

UNIFIED SCALARIZATION FOR SETS IN SET-VALUED OPTIMIZATION*

(集合値最適化における集合の統一的なスカラー化)

新潟大学・大学院自然科学研究科 桑野一成, 田中 環, 山田修司
Issei Kuwano, Tamaki Tanaka, Syuuji Yamada[†]
Graduate School of Science and Technology,
Niigata University, Japan

Abstract

In the paper, we introduce several types of set-valued optimization problems and investigate optimality conditions for them to use unified types of scalarizing functions for set-valued maps.

1 Introduction

In recent years, nonlinear scalarization methods for sets are studied as one of important tools in set-valued optimization. In [1], they introduce sublinear scalarizing functions for vectors and show several optimality conditions for vector-valued optimization. In [6], they extend these scalarizing functions to four types of nonlinear scalarizing functions for set-valued maps, and show several useful properties of them. Moreover, in [8], they introduce several optimality conditions for set-valued optimization to use these four types of nonlinear scalarizing functions. In [2], certain interesting nonlinear scalarizing functions for sets are proposed and they give generalized results on Ekeland variational principle in an abstract space like topological vector space without such strong assumption as convexity. Moreover, a modified scalarizing function in [7] gives a similar result to a minimal element theorem in [2] under different assumptions. In [3], they introduced several optimality conditions for set-valued optimization to use nonlinear scalarizing functions for sets. As seen from the above, there are several types of nonlinear scalarizing functions for set-valued maps. In [5], we introduce new unified approach on such scalarization for sets and some properties of these functions. The aim of this paper is to investigate some properties of unified types of scalarizing functions proposed in [5] and optimality conditions for set-valued

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[†] E-mail: kuwano@m.sc.niigata-u.ac.jp, {tamaki,yamada}@math.sc.niigata-u.ac.jp
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optimization to use these functions.

The organization of the paper is as follows. In Section 2, we introduce mathematical methodology on comparison between two sets in an ordered vector space proposed in [4] and some definitions of solutions for set-valued optimization problem. In Section 3, we introduce two types of nonlinear scalarizing functions for sets proposed by the unified approach in [5], and investigate their properties including the monotonicity. In Section 4, we investigate several optimality conditions for set-valued optimization.

2 Mathematical Preliminaries

Let Y be a real topological vector space with the partial ordering \leq_C induced by a nonempty convex cone C ($C + C = C$ and $\lambda C \subset C$ for all $\lambda \geq 0$) as follows:

$$x \leq_C y \text{ if } y - x \in C \text{ for } x, y \in Y.$$

It is well known that \leq_C is reflexive and transitive when C is a convex cone, moreover, \leq_C has invariant properties to vector space structure as translation and scalar multiplication. Then, the space Y is called a *partially ordered topological vector space*, and if \leq_C is antisymmetric it becomes an *ordered topological vector space*.

Throughout the paper, X is a real topological vector space, Y a real ordered topological vector space and F a set-valued map from X into $2^Y \setminus \{\emptyset\}$. Moreover, for any $A \subset Y$ we denote the interior, closure of A by $\text{int}(A)$, $\text{cl}(A)$, respectively.

Let us recall some definitions. It is said that A is C -closed if $A + C$ is a closed set, C -bounded if for each neighborhood U of zero in Y there is some positive number t such that $A \subset tU + C$.

At first, we review some basic concepts of set-relation.

