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A HISTORY OF THE NASH EQUILIBRIUM THEOREM IN THE KKM THEORY

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ABSTRACT. In 1966, Ky Fan first applied the KKM theorem to the Nash equilibrium theorem. Since then there have appeared several generalizations of the Nash theorem on various types of abstract convex spaces satisfying abstract forms of the KKM theorem. In this review, we introduce the most general results with examples appeared in each of several stages of such developments.

1. Introduction

In 1928, John von Neumann found his celebrated minimax theorem [V1] and, in 1937, his intersection lemma [V2], which was intended to establish his minimax theorem and his theorem on optimal balanced growth paths. In 1941, Kakutani [K] obtained a fixed point theorem for multimaps on a simplex, from which von Neumann's minimax theorem and intersection lemma were easily deduced. In 1950, John Nash [N1,2] established his celebrated equilibrium theorem by applying the Brouwer or the Kakutani fixed point theorem. Later Kakutani's theorem was extended to locally convex Hausdorff topological vector spaces by Fan [F1] and Glicksberg [G] in 1952 and by Himmelberg [H] in 1972. Those were applied to generalize the above mentioned theorems.

In 1961, Fan [F2] obtained his own KKM lemma and, in 1964 [F3], applied it to another intersection theorem for a finite family of sets having convex sections. This was applied in 1966 [F4] to a proof of the Nash equilibrium theorem. This is the origin of the application of the KKM theory to the Nash theorem. Moreover, in 1969, Ma [M] extended Fan's intersection theorem [F3] to infinite families and the Nash theorem for arbitrary families.

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Note that all of the above results are mainly concerned with convex subsets of topological vector spaces; see Granas [Gr]. Later, many authors tried to generalize them to various types of abstract convex spaces. The present author also extended them in [P3,4,7-10,12-14,PP,IP] by developing theory of generalized convex spaces (simply, G-convex spaces) related to the KKM theory and analytical fixed point theory. In the framework of G-convex spaces, we obtained some minimax theorems and the Nash equilibrium theorems in [P7,8,12] based on coincidence theorems or intersection theorems for finite families of sets; and in [P13] based on continuous selection theorems for Fan-Browder maps.

Furthermore, in our recent works [P15-17], we studied the foundations of the KKM theory on abstract convex spaces. The partial KKM principle for an abstract convex space is an abstract form of the classical KKM theorem [KKM]. We noticed that many important results in the KKM theory are closely related to abstract convex spaces satisfying the partial KKM principle and that a number of such results are equivalent to each other.

On the other hand, some other authors studied particular types of abstract convex spaces and deduced some Nash type equilibrium theorem from the corresponding partial KKM principle; for example, [Bi,BH,GKR,KSY,Lu,P7,12], explicitly, and many more in the literature, implicitly. Therefore, in order to avoid unnecessary repetitions for each particular type of abstract convex spaces, it would be necessary to state them clearly for general abstract convex spaces. This was simply done in [P18].

In this review, we introduce several stages of such developments of generalizations of the Nash theorem and related results within the frame of the KKM theory. Section 2 deals with a brief history from the von Neumann minimax theorem to the Nash theorem. In Section 3, we review the KKM theorem and its direct applications. Section 4 deals with basic concepts on our new abstract convex spaces and their fundamental properties. In Section 5, two methods leading to the Nash theorem — continuous selection method in [P13] and the KKM method in [P18] — are introduced. More precisely, results in these two papers are compared step-by-step. We will note that results in [P13] work for any, finite or infinite, families of Hausdorff G-convex spaces and, on the other hand, results in [P18] work for finite families of abstract convex spaces whose products satisfy the partial KKM principle.

More detailed version of this preview will appear elsewhere.

2. From von Neumann to Nash

In 1928, J. von Neumann [V1] obtained the following minimax theorem, which is one of the fundamental results in the theory of games developed by himself. We adopt Kakutani's formulation in 1941 [K]:

Theorem [V1]. *Let $f(x,y)$ be a continuous real-valued function defined for $x \in K$ and $y \in L$, where K and L are arbitrary bounded closed convex sets in two Euclidean spaces $\mathbf{R}^m$ and $\mathbf{R}^n$. If for every $x_0 \in K$ and for every real number α, the set of all $y \in L$ such that $f(x_0, y) \leq \alpha$ is convex, and if for every $y_0 \in L$ and for every real number β, the set of all $x \in K$ such that $f(x, y_0) \geq \beta$ is convex, then we have*

$$\max_{x\in K}\min_{y\in L} f(x,y) = \min_{y\in L}\max_{x\in K} f(x,y).$$

The minimax theorem is later extended by von Neumann [V2] in 1937 to the following intersection lemma. We also adopt Kakutani's formulation:

