r) \in \mathcal{V}} \langle e_{i_l} | e_{i_r} \rangle \\ &= t(\mathcal{V}). \end{aligned}$$

Here $t(\cdot)$ is the positive definite function $t : \bigcup_{p=1}^{\infty} \mathcal{P}_2(p) \to \mathbb{C}$ associated to the Fock space $\mathcal{F}_q^{(\nu)}(\mathcal{H})$ as a generalized Brownian motion, in the sense of Bożejko and Speicher [4], and $Q(\cdot,\cdot)$ is defined by $Q(-,+) := 1$ and $Q(+,+) = Q(-,-) = Q(+,-) := 0$. Also we used here the Wick formula for a generalized Brownian motion.

Therefore we get for large N

$$\begin{aligned} &\Big\langle \Big(\frac{1}{\sqrt{N}} \big\{ X_1 + X_2 + \cdots + X_N \big\} \Big)^p \Big\rangle \\ &\sim \sum_{\mathcal{V} \in \mathcal{P}_2(p)} \Big(\frac{1}{\sqrt{N}} \Big)^p \sum_{\substack{(i_1 \cdots i_p) \in \{1,2,\cdots,N\}^p \\ (i_1 \cdots i_p) \eta \mathcal{V}}} \langle B_{i_1}^{\varepsilon_1} \cdots B_{i_p}^{\varepsilon_p} \rangle \\ &\sim \sum_{\mathcal{V} \in \mathcal{P}_2(p)} t(\mathcal{V}). \end{aligned}$$

This means that, under the vacuum state, the p^{th} moment of $\dfrac{1}{\sqrt{N}} \{X_1 + X_2 + \cdots + X_N\}$ converges to $\displaystyle\int_{\mathbb{R}} x^p \, d\nu(x)$ for all $p \in \mathbb{N}^*$. $\diamond$

4.2 Functional central limit theorem and inequivalent Brownian motions

For a family $\{X^{(q)}\}_{q \in Q}$ of models $X^{(q)} = \{X_l^{(q)}\}_{l=1}^{\infty}$ parametrized by $q = \{q_n\}_{n=2}^{\infty} \in Q = \prod_{n=2}^{\infty}(-1, 1)$, let us show that, in the functional central limit, the resulting limit processes

(= Brownian motions) $\{B_t^{(q)}\}_{t\geq 0}$ are mutually inequivalent for different values of $q = \{q_n\}_{n=2}^{\infty}$ although the limit measure (= one dimensional distribution) always equals to the same measure ν.

On the (ν, q)-Fock space $\mathcal{F}_q^{(\nu)}(\mathcal{H})$ $(= \mathcal{F}_{\lambda,q}(\mathcal{H})$) with $\mathcal{H} := L^2(\mathbb{R}_+)$, we define the operator process $\{B_t^{(q)}\}_{t\geq 0}$ by

$$B_t^{(q)} := Q^{(q)}_{\chi_{(0,t]}} = A^{(q)}_{\chi_{(0,t]}}{}^{+} + A^{(q)}_{\chi_{(0,t]}}{}^{-}$$

where $A_f^{(q)\pm}$ is the creation and annihilation operators on $\mathcal{F}_q^{(\nu)}(\mathcal{H})$, and χ_I denotes the indicator function of an interval $I \subset \mathbb{R}_+$.

Theorem 4.2 *(functional central limit theorem). Suppose that the same assumptions as in Theorem 4.1 hold. Then, for each q, the sequence of processes*

$$Y_t^{(q,N)} := \frac{1}{\sqrt{N}}\{X_1^{(q)} + X_2^{(q)} + \cdots + X_{[Nt]}^{(q)}\}$$

converges in the limit $N \to \infty$ to the Brownian motion $\{B_t^{(q)}\}_{t\geq 0}$ in the sense that

$$\left\langle Y_{t_1}^{(q,N)} Y_{t_2}^{(q,N)} \cdots Y_{t_p}^{(q,N)} \right\rangle \longrightarrow \left\langle B_{t_1}^{(q)} B_{t_2}^{(q)} \cdots B_{t_p}^{(q)} \right\rangle .$$

for all $t_1, t_2, \cdots, t_p \geq 0$ and all $p \in \mathbb{N}^*$. Besides, for different $q \neq q'$, the corresponding two Brownian motions $\{B_t^{(q)}\}_{t\geq 0}$ and $\{B_t^{(q')}\}_{t\geq 0}$ are not stochastically equivalent in the sense of Accardi-Frigerio-Lewis [3].

The proof of convergence is given by the same method as in the proof of Theorem 4.1. The inequivalence between $\{B_t^{(q)}\}_{t\geq 0}$ and $\{B_t^{(q')}\}_{t\geq 0}$ can be easily shown by the calculation of joint moments (see [8]).

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