Definition 2.1. (set-relation, [4]) For nonempty sets $A, B \subset Y$ and convex cone C in Y , we write

$$\begin{aligned} A \leq_C^{(1)} B & \text{ by } A \subset \bigcap_{b \in B} (b - C), \text{ equivalently } B \subset \bigcap_{a \in A} (a + C); \\ A \leq_C^{(2)} B & \text{ by } A \cap \left(\bigcap_{b \in B} (b - C) \right) \neq \emptyset; \\ A \leq_C^{(3)} B & \text{ by } B \subset (A + C); \\ A \leq_C^{(4)} B & \text{ by } \left(\bigcap_{a \in A} (a + C) \right) \cap B \neq \emptyset; \\ A \leq_C^{(5)} B & \text{ by } A \subset (B - C); \\ A \leq_C^{(6)} B & \text{ by } A \cap (B - C) \neq \emptyset, \text{ equivalently } (A + C) \cap B \neq \emptyset. \end{aligned}$$

Proposition 2.1. ([4]) For nonempty sets $A, B \subset Y$, the following statements hold.

$$\begin{aligned} A \leq_C^{(1)} B & \text{ implies } A \leq_C^{(2)} B; & A \leq_C^{(1)} B & \text{ implies } A \leq_C^{(4)} B; \\ A \leq_C^{(2)} B & \text{ implies } A \leq_C^{(3)} B; & A \leq_C^{(4)} B & \text{ implies } A \leq_C^{(5)} B; \\ A \leq_C^{(3)} B & \text{ implies } A \leq_C^{(6)} B; & A \leq_C^{(5)} B & \text{ implies } A \leq_C^{(6)} B. \end{aligned}$$

Proposition 2.2. ([5]) For nonempty sets $A, B \subset Y$, the following statements hold.

- (i) For each $j = 1, \dots, 6$,
 $A \leq_C^{(j)} B$ implies $(A + y) \leq_C^{(j)} (B + y)$ for $y \in Y$, and
 $A \leq_C^{(j)} B$ implies $\alpha A \leq_C^{(j)} \alpha B$ for $\alpha \geq 0$;
- (ii) For each $j = 1, \dots, 5$, $\leq_C^{(j)}$ is transitive;
- (iii) For each $j = 3, 5, 6$, $\leq_C^{(j)}$ is reflexive.

Next, we consider the following six kinds of set-valued optimization problems:

$$(j\text{-SVOP}) \begin{cases} j\text{-Min} & F(x) \\ \text{Subject to} & x \in X \end{cases}$$

and we introduce the concepts of solutions for these problems under six kinds of set-relations in Definition 2.1.

Definition 2.2. (solution and weak solution of j -SVOP) Let $x_0 \in X$. For each $j = 1, \dots, 6$, x_0 is a solution of (j -SVOP) if for any $x \in X \setminus \{x_0\}$,

$$F(x) \leq_C^{(j)} F(x_0) \quad \text{implies} \quad F(x_0) \leq_C^{(j)} F(x).$$

Moreover, x_0 is a weak solution of (j -SVOP) if for any $x \in X \setminus \{x_0\}$,

$$F(x) \leq_{\text{int}C}^{(j)} F(x_0) \quad \text{implies} \quad F(x_0) \leq_{\text{int}C}^{(j)} F(x).$$

We denote the solution sets of (j -SVOP) by (j)-Min $F(X)$ and the weak solution sets of (j -SVOP) by (j)-WMin $F(X)$.

Example 2.1. Let $X = \mathbb{R}_+$, $Y = \mathbb{R}^2$ and $C = \mathbb{R}_+^2$. We consider a set-valued map $F : X \rightarrow 2^Y$

$$F(x) := \begin{cases} \left[\begin{pmatrix} x \\ \frac{1}{x} \end{pmatrix}, \begin{pmatrix} x \\ \frac{1}{x} + 1 \end{pmatrix} \right] & (0 \leq x \leq 1), \\ \left[\begin{pmatrix} x \\ 1 \end{pmatrix}, \begin{pmatrix} x \\ 2 \end{pmatrix} \right] & (1 \leq x), \end{cases}$$

where $[a, b] := \{c \in Y \mid a \leq_C c \text{ and } c \leq_C b\}$. Then (1)-Min $F(x) = (1)$ -WMin $F(x) = X$. For each $j = 2, \dots, 5$, (j)-Min $F(x) = [0, 1]$, (j)-WMin $F(x) = X$.