Lemma [V2]. *Let K and L be two bounded closed convex sets in the Euclidean spaces $\mathbf{R}^m$ and $\mathbf{R}^n$ respectively, and let us consider their Cartesian product $K \times L$ in $\mathbf{R}^{m+n}$. Let U and V be two closed subsets of $K \times L$ such that for any $x_0 \in K$ the set U_{x_0}, of $y \in L$ such that $(x_0, y) \in U$, is nonempty, closed and convex and such that for any $y_0 \in L$ the set V_{y_0}, of all $x \in K$ such that $(x, y_0) \in V$, is nonempty, closed and convex. Under these assumptions, U and V have a common point.*

Von Neumann proved this by using a notion of integral in Euclidean spaces and applied this to the problems of mathematical economics.

Recall that a multimap $F : X \multimap Y$, where X and Y are topological spaces, is *upper semicontinuous* (u.s.c.) whenever, for any $x \in X$ and any neighborhood U of $F(x)$, there exists a neighborhood V of x satisfying $F(V) \subset U$.

In order to give simple proofs of von Neumann's Lemma and the minimax theorem, Kakutani in 1941 [K] obtained the following generalization of the Brouwer fixed point theorem to multimaps:

Theorem [K]. *If $x \mapsto \Phi(x)$ is an upper semicontinuous point-to-set mapping of an r-dimensional closed simplex S into the family of nonempty closed convex subset of S, then there exists an $x_0 \in S$ such that $x_0 \in \Phi(x_0)$.*

Equivalently,

Corollary [K]. *Theorem is also valid even if S is an arbitrary bounded closed convex set in a Euclidean space.*

As Kakutani noted, Corollary readily implies von Neumann's Lemma, and later Nikaido [Ni2] noted that those two results are directly equivalent.

This was the beginning of the fixed point theory of multimaps having a vital connection with the minimax theory in game theory and the equilibrium theory in economics.

In the 1950's, Kakutani's theorem was extended to Banach spaces by Bohnenblust and Karlin [BK] and to locally convex Hausdorff topological vector spaces by Fan [F1] and Glicksberg [G]. These extensions were mainly applied to extend von Neumann's works in the above.

The first remarkable one of generalizations of von Neumann's minimax theorem was Nash's theorem [N1,2] on equilibrium points of non-cooperative games. The following formulation is given by Fan [F4, Theorem 4]:

Theorem. *Let $X_1, X_2, \cdots, X_n$ be n (≥ 2) nonempty compact convex sets each in a real Hausdorff topological vector space. Let $f_1, f_2, \cdots, f_n$ be n real-valued continuous functions defined on $\prod_{i=1}^n X_i$. If for each $i = 1, 2, \cdots, n$ and for any given point $(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n) \in \prod_{j\neq i} X_j$, $f_i(x_1, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_n)$ is a quasiconcave function on X_i, then there exists a point $(\widehat{x}_1, \widehat{x}_2, \cdots, \widehat{x}_n) \in \prod_{i=1}^n X_i$ such that*

$$f_i(\widehat{x}_1, \widehat{x}_2, \cdots, \widehat{x}_n) = \operatorname*{Max}_{y_i \in X_i} f_i(\widehat{x}_1, \cdots, \widehat{x}_{i-1}, y_i, \widehat{x}_{i+1}, \cdots, \widehat{x}_n) \quad (1 \leq i \leq n).$$

3. From KKM to Fan-Browder

In 1929, Knaster, Kuratowski, and Mazurkiewicz [KKM] obtained the following celebrated KKM theorem from the Sperner combinatorial lemma in 1928:

Theorem [KKM]. *Let A_i $(0 \leq i \leq n)$ be $n+1$ closed subsets of an n-simplex $p_0 p_1 \cdots p_n$. If the inclusion relation*

$$p_{i_0} p_{i_1} \cdots p_{i_k} \subset A_{i_0} \cup A_{i_1} \cup \cdots \cup A_{i_k}$$

holds for all faces $p_{i_0} p_{i_1} \cdots p_{i_k}$ $(0 \leq k \leq n,\ 0 \leq i_0 < i_1 < \cdots < i_k \leq n)$, then $\bigcap_{i=0}^n A_i \neq \emptyset$.

