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The modified Mann's iteration methods for a family of strict pseudo-contractions

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Abstract

In this paper, we first propose a modification of Mann's iteration method for a family of strict pseudo-contractions in Hilbert spaces. Next we study the weak and strong convergence of Mann type algorithms for such a family, which extend and improve the corresponding ones due to Acedo and Xu [*Nonlinear Anal.* 67 (2007), 2258–2271] for a finite family of strict pseudo-contractions.

Keywords: Strict pseudo-contraction, modified Mann's iteration method, weak (strong) convergence, fixed point, projection.

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1 Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a mapping. We use $F(T)$ to denote the set of fixed points of T; that is, $F(T) = \{x \in C : Tx = x\}$. (Throughout this paper, we always assume that $F(T) \neq \emptyset$.)

Iterative methods are often used to solve the fixed point equation $Tx = x$. The most well-known method is perhaps the Picard successive iteration method when T is a contraction. Picard's method generates a sequence $\{x_n\}$ successively as $x_n = Tx_{n-1}$ for $n \geq 2$ with $x_1 := x$ arbitrary, and this sequence converges in norm to the unique fixed point of T. However, if T is not a contraction (for instance, if T is nonexpansive), then Picard's successive iteration fails, in general, to converge. Instead, Mann's iteration method [6] prevails.

The Mann's algorithm, an averaged process in nature, generates a sequence $\{x_n\}$ recursively by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad n \geq 1, \tag{1.1}$$

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where the initial guess $x_1 := x \in C$ is arbitrarily chosen and the sequence $\{\alpha_n\}$ lies in the interval $[0, 1]$.

Recall that a mapping $T : C \to C$ is said to be a *strict pseudo-contraction* [1] if there exists a constant $0 \leq \kappa < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \kappa\|(I - T)x - (I - T)y\|^2 \tag{1.2}$$

for all $x, y \in C$. For such a case, T is said to be a κ-strict pseudo-contraction. A 0-strict pseudo-contraction T is nonexpansive; that is, T is nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in C$.

The Mann's algorithm for nonexpansive mappings has been extensively investigated; see [1, 3, 4, 11, 12, 13, 14, 15] and the references therein. One of the well known results is proven by Reich [11] for a nonexpansive mapping $T : C \to C$, which asserts the weak convergence of the sequence $\{x_n\}$ generated by (1.1) in a uniformly convex Banach space with a Frechet differentiable norm under the control condition $\sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty$. However iterative methods for strict pseudo-contractions are far less developed though Browder and Petryshyn [1] initiated their work in 1967. Recently, Marino and Xu [7] developed and extended Reich's result to strict pseudo-contractions in the Hilbert space setting. More precisely, they proved the weak convergence of Mann's iteration process (1.1) for a κ-strict pseudo-contraction T of C.

It is known that Mann's iteration method (1.1) is in general not strongly convergent [2] for either nonexpansive mappings or strict pseudo-contractions. In 2003, a method (called hybrid method) to modify the Mann's iteration method (1.1) so that strong convergence is guaranteed has been proposed by Nakajo and Takahashi [10] for a single nonexpansive mapping T with $F(T) \neq \emptyset$ in a Hilbert space H:

$$\begin{cases} x_1 := x \in C \text{ chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1 - \alpha_n)Tx_n, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x, \quad n \geq 1, \end{cases} \tag{1.3}$$

where P_K denotes the metric projection from H onto a nonempty closed convex subset K of H. They proved that if the sequence $\{\alpha_n\}$ is bounded above from one, then the sequence $\{x_n\}$ generated by (1.3) converges strongly to $P_{F(T)}x$. This result has been extended to the class of κ-strict pseudo-contractions by Marino and Xu [8] as follows.

Theorem MX (see Theorem 4.1 of [8]) *Let C be a closed convex subset of a Hilbert space H. Let $T : C \to C$ be a κ-strict pseudo-contraction for some $0 \leq \kappa < 1$ and assume that the fixed point set $F(T)$ of T is nonempty. Let $\{x_n\}$ be the sequence* *generated by the following hybrid algorithm:*

$$\begin{cases} x_1 := x \in C \text{ chosen arbitrarily,} \\ y_n = \alpha_n x_n + (1 - \alpha_n) T x_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 + (1 - \alpha_n)(\kappa - \alpha_n)\|x_n - Tx_n\|^2\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n \rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x, \quad n \ge 1. \end{cases} \tag{1.4}$$

Assume that the control sequence $\{\alpha_n\}$ is chosen so that $\alpha_n < 1$ for all n. Then $\{x_n\}$ converges strongly to $P_{F(T)}x$.

In this paper, motivated by definition of (1.2), we say that a family $\Im = \{S_n : C \to C\}$ of self-mappings of C is *κ-strict pseudo-contraction*(in brief, κ-SPC) on C if there exist a constant $\kappa \in [0, 1)$ such that

$$\|S_n x - S_n y\|^2 \le \|x - y\|^2 + \kappa\|(I - S_n)x - (I - S_n)y\|^2 \tag{1.5}$$

for all $x, y \in C$ and all integers $n \ge 1$. In particular, note that taking $S_n := T$ for a strict pseudo-contraction $T : C \to C$ in (1.5) reduces to (1.2). We propose the following modification of the algorithm (1.1) for this family $\Im = \{S_n : C \to C\}$:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S_n x_n, \quad n \ge 1, \tag{1.6}$$

where the initial guess $x_1 := x \in C$ is arbitrarily chosen and the sequence $\{\alpha_n\}$ lies in the interval $[0, 1]$.