It is clear that if x_0 is a solution of (j -SVOP) then x_0 is a weak solution of (j -SVOP).

3 Unified Scalarization Methods for Sets

At first, we introduce the definition of two types of nonlinear scalarizing functions for sets proposed by a unified approach in [5]

Definition 3.1. (unified types of scalarizing functions, [5].) Let V and V' be nonempty subsets of Y , and direction $k \in \text{int}C$. For each $j = 1, \dots, 6$, $I_{k,V}^{(j)} :$

$2^Y \setminus \{\emptyset\} \rightarrow \mathbb{R} \cup \{\pm\infty\}$ and $S_{k,V'}^{(j)} : 2^Y \setminus \{\emptyset\} \rightarrow \mathbb{R} \cup \{\pm\infty\}$ are defined by

$$I_{k,V'}^{(j)}(V) := \inf \left\{ t \in \mathbb{R} \mid V \leq_C^{(j)} (tk + V') \right\},$$

$$S_{k,V'}^{(j)}(V) := \sup \left\{ t \in \mathbb{R} \mid (tk + V') \leq_C^{(j)} V \right\},$$

respectively.

In this section, we introduce some properties of unified types of scalarizing functions.

Proposition 3.1. ([5]) *Let $V \in 2^Y \setminus \{\emptyset\}$. For each $j = 1, \dots, 6$, the following statements hold.*

$$V \leq_C^{(j)} (tk + V') \quad \text{implies} \quad V \leq_C^{(j)} (sk + V') \quad \text{for any } s \geq t;$$

$$(tk + V') \leq_C^{(j)} V \quad \text{implies} \quad (sk + V') \leq_C^{(j)} V \quad \text{for any } s \leq t.$$

Proposition 3.2. ([5]) *For nonempty subsets $A, B, V \subset Y$, $I_{k,V'}^{(j)}$ and $S_{k,V'}^{(j)}$ satisfy the following properties:*

(i) *For each $j = 1, \dots, 6$ and $\alpha \in \mathbb{R}_+$,*

$$I_{k,V'}^{(j)}(V + \alpha k) = I_{k,V'}^{(j)}(V) + \alpha;$$

$$S_{k,V'}^{(j)}(V + \alpha k) = S_{k,V'}^{(j)}(V) + \alpha.$$

(ii) *For each $j = 1, \dots, 5$,*

$$A \leq_C^{(j)} B \quad \text{implies} \quad I_{k,V'}^{(j)}(A) \leq I_{k,V'}^{(j)}(B) \quad \text{and} \quad S_{k,V'}^{(j)}(A) \leq S_{k,V'}^{(j)}(B).$$

Proposition 3.3. *For each $j = 1, \dots, 5$, $I_{k,V'}^{(j)}(V') \geq 0$ and $S_{k,V'}^{(j)}(V') \leq 0$, in particular,*

$$V' \leq_C^{(j)} V' \quad \text{implies} \quad I_{k,V'}^{(j)}(V') = S_{k,V'}^{(j)}(V') = 0;$$

Proof. The case of $j = 3, 5$, by Proposition 2.2 (iii), $V' \leq_C^{(j)} V'$. Hence we obtain $I_{k,V'}^{(j)}(V') \geq 0$ and $S_{k,V'}^{(j)}(V') \leq 0$. We consider the case of $j = 1, 2, 4$. Let $I_{k,V'}^{(j)}(V') = t_j$ and assume that $t_j < 0$. Then, there exists $\epsilon > 0$ and $t(\epsilon) \in \mathbb{R}$ such that

$$t_j < t(\epsilon) < t + \epsilon < 0 \quad \text{and} \quad V' \leq_C^{(j)} t(\epsilon)k + V'. \quad (3.1)$$