In 1958, von Neumann's minimax theorem was extended by Sion [Si] to arbitrary topological vector spaces as follows:

Theorem [Si]. *Let X, Y be a compact convex set in a topological vector space. Let f be a real-valued function defined on $X \times Y$. If*

(1) *for each fixed $x \in X$, $f(x, y)$ is a lower semicontinuous, quasiconvex function on Y, and*
(2) *for each fixed $y \in Y$, $f(x, y)$ is an upper semicontinuous, quasiconcave function on X,*

then we have

$$\operatorname*{Min}_{y \in Y} \operatorname*{Max}_{x \in X} f(x, y) = \operatorname*{Max}_{x \in X} \operatorname*{Min}_{y \in Y} f(x, y).$$

Sion's proof was based on the KKM theorem and this is the first application of the theorem after [KKM] in 1929.

A milestone of the history of the KKM theory was erected by Ky Fan in 1961 [F2]. He extended the KKM theorem to arbitrary topological vector spaces and applied it to coincidence theorems generalizing the Tychonoff fixed point theorem and a large number of problems in a sequence of papers; see [P6].

Lemma [F2]. *Let X be an arbitrary set in a Hausdorff topological vector space Y. To each $x \in X$, let a closed set $F(x)$ in Y be given such that the following two conditions are satisfied:*

(i) *The convex hull of any finite subset $\{x_1, x_2, \cdots, x_n\}$ of X is contained in $\bigcup_{i=1}^n F(x_i)$.*
(ii) *$F(x)$ is compact for at least one $x \in X$.*

Then $\bigcap_{x \in X} F(x) \neq \emptyset$.

In 1968, Browder [Br] restated Fan's geometric lemma [F2] in the convenient form of a fixed point theorem by means of the Brouwer fixed point theorem and the partition of unity argument. Since then the following is known as the Fan-Browder fixed point theorem:

Theorem [Br]. *Let K be a nonempty compact convex subset of a Hausdorff topological vector space. Let T be a map of K into 2^K, where for each $x \in K, T(x)$ is a nonempty convex subset of K. Suppose further that for each y in $K, T^{-1}(y) = \{x \in K : y \in T(x)\}$ is open in K. Then there exists x_0 in K such that $x_0 \in T(x_0)$.*

Later the Hausdorffness in the Fan lemma and Browder's theorem was known to be redundant. It is well-known that this theorem is equivalent to the KKM theorem.

4. Abstract convex spaces

A *multimap* or *map* $T : X \multimap Y$ is a function from X into the power set of Y, and $x \in T^-(y)$ if and only if $y \in T(x)$.

Let $\langle D \rangle$ denote the set of all nonempty finite subsets of a set D.

Definition. A *generalized convex space* or a *G-convex space* $(E, D; \Gamma)$ consists of a topological space E, a nonempty set D, and a multimap $\Gamma : \langle D \rangle \multimap E$ such that for each $A \in \langle D \rangle$ with the cardinality $|A| = n + 1$, there exists a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$.

Here, Δ_n is a standard n-simplex with vertices $\{e_i\}_{i=0}^n$, and Δ_J the face of Δ_n corresponding to $J \in \langle A \rangle$; that is, if $A = \{a_0, a_1, \dots, a_n\}$ and $J = \{a_{i_0}, a_{i_1}, \dots, a_{i_k}\} \subset A$, then $\Delta_J = \mathrm{co}\{e_{i_0}, e_{i_1}, \dots, e_{i_k}\}$.

For details on G-convex spaces; see [P5-8,11-13] and references therein.

Example. Typical examples of G-convex spaces are convex subsets of topological vector spaces, Lassonde type convex spaces [L1], C-spaces or H-spaces due to Horvath [H1,2].

Recall the following in [P15-18]:

Definition. An *abstract convex space* $(E, D; \Gamma)$ consists of a topological space E, a nonempty set D, and a multimap $\Gamma : \langle D \rangle \multimap E$ with nonempty values $\Gamma_A := \Gamma(A)$ for $A \in \langle D \rangle$.

For any $D' \subset D$, the Γ-*convex hull* of D' is denoted and defined by

$$\mathrm{co}_\Gamma D' := \bigcup \{\Gamma_A \mid A \in \langle D' \rangle\} \subset E.$$

A subset X of E is called a Γ-*convex subset* of $(E, D; \Gamma)$ relative to $D' \subset D$ if for any $N \in \langle D' \rangle$, we have $\Gamma_N \subset X$, that is, $\mathrm{co}_\Gamma D' \subset X$. Then $(X, D'; \Gamma|_{\langle D' \rangle})$ is called a Γ-*convex subspace* of $(E, D; \Gamma)$.