This paper is constructed as follows. In section 2, we present some prerequisites which are useful in our discussion. In section 3, motivated and inspired by the research works in [7], [5] and [8], we study the weak and strong convergence of the above algorithm (1.6) for the family $\Im = \{S_n : C \to C\}$ stated as in (1.5). Finally, in section 4, some applications for the parallel algorithm (4.1) and the cyclic algorithm (4.11) relating to our main results are added, which extend and improve the corresponding ones due to Acedo and Xu [5] for a finite family $\{T_i\}_{i=1}^N$ of κ_i-strict pseudo-contractions.

2 Preliminaries

Let H be a real Hilbert space with the duality product $\langle \cdot, \cdot \rangle$. When $\{x_n\}$ is a sequence in H, we denote the strong convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. We also denote the weak ω-limit set of $\{x_n\}$ by

$$\omega_w(x_n) = \{x : \exists\, x_{n_j} \rightharpoonup x\}.$$

We now need some facts and tools in a real Hilbert space H which are listed as lemmas below (see [9] for necessary proofs of Lemmas 2.2 and 2.5).

Lemma 2.1. *Let H be a real Hilbert space. There hold the following identities (which will be used in the various places in the proofs of the results of this paper).*

(i) $\|x-y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x-y, y\rangle, \quad x, y \in H.$

(ii) *For all* $\lambda_i \in [0,1]$ *with* $\sum_{i=1}^n \lambda_i = 1$, *and* $x, y \in H$, *the following equality holds:*

$$\|\sum_{i=1}^n \lambda_i x_i\|^2 = \sum_{i=1}^n \lambda_i \|x_i\|^2 - \sum_{i \neq j}^n \lambda_i \lambda_j \|x_i - x_j\|^2. \tag{2.1}$$

In particular, for $n = 2$ *we have*

$$\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2, \quad t \in [0,1]. \tag{2.2}$$

Lemma 2.2. ([9]) *Let* H *be a real Hilbert space. Given a closed convex subset* $C \subset H$ *and points* $x, y, z \in H$. *Given also a real number* $a \in \mathbb{R}$. *The set*

$$\{v \in C : \|y-v\|^2 \leq \|x-v\|^2 + \langle z, v\rangle + a\}$$

is convex (and closed).

Recall that given a closed convex subset K of a real Hilbert space H, the nearest point projection P_K from H onto K assigns to each $x \in H$ its nearest point denoted $P_K x$ in K from x to K; that is, $P_K x$ is the unique point in K with the property

$$\|x - P_K x\| \leq \|x - y\|, \quad y \in K.$$

Lemma 2.3. *Let* K *be a closed convex subset of real Hilbert space* H. *Given* $x \in H$ *and* $z \in K$. *Then* $z = P_K x$ *if and only if there holds the relation:*

$$\langle x - z, y - z\rangle \leq 0, \quad y \in K.$$

Lemma 2.4. ([5]) *Let* K *be a closed convex subset of* H. *Let* $\{x_n\}$ *be a bounded sequence in* H. *Assume*

(i) *The weak* ω*-limit set* $\omega_w(x_n) \subset K$.

(ii) *For each* $z \in K$, $\lim_{n\to\infty} \|x_n - z\|$ *exists.*

Then $\{x_n\}$ *is weakly convergent to a point in* K.

Lemma 2.5. ([9]) *Let* K *be a closed convex subset of* H. *Let* $\{x_n\}$ *be a sequence in* H *and* $x \in H$. *Let* $q = P_K x$. *If* $\{x_n\}$ *is such that* $\omega_w(x_n) \subset K$ *and satisfies the condition*

$$\|x_n - x\| \leq \|q - x\|, \quad n \geq 1. \tag{2.3}$$

Then $x_n \to q$.

3 Convergence theorems

We begin with the following lemmas which are useful in our further discussion.

Lemma 3.1. *Let C be a nonempty closed convex subset of a Hilbert space H. Let a family $\Im = \{S_n : C \to C\}$ be κ-SPC on C. Then,*

(a) *For each $n \geq 1$, S_n satisfies the Lipschitz condition, namely,*

$$\|S_n x - S_n y\| \leq L_n \|x - y\|,$$

where $L_n = \frac{1+\kappa}{1-\kappa}$.