By Proposition 3.2 (ii),

$$I_{k,V'}^{(j)}(V') \leq I_{k,V'}^{(j)}(t(\epsilon)k + V').$$

Moreover, by Proposition 3.2 (i),

$$I_{k,V'}^{(j)}(t(\epsilon)k + V') = I_{k,V'}^{(j)}(V') + t(\epsilon).$$

Hence, we obtain $t_j \leq t_j + t(\epsilon)$ and so $t(\epsilon) \geq 0$. This contradicts (3.1). Consequently, we have $I_{k,V'}^{(j)}(V') \geq 0$. The case of $S_{k,V'}^{(j)}(V')$ are proved in the similar way. Next, we assume that $V' \leq_C^{(j)} V'$. By Proposition 3.2 (ii), we obtain $I_{k,V'}^{(j)}(V') = S_{k,V'}^{(j)}(V') = 0$. \square

Proposition 3.4. *Let $A \in 2^Y \setminus \{\emptyset\}$. Then, the following statements hold:*

(i) *For each $j = 1, \dots, 3$, A and V' are C -bounded sets if and only if*

$$I_{k,V'}^{(j)}(A) > -\infty \quad \text{and} \quad S_{k,V'}^{(j)}(A) < \infty,$$

(ii) *For each $j = 4, 5$, A and V' are $(-C)$ -bounded sets if and only if*

$$I_{k,V'}^{(j)}(A) > -\infty \quad \text{and} \quad S_{k,V'}^{(j)}(A) < \infty,$$

Proof. In the case of $j = 3, 5$, they are shown in [Theorem 3.6, 3]. The others can be proved by similar ways in the case of $j = 3, 5$, respectively. \square

Proposition 3.5. *Let $A \in 2^Y \setminus \{\emptyset\}$. Then, the following statements hold:*

(i) *For each $j = 1, \dots, 3$, if A is C -closed, C -bounded and V' is C -bounded then*

$$I_{k,V'}^{(j)}(A) = \min\{t \in \mathbb{R} \mid A \leq_C^{(j)} tk + V'\},$$

$$S_{k,V'}^{(j)}(A) = \max\{t \in \mathbb{R} \mid tk + V' \leq_C^{(j)} A\},$$

(ii) *For each $j = 4, 5$, if A is $(-C)$ -closed, $(-C)$ -bounded and V' is $(-C)$ -bounded then*

$$I_{k,V'}^{(j)}(A) = \min\{t \in \mathbb{R} \mid A \leq_C^{(j)} tk + V'\},$$

$$S_{k,V'}^{(j)}(A) = \max\{t \in \mathbb{R} \mid tk + V' \leq_C^{(j)} A\}.$$

Proof. In the case of $j = 3, 5$, they are shown in [Proposition 3.2, 3]. The others can be proved by similar ways in the case of $j = 3, 5$, respectively. \square

Proposition 3.6. *Let $A, B \in 2^Y \setminus \{\emptyset\}$. Then, the following statements hold:*

(i) *For each $j = 1, 2, 3$, if B is C -closed and $A \leq_{\text{int}C}^{(j)} B$ then*

$$I_{k,V'}^{(j)}(A) < I_{k,V'}^{(j)}(B) \quad \text{and} \quad S_{k,V'}^{(j)}(A) < S_{k,V'}^{(j)}(B),$$

(ii) *For each $j = 4, 5$, if A is $(-C)$ -closed and $A \leq_{\text{int}C}^{(j)} B$ then*

$$I_{k,V'}^{(j)}(A) < I_{k,V'}^{(j)}(B) \quad \text{and} \quad S_{k,V'}^{(j)}(A) < S_{k,V'}^{(j)}(B).$$

Proof. First, we prove (i). Assume that B is C -closed and $A \leq_{\text{int}C}^{(j)} B$. We consider the case of $j = 3$. Let $t_A := I_{k,V'}^{(3)}(A)$ and $t_B := I_{k,V'}^{(3)}(B)$. Then, for any $\epsilon > 0$ there exists $t(\epsilon) \in \mathbb{R}$ such that

$$t_B < t(\epsilon) < t_B + \epsilon \quad \text{and} \quad B \leq_C^{(3)} t(\epsilon)k + V'.$$