When $D \subset E$, the space is denoted by $(E \supset D; \Gamma)$. In such case, a subset X of E is said to be Γ-*convex* if $\mathrm{co}_\Gamma(X \cap D) \subset X$; in other words, X is Γ-convex relative to $D' := X \cap D$. In case $E = D$, let $(E; \Gamma) := (E, E; \Gamma)$.

Example. Every G-convex space is an abstract convex space. For other examples, see [P15-18].

Definition. Let $(E, D; \Gamma)$ be an abstract convex space. If a multimap $G : D \multimap E$ satisfies

$$\Gamma_A \subset G(A) := \bigcup_{y \in A} G(y) \qquad \text{for all } A \in \langle D \rangle,$$

then G is called a *KKM map*.

Definition. The *partial KKM principle* for an abstract convex space $(E, D; \Gamma)$ is the statement that, for any closed-valued KKM map $G : D \multimap E$, the family $\{G(y)\}_{y \in D}$ has the finite intersection property.

Definition. For a topological space X and an abstract convex space $(E, D; \Gamma)$, a multimap $T : X \multimap E$ is called a Φ-*map* or a *Fan-Browder map* provided that there exists a companion map $S : X \multimap D$ satisfying

(a) for each $x \in X$, $\mathrm{co}_\Gamma S(x) \subset T(x)$; and
(b) $X = \bigcup \{\mathrm{Int}\, S^-(y) \mid y \in D\}$.

Lemma 1. [P5] *Let X be a Hausdorff space, $(E, D; \Gamma)$ a G-convex space, and $T : X \multimap E$ a Φ-map. Then for any nonempty compact subset K of X, $T|_K$ has a continuous selection $f : K \to E$ such that $f(K) \subset \Gamma_A$ for some $A \in \langle D \rangle$. More precisely, there exist two continuous functions $p : K \to \Delta_n$ and $\phi_A : \Delta_n \to \Gamma_A$ such that $f = \phi_A \circ p$ for some $A \in \langle D \rangle$ with $|A| = n + 1$.*

For an abstract convex space $(E \supset D; \Gamma)$, an extended real-valued function $f : E \to \overline{\mathbb{R}}$ is said to be *quasiconcave* [resp., *quasiconvex*] if $\{x \in E \mid f(x) > r\}$ [resp., $\{x \in E \mid f(x) < r\}$] is Γ-convex for each $r \in \mathbb{R}$.

Recall that a function $f : X \to \overline{\mathbb{R}}$, where X is a topological space, is *lower* [resp., *upper*] *semicontinuous* (l.s.c.) [resp., u.s.c.] if $\{x \in X \mid f(x) > r\}$ [resp., $\{x \in X \mid f(x) < r\}$] is open for each $r \in \mathbb{R}$.

Let $\{X_i\}_{i \in I}$ be a family of sets, and let $i \in I$ be fixed. Let

$$X = \prod_{j \in I} X_j, \quad X^i = \prod_{j \in I \setminus \{i\}} X_j.$$

If $x^i \in X^i$ and $j \in I \setminus \{i\}$, let x^i_j denote the jth coordinate of x^i. If $x^i \in X^i$ and $x_i \in X_i$, let $[x^i, x_i] \in X$ be defined as follows: its ith coordinate is x_i and, for $j \neq i$ the jth coordinate is x^i_j. Therefore, any $x \in X$ can be expressed as $x = [x^i, x_i]$ for any $i \in I$, where x^i denotes the projection of x in X^i.

The following is known:

Lemma 2. *Let $\{(X_i, D_i; \Gamma_i)\}_{i \in I}$ be any family of abstract convex spaces. Let $X := \prod_{i \in I} X_i$ be equipped with the product topology and $D = \prod_{i \in I} D_i$. For each $i \in I$, let $\pi_i : D \to D_i$ be the projection. For each $A \in \langle D \rangle$, define $\Gamma(A) := \prod_{i \in I} \Gamma_i(\pi_i(A))$. Then $(X, D; \Gamma)$ is an abstract convex space.*

Let $\{(X_i, D_i; \Gamma_i)\}_{i \in I}$ be a family of G-convex spaces. Then $(X, D; \Gamma)$ is a G-convex space and hence a KKM space.