(b) *$F := \cap_{n=1}^{\infty} F(S_n)$ is closed.*

Proof. Similarly, we can derive (a) by replacing T in the proof of Proposition 2.1 (i) in [8] with S_n. Also, the continuity of S_n for each $n \geq 1$ by (a) immediately yields the closedness of F. □

Lemma 3.2. *Let C be a nonempty closed convex subset of a Hilbert space H. Let a family $\Im = \{S_n : C \to C\}$ be κ-SPC on C. Assume that $F := \cap_{n=1}^{\infty} F(S_n) \neq \emptyset$ and the control sequence $\{\alpha_n\}$ is chosen so that $\kappa + \epsilon \leq \alpha_n \leq 1 - \epsilon$, where $\epsilon \in (0,1)$ is a small enough constant. Starting from an arbitrarily given $x_1 := x \in C$, let $\{x_n\}$ be the sequence generated by the algorithm (1.6). Then there hold the following properties.*

(a) *For each $p \in F$, $\lim_{n\to\infty} \|x_n - p\|$ exists.*

(b) *$\|x_n - S_n x_n\| \to 0$ and, furthermore, $\|x_n - x_{n+1}\| \to 0$ as $n \to \infty$.*

Proof. First to prove (a) let $p \in F$. By virtue of (1.5), we see

$$\|S_n x_n - p\|^2 = \|S_n x_n - S_n p\|^2 \leq \|x_n - p\| + \kappa \|x_n - S_n x_n\|^2.$$

Then this together with the hypothesis (ii) yields

$$\begin{aligned} & \|x_{n+1} - p\|^2 = \|\alpha_n(x_n - p) + (1-\alpha_n)(S_n x_n - p)\|^2 \\ = \quad & \alpha_n \|x_n - p\|^2 + (1-\alpha_n)\|S_n x_n - p\|^2 - \alpha_n(1-\alpha_n)\|x_n - S_n x_n\|^2 \\ \leq \quad & \|x_n - p\|^2 - (1-\alpha_n)(\alpha_n - \kappa)\|x_n - S_n x_n\|^2 \\ \leq \quad & \|x_n - p\|^2 - \epsilon^2 \|x_n - S_n x_n\|^2, \end{aligned} \tag{3.1}$$

in particular,

$$\|x_{n+1} - p\|^2 \leq \|x_n - p\|^2$$

and so $\lim_{n\to\infty} \|x_n - p\|$ exists and (i) is obtained. Since $\{x_n\}$ is bounded, so is $\{S_n x_n\}$. Now rewrite (3.1) in the form

$$\|x_n - S_n x_n\|^2 \leq \frac{1}{\epsilon^2}(\|x_n - p\|^2 - \|x_{n+1} - p\|^2).$$

Then, as $n \to \infty$, we get

$$\|x_n - S_n x_n\| \to 0. \tag{3.2}$$

¿From definition of x_{n+1}, it follows that

$$\|x_{n+1} - x_n\| = (1 - \alpha_n)\|x_n - S_n x_n\| \to 0. \tag{3.3}$$

Hence (b) is obtained. □

Lemma 3.3. *Let C be a nonempty closed convex subset of a Hilbert space H. Let a family $\Im = \{S_n : C \to C\}$ be κ-SPC on C. Assume that $F := \cap_{n=1}^{\infty} F(S_n) \neq \emptyset$, and also that the control sequence $\{\alpha_n\}$ is chosen so that $0 \leq \alpha_n < 1$ for $n \geq 1$. Let $\{x_n\}$ be the sequence generated by the following modified algorithm:*

$$\begin{cases} x_1 := x \in C \text{ chosen arbitrarily,} \\ y_n = \alpha_n x_n + (1 - \alpha_n) S_n x_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \leq \|x_n - z\|^2 + (1 - \alpha_n)(\kappa - \alpha_n)\|x_n - S_n x_n\|^2\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x, \quad n \geq 1. \end{cases}$$

There hold the following properties.

(a) $\|x_n - x\| \leq \|q - x\|$ *for all* $n \geq 1$, *where* $q := P_F x$.

(b) $\|x_n - x_{n+1}\| \to 0$ *and, furthermore,* $\|x_n - S_n x_n\| \to 0$ *as* $n \to \infty$.

Proof. First observe that C_n is convex by Lemma 2.2. Next we show that $F \subset C_n$ for $n \geq 1$. Indeed, we have, for all $p \in F$, replacing x_{n+1} in (3.1) with y_n we have

$$\begin{aligned} \|y_n - p\|^2 &= \|\alpha_n(x_n - p) + (1 - \alpha_n)(S_n x_n - p)\|^2 \\ &\leq \|x_n - p\|^2 - (1 - \alpha_n)(\alpha_n - \kappa)\|x_n - S_n x_n\|^2 \\ &\leq \|x_n - p\|^2 + (1 - \alpha_n)(\kappa - \alpha_n)\|x_n - S_n x_n\|^2 \end{aligned}$$

and thus $p \in C_n$ for all n. This shows $F \subset C_n$ for each $n \geq 1$.

Next we show that

$$F \subset Q_n, \quad n \geq 1. \tag{3.4}$$

We prove this by induction. For $n = 1$, we have $F \subset C = Q_1$. Assume that $F \subset Q_k$. Since x_{k+1} is the projection of x onto $C_k \cap Q_k$, by Lemma 2.3 we have

$$\langle x_{k+1} - z, x - x_{k+1} \rangle \geq 0, \quad z \in C_k \cap Q_k.$$

As $F \subset C_k \cap Q_k$ by the induction assumption, the last inequality holds, in particular, for all $z \in F$. This together with the definition of Q_{k+1} implies that $F \subset Q_{k+1}$. Hence (3.4) holds for all $n \geq 1$, and x_n is well defined for all n.

Notice that the definition of Q_n actually implies $x_n = P_{Q_n} x$. This together with the fact $F \subset Q_n$ further implies

$$\|x_n - x\| \leq \|p - x\|, \quad p \in F.$$

In particular, $\{x_n\}$ is bounded and

$$\|x_n - x\| \leq \|q - x\|, \quad \text{where } q := P_F x. \tag{3.5}$$

Hence (a) is obtained.

The fact $x_{n+1} \in Q_n$ asserts that $\langle x_{n+1} - x_n, x_n - x\rangle \geq 0$. This together with Lemma 2.1 (i) implies

$$\begin{aligned} \|x_{n+1} - x_n\|^2 &= \|(x_{n+1} - x) - (x_n - x)\|^2 \\ &= \|x_{n+1} - x\|^2 - \|x_n - x\|^2 - 2\langle x_{n+1} - x_n, x_n - x\rangle \\ &\leq \|x_{n+1} - x\|^2 - \|x_n - x\|^2. \end{aligned} \tag{3.6}$$

This implies that the sequence $\{\|x_n - x\|\}$ is increasing. Since it is also bounded, we see that $\lim_{n\to\infty} \|x_n - x\|$ exists. Note that since $\{x_n\}$ is bounded, so is $\{S_n x_n\}$. Then it turns out from (3.6) that

$$\|x_{n+1} - x_n\| \to 0. \tag{3.7}$$

To prove the second part of (b), i.e., $\|x_n - S_n x_n\| \to 0$, use the fact $x_{n+1} \in C_n$ to get

$$\begin{aligned} &\|y_n - x_{n+1}\|^2 \\ \leq\ &\|x_n - x_{n+1}\|^2 + (1 - \alpha_n)(\kappa - \alpha_n)\|x_n - S_n x_n\|^2. \end{aligned} \tag{3.8}$$

On the other hand, by virtue of $y_n = \alpha_n x_n + (1 - \alpha_n)S_n x_n$ and (2.2) in Lemma 2.1, we have

$$\begin{aligned} \|y_n - x_{n+1}\|^2 &= \|\alpha_n(x_n - x_{n+1}) + (1 - \alpha_n)(S_n x_n - x_{n+1})\|^2 \\ &= \alpha_n\|x_n - x_{n+1}\|^2 + (1 - \alpha_n)\|S_n x_n - x_{n+1}\|^2 \\ &\quad -\alpha_n(1 - \alpha_n)\|x_n - S_n x_n\|^2. \end{aligned}$$

After substituting this equality into (3.8), by simplifying and dividing both sides by $(1 - \alpha_n)$ (note that $\alpha_n < 1$ for all $n \geq 1$), we arrive at

$$\|x_{n+1} - S_n x_n\|^2 \leq \|x_{n+1} - x_n\|^2 + \kappa\|x_n - S_n x_n\|^2. \tag{3.9}$$

Also, since

$$\begin{aligned} &\|x_{n+1} - S_n x_n\|^2 = \|(x_{n+1} - x_n) + (x_n - S_n x_n)\|^2 \\ =\ &\|x_{n+1} - x_n\|^2 + \|x_n - S_n x_n\|^2 - 2\langle x_n - x_{n+1}, x_n - S_n x_n\rangle \end{aligned}$$

by the parallelogram law, substituting this equality into (3.9) and simplifying, we have

$$\begin{aligned} (1 - \kappa)\|x_n - S_n x_n\|^2 &\leq 2\langle x_n - x_{n+1}, x_n - S_n x_n\rangle \\ &\leq 2\|x_n - x_{n+1}\|\,\|x_n - S_n x_n\| \end{aligned}$$

or

$$(1 - \kappa)\|x_n - S_n x_n\| \leq 2\|x_n - x_{n+1}\| \to 0$$

by (3.7), and so $\lim_{n\to\infty} \|x_n - S_n x_n\| = 0$. □

Now we present the weak and strong convergence of the algorithm (1.6) for a κ-SPC family $\Im = \{S_n : C \to C\}$.

Theorem 3.4. *Under the same hypotheses with Lemma 3.2, assume, in addition, that $\omega_w(x_n) \subset F$ and F is convex. Then $\{x_n\}$ converges weakly to a common fixed point of $\Im$.*

Proof. By (a) of Lemma 3.2, $\lim_{n\to\infty} \|x_n - p\|$ exists for $p \in F$. Also, by the assumption, $\omega_w(x_n) \subset F$. Note also that F is a nonempty closed convex subset of C. Hence an application of Lemma 2.4 with $K := F$ ensures that $\{x_n\}$ converges weakly to a point in F. □

Theorem 3.5. *Under the same hypotheses with Lemma 3.3, assume, in addition, that $\omega_w(x_n) \subset F$ and F is convex. Then $x_n \to P_F x$.*

Proof. By virtue of the assumption $\omega_w(x_n) \subset F$ and (3.5), an application of Lemma 2.5 ensures that $x_n \to q$, where $q = P_F x$. □

4 Applications

Let C be a nonempty closed convex subset of a Hilbert space H. Unless other specified throughout this section, we always assume that

(c_1) for each $1 \le i \le N$, $T_i : C \to C$ be a κ_i-strict pseudo-contraction for some $0 \le \kappa_i < 1$,

(c_2) for each $n \ge 1$, $\{\lambda_i^{(n)}\}$ is a finite sequence of positive numbers such that $\sum_{i=1}^N \lambda_i^{(n)} = 1$ for all n, and $\bar{\lambda}_i := \inf\{\lambda_i^{(n)} : n \ge 1\} > 0$ for $1 \le i \le N$.