As we have seen in Sections 1-3, we have three methods in our subject as follows:

(1) Fixed point method — Applications of the Kakutani theorem and its various generalizations (for example, for acyclic valued multimaps, admissible maps, or better admissible maps in the sense of Park); see [BK,D,F1,3,G,H,IP,K,L2,M,N1,2, Ni1,P3,4,9-11,14,PP] and others.

(2) Continuous selection method — Applications of the fact that Fan-Browder type maps have continuous selections under certain assumptions like Hausdorffness and compactness of relevant spaces; see [BDG,Br,H1,HL,P5,7,13,T] and others.

(3) The KKM method — As for the Sion theorem, direct applications of the KKM theorem or its equivalents like as the Fan-Browder fixed point theorem for which we do not need the Hausdorffness; see [BH,CG,C,CKL,F2,4,5,GKR,Gr,H1-3,HL,Kh,KSY,Ko,L1,Lu,P2,7,12,15-17,S,Si] and others.

For Case (1), we will study elsewhere and, in this paper, we are mainly concerned with Cases (2) and (3).

An abstract convex space $(E, D; \Gamma)$ is said to be *compact* if E is a compact topological space.

¿From now on, for simplicity, we are mainly concerned with compact abstract convex spaces $(E; \Gamma)$ satisfying the partial KKM principle. For example, any compact G-convex space, any compact H-space, or any compact convex space is such a space.

5. From collective fixed points to Nash equilibria

In this section, we compare consequences of Cases (2) and (3) which lead to the Nash theorem. In fact, such results in Case (2) are for infinite families of Hausdorff compact G-convex spaces; and, in Case (3) for finite families of compact abstract convex spaces whose products satisfy the partial KKM principle.

We have the following collective fixed point theorem:

Theorem 1. Collective fixed point theorem. [P5] *Let $\{(X_i; \Gamma_i)\}_{i\in I}$ be a family of Hausdorff compact G-convex spaces, $X = \prod_{i\in I} X_i$, and for each $i \in I$, $T_i : X \multimap X_i$ a Φ-map. Then there exists a point $x \in X$ such that $x \in T(x) := \prod_{i\in I} T_i(x)$; that is, $x_i = \pi_i(x) \in T_i(x)$ for each $i \in I$.*

Example. In case when $(X_i; \Gamma_i)$ are all H-spaces, Theorem 1 reduces to Tarafdar [T, Theorem 2.3]. This is applied to sets with H-convex sections [T, Theorem 3.1] and to existence of equilibrium point of an abstract economy [T, Theorem 4.1 and Corollary 4.1]. These results also can be extended to G-convex spaces and we will not repeat here.

But, the following is possible:

Theorem 1′. Collective fixed point theorem. *Let $\{(X_i; \Gamma_i)\}_{i=1}^n$ be a finite family of compact abstract convex spaces such that $(E; \Gamma) = (\prod_{i=1}^n X_i; \Gamma)$ satisfies the partial KKM principle, and for each i, $T_i : E \multimap X_i$ a Φ-map. Then there exists a point $x \in X$ such that $x \in T(x) := \prod_{i=1}^n T_i(x)$; that is, $x_i = \pi_i(x) \in T_i(x)$ for each i.*

Comparing Theorems 1 and 1′, the former assumes the Hausdorffness of the G-convex spaces and is a consequence of a selection theorem based on the character of G-convex spaces. However, Theorem $1'$ assumes the finiteness of the family and follows from the Fan-Browder fixed point equivalent to the partial KKM principle.

Example. 1. If I is a singleton, X is a convex space, and $S_i = T_i$, then Theorem $1'$ reduces to the Fan-Browder fixed point theorem.

2. For the case I is a singleton, Theorem $1'$ for a convex space X was obtained by Ben-El-Mechaiekh et al. [BDG, Theorem 1] and Simons [S, Theorem 4.3]. This was extended by many authors; see Park [P2].

The collective fixed point theorems can be reformulated to generalizations of various Fan type intersection theorems for sets with convex sections as follows:

Theorem 2. The von Neumann-Fan-Ma intersection theorem. [P13] *Let $\{(X_i;\Gamma_i)\}_{i\in I}$ be a family of Hausdorff compact G-convex spaces and, for each $i \in I$, let A_i and B_i are subsets of $X = \prod_{i\in I} X_i$ satisfying the following:*

(2.1) *for each $x^i \in X^i$, $\emptyset \neq \mathrm{co}_{\Gamma_i} B_i(x^i) \subset A_i(x^i) := \{y_i \in X_i \mid [x^i, y_i] \in A_i\}$; and*

(2.2) *for each $y_i \in X_i$, $B_i(y_i) := \{x^i \in X^i \mid [x^i, y_i] \in B_i\}$ is open in X^i.*

Then we have $\bigcap_{i\in I} A_i \neq \emptyset$.