Recently, Lopez Acedo and Xu [5] considered the problem of finding a point x such that

$$x \in \cap_{i=1}^N F(T_i),$$

where $\{T_i\}_{i=1}^N$ are κ_i-strict pseudo-contractions defined on C under the condition (c_2). As $F := \cap_{i=1}^N F(T_i) \neq \emptyset$, they investigated the weak and strong convergence problems of the sequence $\{x_n\}$ generated explicitly by the following parallel algorithm:

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)\sum_{i=1}^N \lambda_i^{(n)} T_i x_n, \quad n \ge 1, \tag{4.1}$$

where the initial guess $x_1 := x \in C$ is arbitrarily chosen and $\{\alpha_n\} \subset [0,1]$.

For each $n \ge 1$, let a mapping $S_n : C \to C$ defined by

$$S_n x = \sum_{i=1}^N \lambda_i^{(n)} T_i x \tag{4.2}$$

for all $x \in C$, Then the parallel algorithm (4.1) can be written simply as

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n) S_n x_n, \quad n \ge 1 \tag{4.3}$$

and it is not hard to see that

$$F_N \subset F := \cap_{n=1}^{\infty} F(S_n), \tag{4.4}$$

where $F_N := \cap_{i=1}^{N} F(T_i)$.

Put $\kappa := \max\{\kappa_i : 1 \le i \le N\}$. Obviously, $0 \le \kappa < 1$ and we therefore obtain the following properties of the mapping S_n.

Lemma 4.1. *Let $x, y \in C$ and $1 \le i \le N$. Then the following properties are satisfied.*

(i) $\|T_i x - T_i y\|^2 \le \|x - y\|^2 + \kappa\|(I - T_i)x - (I - T_i)y\|^2$.

(ii) $\|S_n x - S_n y\|^2 \le \|x - y\|^2 + \kappa\|(I - S_n)x - (I - S_n)y\|^2$. *In other words, the family $\Im = \{S_n : C \to C\}$ is κ-SPC on C.*

(iii) *If $F_N := \cap_{i=1}^{N} F(T_i) \ne \emptyset$, then $F_N = F := \cap_{n=1}^{\infty} F(S_n)$. (In this case, note that F in Theorem 3.4 and 3.5 is closed convex so that the projection P_F is well defined.)*

Proof. (i) is obvious from the definition of strict pseudo-contraction. To prove (ii), use (2.1) of Lemma 2.1 to derive

$$\begin{aligned}
&\|(I - S_n)x - (I - S_n)y\|^2 \\
&= \Big\|\sum_{i=1}^{N} \lambda_i^{(n)}[(I - T_i)x - (I - T_i)y]\Big\|^2 \\
&= \sum_{i=1}^{N} \lambda_i^{(n)}\|(I - T_i)x - (I - T_i)y\|^2 - \sum_{i \ne j}^{N} \lambda_i^{(n)}\lambda_j^{(n)}\|(T_i x - T_i y) - (T_j x - T_j y)\|^2.
\end{aligned}$$

This yields a simple form:

$$\sum_{i=1}^{N} \lambda_i^{(n)}\|(I - T_i)x - (I - T_i)y\|^2 = \|(I - S_n)x - (I - S_n)y\|^2 + J, \tag{4.5}$$

where $J := \sum_{i \ne j}^{N} \lambda_i^{(n)}\lambda_j^{(n)}\|(T_i x - T_i y) - (T_j x - T_j y)\|^2 \ge 0$. Use (2.1), (i) and (4.5) in turn to get

$$\begin{aligned}
\|S_n x - S_n y\|^2 &= \Big\|\sum_{i=1}^{N} \lambda_i^{(n)}(T_i x - T_i y)\Big\|^2 \\
&= \sum_{i=1}^{N} \lambda_i^{(n)}\|T_i x - T_i y\|^2 - J \\
&\le \sum_{i=1}^{N} \lambda_i^{(n)}\{\|x - y\|^2 + \kappa\|(I - T_i)x - (I - T_i)y\|^2\} - J
\end{aligned}$$

$$\begin{aligned}
&= \|x-y\|^2 + \kappa \sum_{i=1}^N \lambda_i^{(n)} \|(I-T_i)x - (I-T_i)y\|^2 - J \\
&= \|x-y\|^2 + \kappa \|(I-S_n)x - (I-S_n)y\|^2 - (1-\kappa)J \\
&\leq \|x-y\|^2 + \kappa \|(I-S_n)x - (I-S_n)y\|^2.
\end{aligned}$$

Hence (ii) is proven.