Example. For convex subset X_i of Hausdorff topological vector spaces, particular forms of Theorem 2 have appeared as follows:

1. Ma [M, Theorem 2]: $A_i = B_i$ for all $i \in I$. The proof is different from ours.

2. Chang [C, Theorem 4.2] obtained Theorem 2 with a different proof. She also obtained a noncompact version of Theorem 2 as [C, Theorem 4.3].

3. Park [P9, Theorem 4.2]: A related result.

Theorem 2′. The von Neumann-Fan intersection theorem. [P18] *Let $\{(X_i;\Gamma_i)\}_{i=1}^n$ be a finite family of compact abstract convex spaces such that $(X;\Gamma) = (\prod_{i=1}^n X_i;\Gamma)$ satisfies the partial KKM principle and, for each i, let A_i and B_i are subsets of E satisfying*

$(2.1)'$ *for each $x^i \in X^i$, $\emptyset \neq \mathrm{co}_{\Gamma_i} B_i(x^i) \subset A_i(x^i) := \{y \in X \mid [x^i, y_i] \in A_i\}$; and*

$(2.2)'$ *for each $y_i \in X_i$, $B_i(y_i) := \{x^i \in X^i \mid [x^i, y_i] \in B_i\}$ is open in X^i.*

Then we have $\bigcap_{i=1}^n A_i \neq \emptyset$.

Example. For convex spaces X_i, particular forms of Theorem $2'$ have appeared as follows:

1. Fan [F3, Théorème 1]: $A_i = B_i$ for all i.

2. Fan [F4, Theorem $1'$]: $I = \{1, 2\}$ and $A_i = B_i$ for all $i \in I$.

¿From these results, Fan [F4] deduced an analytic formulation, fixed point theorems, extension theorems of monotone sets, and extension theorems for invariant vector subspaces.

For particular types of G-convex spaces, Theorem $2'$ was known as follows:

3. Bielawski [Bi, Proposition (4.12) and Theorem (4.15)]: X_i has the finitely local convexity.

4. Kirk, Sims, and Yuan [KSY, Theorem 5.2]: X_i are hyperconvex metric spaces.

5. Park [P7, Theorem 4], [P8, Theorem 19]: X_i are G-convex spaces.

From the above intersection theorems, resp., we can deduce the following equivalent forms, resp., of a generalized Fan type minimax theorem or an analytic alternative:

Theorem 3. The Fan type analytic alternative. [P13] *Let $\{(X_i; \Gamma_i)\}_{i\in I}$ be a family of Hausdorff compact G-convex spaces and, for each $i \in I$, let $f_i, g_i : X = X^i \times X_i \to \mathbb{R}$ be real functions satisfying*

(3.1) *$f_i(x) \leq g_i(x)$ for each $x \in X$;*

(3.2) *for each $x^i \in X^i$, $x_i \mapsto g_i[x^i, x_i]$ is quasiconcave on X_i; and*

(3.3) *for each $x_i \in X_i$, $x^i \mapsto f_i[x^i, x_i]$ is l.s.c. on X^i.*

Let $\{t_i\}_{i\in I}$ be a family of real numbers. Then either

(a) *there exist an $i \in I$ and an $x^i \in X^i$ such that*

$$f_i[x^i, y_i] \leq t_i \quad \text{for all } y_i \in X_i; \quad \text{or}$$

(b) *there exists an $x \in X$ such that*

$$g_i(x) > t_i \quad \text{for all } i \in I.$$

Example. 1. Ma [M, Theorem 3]: Each X_i is a compact convex subsets each in a Hausdorff topological vector spaces and $f_i = g_i$ for all $i \in I$.

3. Park [P9, Theorem 8.1]: X_i are convex spaces.

Theorem 3$'$. The Fan type analytic alternative. *Let $\{(X_i; \Gamma_i)\}_{i=1}^n$ be a finite family of compact abstract convex spaces such that $(X; \Gamma) = (\prod_{i=1}^n X_i; \Gamma)$ satisfies the partial KKM principle and, for each $i \in I$, let $f_i, g_i : X = X^i \times X_i \to \mathbb{R}$ be real functions satisfying* (3.1) – (3.3). *Then the conclusion of Theorem 3 holds for $I = \{1, 2, \ldots, n\}$.*

Example. Fan [F3, Théorème 2], [F4, Theorem 3]: X_i are convex subsets, and $f_i = g_i$ for all $i \in I$. From this, Fan [F2,3] deduced Sion's minimax theorem [Si], the Tychonoff fixed point theorem, solutions to systems of convex inequalities, extremum problems for matrices, and a theorem of Hardy-Littlewood-Pólya.