Finally to prove (iii), by (4.4), it suffices to show that $F \subset F_N$. Indeed, let $x = S_n x$ for all $n \geq 1$. Since $F_N \neq \emptyset$, for $p \in F_N$, use (2.1) and (i) to derive

$$\begin{aligned}
\|p-x\|^2 &= \|p - S_n x\|^2 = \Big\| \sum_{i=1}^N \lambda_i^{(n)} (p - T_i x) \Big\|^2 \\
&= \sum_{i=1}^N \lambda_i^{(n)} \|p - T_i x\|^2 - \sum_{i \neq j}^N \lambda_i^{(n)} \lambda_j^{(n)} \|T_i x - T_j x\|^2 \\
&\leq \sum_{i=1}^N \lambda_i^{(n)} \{\|p-x\|^2 + \kappa \|x - T_i x\|^2\} - \delta \\
&= \|p-x\|^2 + \kappa \sum_{i=1}^N \lambda_i^{(n)} \|x - T_i x\|^2 - \delta
\end{aligned}$$

where $\delta := \sum_{i \neq j}^N \lambda_i^{(n)} \lambda_j^{(n)} \|T_i x - T_j x\|^2$. Therefore, we have

$$\delta \leq \gamma_n \|p-x\|^2 + \kappa \sum_{i=1}^N \lambda_i^{(n)} \|x - T_i x\|. \tag{4.6}$$

On the other hand, since $S_n x = x$ for all $n \geq 1$, it follows from (2.1) that

$$\begin{aligned}
0 &= \|S_n x - x\| = \Big\| \sum_{i=1}^N \lambda_i^{(n)} (T_i x - x) \Big\|^2 \\
&= \sum_{i=1}^N \lambda_i^{(n)} \|T_i x - x\|^2 - \delta.
\end{aligned} \tag{4.7}$$

Substituting (4.7) into (4.6) and simplifying, we have

$$\begin{aligned}
0 &\leq (1-\kappa) \sum_{i=1}^N \bar{\lambda}_i \|T_i x - x\|^2 \\
&\leq (1-\kappa) \sum_{i=1}^N \lambda_i^{(n)} \|T_i x - x\|^2 \\
&\leq 0.
\end{aligned}$$

This implies that, for $1 \leq i \leq N$, $T_i x = x$ and so $x \in F_N = \cap_{i=1}^N F(T_i)$, which proves (iii). □

Lemma 4.2. *Assume the common fixed point set* $F_N := \cap_{i=1}^N F(T_i)$ *is nonempty. Let* $1 \le i \le N$, $x \in C$ *and* $p \in F_N$. *Then,*

(i) $(1-\kappa)\sum_{i=1}^N \lambda_i^{(n)}\|x - T_i x\|^2 \le 2\|p - x\|\|x - S_n x\|$.

(ii) *Let* $\{x_n\} \subset C$ *such that* $x_n \rightharpoonup z$ *and* $\|x_n - S_n x_n\| \to 0$. *Assume, in addition,* $\|x_n - x_{n+1}\| \to 0$. *Then* $z \in F_N$.

Proof. Put $I := \sum_{i=1}^N \lambda_i^{(n)}\|x - T_i x\|^2$ and $J := \sum_{i \ne j}^N \lambda_i^{(n)}\lambda_j^{(n)}\|T_i x - T_j x\|^2$. Use (2.1) to get

$$\|x - S_n x\|^2 = \Big\|\sum_{i=1}^N \lambda_i^{(n)}(x - T_i x)\Big\|^2 = I - J.$$

Observe

$$\begin{aligned}
\|p - S_n x\|^2 &= \|(p - x) + (x - S_n x)\|^2 \\
&= \|p - x\|^2 + \|x - S_n x\|^2 - 2\langle x - p, x - S_n x\rangle \\
&= \|p - x\|^2 + I - J - 2\langle x - p, x - S_n x\rangle \qquad (4.8)
\end{aligned}$$

by parallelogram law. Using (2.1) and (i) of Lemma 4.1 we have

$$\begin{aligned}
\|p - S_n x\|^2 &= \Big\|\sum_{i=1}^N \lambda_i^{(n)}(p - T_i x)\Big\|^2 = \sum_{i=1}^N \lambda_i^{(n)}\|p - T_i x\|^2 - J \\
&\le \sum_{i=1}^N \lambda_i^{(n)}[\|p - x\|^2 + \kappa\|x - T_i x\|^2] - J \\
&\le \|p - x\|^2 + \kappa I - J. \qquad (4.9)
\end{aligned}$$

Substituting (4.8) into (4.9) and simplifying we have

$$\begin{aligned}
(1-\kappa)I &\le 2\langle x - p, x - S_n x\rangle \\
&\le 2\|p - x\|\|x - S_n x\|,
\end{aligned}$$

which proves (i). To show (ii), replacing x with x_n in (i) gives

$$(1-\kappa)\sum_{i=1}^N \lambda_i^{(n)}\|x_n - T_i x_n\|^2 \le 2\|p - x_n\|\|x - S_n x_n\|.$$

Since $\{x_n\}$ is bounded and $\|x_n - S_n x_n\| \to 0$, we can easily derive

$$\|x_n - T_i x_n\| \to 0, \quad 1 \le i \le N. \qquad (4.10)$$

Then the demiclosedness principle of $I - T_i$ implies that $z \in F(T_i)$ for all $1 \le i \le N$. Hence $z \in F_N = \cap_{i=1}^N F(T_i)$ and the proof is complete. □

As direct applications of Theorem 3.4, we have following weak convergence for the parallel algorithm (4.1) (or see (4.3) for a compact form) for a finite family $\{T_i\}_{i=1}^N$ of N κ_i-strict pseudo-contractions; compare with Theorem 3.3 in Lopez Acedo and Xu [5].