From Theorems 3 and 3$'$, we obtain the following generalizations of the Nash-Ma type equilibrium theorem, resp.:

Theorem 4. Generalized Nash-Ma type equilibrium theorem. [P13] *Let* $\{(X_i;\Gamma_i)\}_{i\in I}$ *be a family of Hausdorff compact G-convex spaces and, for each* $i \in I$, *let* $f_i, g_i : X = X^i \times X_i \longrightarrow \mathbb{R}$ *be real functions such that*

(4.0) $f_i(x) \leq g_i(x)$ *for each* $x \in X$;
(4.1) *for each* $x^i \in X^i$, $x_i \mapsto g_i[x^i, x_i]$ *is quasiconcave on* X_i;
(4.2) *for each* $x^i \in X^i$, $x_i \mapsto f_i[x^i, x_i]$ *is u.s.c. on* X_i; *and*
(4.3) *for each* $x_i \in X_i$, $x^i \mapsto f_i[x^i, x_i]$ *is l.s.c. on* X^i.

Then there exists a point $\widehat{x} \in X$ *such that*

$$g_i(\widehat{x}) \geq \max_{y_i\in X_i} f_i[\widehat{x}^i, y_i] \quad \textit{for all } i \in I.$$

Example. Park [P9, Theorem 8.2]: X_i are convex spaces.

Theorem 4′. Generalized Nash-Fan type equilibrium theorem. [P18] *Let* $\{(X_i;\Gamma_i)\}_{i=1}^n$ *be a finite family of compact abstract convex spaces such that* $(X;\Gamma) = (\prod_{i=1}^n X_i;\Gamma)$ *satisfies the partial KKM principle and, for each* i, *let* $f_i, g_i : X = X^i \times X_i \longrightarrow \mathbb{R}$ *be real functions satisfying* $(4.0) - (4.3)$. *Then there exists a point* $\widehat{x} \in X$ *such that*

$$g_i(\hat{x}) \geq \max_{y_i\in X_i} f_i[\hat{x}^i, y_i] \quad \textit{for all} \quad i = 1, 2, \ldots, n.$$

Example. In case when X_i are convex spaces, $f_i = g_i$, Theorem 4′ reduces to Tan et al. [TYY, Theorem 2.1].

From Theorems 4 and 4′, we obtain the following generalization of the Nash equilibrium theorem, resp.:

Theorem 5. Generalized Nash-Ma type equilibrium theorem. [P13] *Let* $\{(X_i;\Gamma_i)\}_{i\in I}$ *be a family of Hausdorff compact G-convex spaces and, for each* $i \in I$, *let* $f_i : X \longrightarrow \mathbb{R}$ *be a function such that*

(5.1) *for each* $x^i \in X^i$, $x_i \mapsto f_i[x^i, x_i]$ *is quasiconcave on* X_i;
(5.2) *for each* $x^i \in X^i$, $x_i \mapsto f_i[x^i, x_i]$ *is u.s.c. on* X_i; *and*
(5.3) *for each* $x_i \in X_i$, $x^i \mapsto f_i[x^i, x_i]$ *is l.s.c. on* X^i.

Then there exists a point $\hat{x} \in X$ *such that*

$$f_i(\hat{x}) = \max_{y_i\in X_i} f_i[\hat{x}^i, y_i] \quad \textit{for all } i \in I.$$

Example. Ma [M, Theorem 4]: Each X_i is a compact convex subsets each in a Hausdorff topological vector spaces.

Theorem 5′. Generalized Nash-Fan type equilibrium theorem. [P18] *Let $\{(X_i;\Gamma_i)\}_{i=1}^n$ be a finite family of compact abstract convex spaces such that $(X;\Gamma) = (\prod_{i=1}^n X_i;\Gamma)$ satisfies the partial KKM principle and, for each i, let $f_i : E \longrightarrow \mathbb{R}$ be a function satisfying* (5.1) – (5.3). *Then there exists a point $\widehat{x} \in X$ such that*

$$f_i(\widehat{x}) = \max_{y_i \in X_i} f_i[\widehat{x}^i, y_i] \quad \textit{for all} \quad i = 1, 2, \ldots, n.$$

Example. For continuous functions f_i, a number of particular forms of Theorem 5′ have appeared for convex subsets X_i of topological vector spaces as follows:

1. Nash [N2, Theorem 1]: X_i are subsets of Euclidean spaces.
2. Nikaido and Isoda [NI, Theorem 3.2].
3. Fan [F4, Theorem 4].