Theorem 4.3. *Let C be a nonempty closed convex subset of a Hilbert space H. Let $\{T_i\}_1^N$ and $\{\lambda_i^{(n)}\}$ be as in (c_1) and (c_2), respectively. Let $\kappa := \max\{\kappa_i : 1 \le i \le N\}$. Assume that $F_N := \cap_{i=1}^N F(T_i) \neq \emptyset$ and the control sequence $\{\alpha_n\}$ are chosen so that $\kappa + \epsilon \le \alpha_n \le 1 - \epsilon$, where $\epsilon \in (0,1)$ is a small enough constant. Starting from an arbitrarily given $x_1 := x \in C$, let $\{x_n\}$ be the sequence generated by the parallel algorithm (4.1) or (4.3). Then $\{x_n\}$ converges weakly to a common fixed point of $\{T_i\}_{i=1}^N$.*

Proof. By (ii) and (iii) of Lemma 4.1, it suffices to show that $\omega_w(x_n) \subset F$. This fact is directly derived from (ii) of Lemma 4.2 by reminding of (b) of Lemma 3.2. Then our conclusion is obtained by Theorem 3.4. □

As direct applications of Theorem 3.5, we have following strong convergence for the parallel algorithm (4.1) (or see (4.3) for a compact form) for a finite family $\{T_i\}_{i=1}^N$ of N κ_i-strict pseudo-contractions due to Lopez Acedo and Xu [5]; see Theorem 5.1 in [5].

Theorem 4.4. ([5]; see Theorem 5.1) *Let C be a nonempty closed convex subset of a Hilbert space H. Let $\{T_i\}_1^N$ and $\{\lambda_i^{(n)}\}$ be as in (c_1) and (c_2), respectively. Let $\kappa := \max\{\kappa_i : 1 \le i \le N\}$. Assume that $F_N := \cap_{i=1}^N F(T_i)$ is a nonempty bounded subset of C, and also that the control sequence $\{\alpha_n\}$ is chosen so that $0 \le \alpha_n < 1$ for $n \ge 1$. Let $\{x_n\}$ be the sequence generated by the following modified parallel algorithm:*

$$\begin{cases} x_1 := x \in C \text{ chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1-\alpha_n)\sum_{i=1}^N \lambda_i^{(n)} T_i x_n = \alpha_n x_n + (1-\alpha_n) S_n x_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 + (1-\alpha_n)(\kappa - \alpha_n)\|x_n - S_n x_n\|^2\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x, \quad n \ge 1. \end{cases}$$

Then $x_n \to P_{F_N} x$.

Proof. By (ii) and (iii) of Lemma 4.1, $\Im = \{S_n : C \to C\}$ is κ-SPC on C and $F = F_N$. Immediately, the fact $\omega(x_n) \subset F$ is required from (ii) of Lemma 4.2 by reminding of (b) of Lemma 3.3. Then our conclusion is achieved by Theorem 3.5. □

Lopez Acedo and Xu [5] also investigated the convergence problems for the following cyclic algorithm:

$$\begin{aligned} x_1 &:= x \in C \text{ chosen arbitrarily}, \\ x_2 &= \alpha_1 x_1 + (1-\alpha_1)T_1 x_1, \\ x_3 &= \alpha_2 x_2 + (1-\alpha_2)T_2 x_2, \\ &\vdots \\ x_{N+1} &= \alpha_N x_N + (1-\alpha_N)T_N x_N, \\ x_{N+2} &= \alpha_{N+1} x_{N+1} + (1-\alpha_{N+1})T_1 x_{N+1}, \\ &\vdots \end{aligned}$$

where $\{\alpha_n\}$ be a sequence in $[0,1]$. The above cyclic algorithm can be written in a more compact form as

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)T_{[n]}x_n, \quad n \geq 1, \tag{4.11}$$

where $T_{[k]} = T_{k \bmod N}$ for integer $k \geq 1$. The mod function takes values in the set $\{1, 2, \cdots, N\}$ as

$$T_{[k]} = \begin{cases} T_N, & \text{if } q = 0; \\ T_q, & \text{if } 0 < q < N \end{cases}$$

for $k = jN + q$ for some integers $j \geq 0$ and $0 \leq q < N$.

Finally, as direct consequences of our main theorems, we obtain the following weak and strong convergence problems for the cyclic algorithm (4.11) for a finite family $\{T_i\}_{i=1}^N$ of κ_i-strict pseudo-contractions due to Lopez Acedo and Xu [5]; see Theorem 4.1 and 5.2, respectively, in [5].

Theorem 4.5. ([5]; see Theorem 4.1) *Under the same hypotheses with Theorem 4.3, the sequence $\{x_n\}$ generated by the cyclic algorithm (4.11) converges weakly to a common fixed point of $\{T_i\}_{i=1}^N$.*

Proof. Replacing all the S_n in the process of the proof of Lemma 3.2 with $T_{[n]}$, we can immediately prove the following facts:

(1) $\lim_{n\to\infty} \|x_n - p\|$ exists for $p \in F_N$;

(2) $\|x_n - T_{[n]}x_n\| \to 0$ (hence $\|x_n - x_{n+1}\| \to 0$) as $n \to \infty$.

By (2), it is not hard to see that, for $1 \leq i \leq N$

$$\|x_n - x_{n+i}\| \to 0 \tag{4.12}$$

and

$$\|T_{[n]}x_n - x_{n+i}\| \to 0, \tag{4.13}$$

that is,

$$\|x_n - T_i x_n\| \to 0, \quad 1 \leq i \leq N. \tag{4.14}$$

Finally to show $\omega_w(x_n) \subset F_N$, use the demiclosedness property of $I - T_i$. Use Lemma 2.4 (with $K = F_N$) to conclude that $\{x_n\}$ converges weakly to a point in F_N. □

Theorem 4.6. ([5]; see Theorem 5.2) *Let C be a nonempty closed convex subset of a Hilbert space H. Let $\{T_i\}_1^N$ and $\{\lambda_i^{(n)}\}$ be as in (c_1) and (c_2), respectively. Let $\kappa := \max\{\kappa_i : 1 \leq i \leq N\}$. Assume that $F_N := \cap_{i=1}^N F(T_i)$ is a nonempty bounded subset of C, and also that the control sequence $\{\alpha_n\}$ is chosen so that $0 \leq \alpha_n < 1$ for all n. Let $\{x_n\}$ be the sequence generated by the following modified cyclic algorithm:*

$$\begin{cases} x_1 := x \in C \text{ chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1-\alpha_n)T_{[n]}^{k(n)}x_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \leq \|x_n - z\|^2 + (1-\alpha_n)(\kappa - \alpha_n)\|x_n - T_{[n]}^{k(n)}x_n\|^2\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}x, \end{cases}$$

where $\theta_n = \gamma_n \cdot \sup\{\|x_n - z\|^2 : z \in F_N\} \to 0$. *Then* $x_n \to P_{F_N}x$.

Proof. First, to claim the following observations (i)-(vi), simply replace S_n in the proof of Lemma 3.3 with $T_{[n]}$.

(i) x_n is well defined for all $n \geq 1$.

(ii) $\|x_n - x\| \leq \|q - x\|$ for all n, where $q = P_{F_N}x$.

(iii) $\|x_{n+1} - x_n\| \to 0$.

(vi) $\|x_n - T_{[n]}x_n\| \to 0$.

To derive $\omega_n(x_n) \subset F_N$, repeat the argument of (4.12)-(4.14) in the proof of Theorem 4.5. Finally use (ii) and Lemma 2.5 to arrive at the our conclusion. □

References

[1] F. E. Browder and W. V. Petryshyn, Construction of fixed points of nonlinear mappings in Hilbert spaces, *J. Math. Anal. Appl.* 20 (1967), 197–228.

[2] A. Genel and J. Lindenstrauss, An example concerning fixed points, *Israel J. Math.* 22 (1975), 81–86.

[3] T. H. Kim and H. K. Xu, Strong convergence of modified Mann iterations, *Nonlinear Anal.* 61 (2005), 51–60.

[4] P. L. Lions, Approximation de points fixes de contractions, *C.R. Acad. Sci. Sèr. A-B Paris* 284 (1977), 1357–1359.

[5] G. Lopez Acedo and H. K. Xu, Iterative methods for strict pseudo-contractions in Hilbert spaces, *Nonlinear Anal.* 67 (2007), 2258–2271.

[6] W. R. Mann, Mean value methods in iteration, *Proc. Amer. Math. Soc.* 4 (1953), 506–510.

[7] G. Marino and H. K. Xu, Convergence of generalized proximal point algorithms, *Comm. Applied Anal.* 3 (2004), 791–808.

[8] G. Marino and H. K. Xu, Weak and strong convergence theorems for strict pseudo-contractions in Hilbert Spaces, *J. Math. Anal. Appl.* 329 (2007) 336–346.

[9] C. Matinez-Yanes and H. K. Xu, Strong convergence of the CQ method for fixed point processes, *Nonlinear Anal.* 64 (2006), 2400-2411.

[10] K. Nakajo and W. Takahashi, Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups, *J. Math. Anal. Appl.* 279 (2003), 372–379.

[11] S. Reich, Weak convergence theorems for nonexpansive mappings in Banach spaces, *J. Math. Anal. Appl.* **67** (1979), 274–276.

[12] R. Wittmann, Approximation of fixed points of nonexpansive mappings, *Arch. Math.* 58 (1992), 486–491.

[13] H. K. Xu, Iterative algorithms for nonlinear operators, *J. London Math. Soc.* 66 (2002), 240–256.

[14] H. K. Xu, Remarks on an iterative method for nonexpansive mappings, *Comm. Applied Nonlinear Anal.* 10 (2003), no. 1, 67–75.

[15] H. K. Xu, Strong convergence of an iterative method for nonexpansive Mappings and accretive operators, *J. Math. Anal. Appl.* 314 (2006), 631–643.