For particular types of G-convex spaces X_i and continuous functions f_i, particular forms of Theorem 5′ have appeared as follows:

4. Bielawski [Bi, Theorem (4.16)]: X_i have the finitely local convexity.
5. Kirk, Sims, and Yuan [KSY, Theorem 5.3]: X_i are hyperconvex metric spaces.
6. Park [P7, Theorem 6], [P8, Theorem 20]: X_i are G-convex spaces.
7. Park [P12, Theorem 4.7]: A variant of Theorem 5′ under the hypothesis that $(X;\Gamma)$ is a compact G-convex space with $X = \prod_{i=1}^n X_i$ and $f_1, \ldots, f_n : X \longrightarrow \mathbb{R}$ are continuous functions such that

(3) for each $x \in X$, each $i = 1, \ldots, n$, and each $r \in \mathbb{R}$, the set $\{(y_i, x^i) \in X \mid f_i(y_i, x^i) > r\}$ is Γ-convex.

8. González et al. [GK]: Each X_i is a compact, sequentially compact L-space and each f_i is continuous as in 7.
9. Briec and Horvath [BH, Theorem 3.2]: Each X_i is a compact $\mathbb{B}$-convex set and each f_i is continuous as in 7.

The point $\widehat{x}$ in the conclusion of Theorem 5 is called a *Nash equilibrium*. This concept is a natural extension of the local maxima and the saddle point as follows.

In case I is a singleton, we obtain the following:

Corollary 5.1. *Let X be a closed bounded convex subset of a reflexive Banach space E and $f : X \longrightarrow \mathbb{R}$ a quasiconcave u.s.c. function. Then f attains its maximum on X; that is, there exists an $\widehat{x} \in X$ such that $f(\widehat{x}) \geq f(x)$ for all $x \in X$.*

Corollary 5.1 is due to Mazur and Schauder in 1936. Some generalized forms of Corollary 1 were known by Park et al. [PK,P1].

For $I = \{1, 2\}$, Theorem 5′ reduces to the following:

Corollary 5.2. The von Neumann-Sion minimax theorem. [P18] *Let* $(X;\Gamma_1)$ *and* $(Y;\Gamma_2)$ *be compact abstract convex spaces and* $f : X \times Y \to \overline{\mathbb{R}}$ *an extended real function such that*

(1) *for each* $x \in X$, $f(x,\cdot)$ *is l.s.c. and quasiconvex on* Y*; and*

(2) *for each* $y \in Y$, $f(\cdot,y)$ *is u.s.c. and quasiconcave on* X.

If $(X \times Y;\Gamma)$ *satisfies the partial KKM principle, then*

(i) f *has a saddle point* $(x_0,y_0) \in X \times Y$*; and*

(ii) *we have*

$$\max_{x\in X}\min_{y\in Y} f(x,y) = \min_{y\in Y}\max_{x\in X} f(x,y).$$

Example. We list historically well-known particular or related forms of Corollary 5.2 in chronological order:

1. von Neumann [V1], Kakutani [K]: X and Y are compact convex subsets of Euclidean spaces and f is continuous.
2. Nikaidô [Ni1]: Euclidean spaces in the above are replaced by Hausdorff topological vector spaces, and f is continuous in each variable.
3. Sion [Si]: X and Y are compact convex subsets in topological vector spaces in Corollary 5.2.
4. Komiya [Ko, Theorem 3]: X and Y are compact convex spaces in the sense of Komiya and Y is Hausdorff.
5. Bielawski [Bi, Theorem (4.13)]: X and Y are compact spaces having certain simplicial convexities.
6. Horvath [H1, Prop. 5.2]: X and Y are C-spaces with Y Hausdorff compact.

In 4 and 6 above, Hausdorffness of Y is assumed since they adopted the partition of unity argument. However, 3 and 5 were based on the corresponding KKM theorems which need not the Hausdorffness of Y.

7. Park [P7, Theorems 2 and 3]: Variants of Corollary 5.2 with different proofs.

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