Since $A \leq_{\text{int}C}^{(3)} B$,

$$t(\epsilon)k + v \in B + C \subset A + \text{int}C = \text{int}(A + C),$$

for all $v \in V'$. Hence, there exists absorbing open neighborhood of zero G such that

$$t(\epsilon)k + v + G \subset \text{int}A + C.$$

Since G is absorbing, there exists $t_0 > 0$ such that $-t_0k \in G$ and so we obtain

$$(t(\epsilon) - t_0)k + v + G \subset \text{int}A + C.$$

Hence we have

$$t_A \leq t(\epsilon) - t_0 < t_B + \epsilon - t_0.$$

Since ϵ is an arbitrary, we obtain $t_A \leq t_B - t_0 < t_B$. The proof of $S_{k,V'}^{(3)}$, and the other cases can be proved in a similar way. Also, the proof of (ii) is shown similarly. \square

Remark 3.1. In Proposition 3.6, the conditions of C -closed or $(-C)$ -closed are necessary. Consider the case of $j = 3$. Let $A, B \subset Y \setminus \{\emptyset\}$ with $A \neq \text{int}A$ and $B + C = \text{int}(A + C)$, and let $t_A := I_{k,V'}^{(3)}(A)$ and $t_B := I_{k,V'}^{(3)}(B)$. Then, since C containing zero and $B + C = \text{int}(A + C)$, we obtain $A \leq_C^{(3)} B$ and so $t_A \leq t_B$ by Proposition 3.2 (ii). We assume that $t_A < t_B$. Then, there exists $\bar{t} \in \mathbb{R}$ such that $t_A < \bar{t} < t_B$ and $A \leq_C^{(3)} \bar{t}k + V'$. Let $t_0 := \frac{1}{2}\bar{t} + \frac{1}{2}t_B$. By Proposition 3.1,

$$A \leq_C^{(3)} t_0k + V' \quad \text{and} \quad B \not\leq_C^{(3)} t_0k + V'.$$

Hence, there exists $t_0k + v \in t_0k + V'$ such that

$$t_0k + v \in A + C \quad \text{and} \quad t_0k + v \notin B + C. \quad (3.2)$$

Since $k \in \text{int}C$ and C is a convex cone, $(t_0 - \bar{t})k \in \text{int}C$. Hence, $(t_0 - \bar{t})k + v \in V' + \text{int}C$ and so we have

$$t_0k + v \in \bar{t}k + V' + \text{int}C \subset A + C + \text{int}C = \text{int}(A + C) = B + C.$$

This contradicts (3.2). Consequently, $I_{k,V'}^{(3)}(A) = I_{k,V'}^{(3)}(B)$ for any $k \in \text{int}C$ although $A \leq_{\text{int}C}^{(3)} B$. The other cases are similar, too.

4 Optimality conditions for set-valued optimization

Let $V' \in 2^Y \setminus \{\emptyset\}$ and direction $k \in \text{int}C$. For any $x \in X$ and for each $j = 1, \dots, 6$, we consider the following composite functions:

$$(I_{k,V'}^{(j)} \circ F)(x) := I_{k,V'}^{(j)}(F(x)),$$

$$(S_{k,V'}^{(j)} \circ F)(x) := S_{k,V'}^{(j)}(F(x)).$$

In this section, we consider some scalar optimization problems where objective functions are unified types of scalarizing functions, and investigate the relation of the solution between these problems and (j -SVOP).

Theorem 4.1. ([3]) *Assume that F is C -closed, C -bounded valued on X and $x_0 \in X$. Let $k \in \text{int}C$. Then x_0 is a solution of (3-SVOP) if and only if x_0 is a solution of the following scalar optimization problem:*

$$(3 - \text{SOP}) \begin{cases} \text{Min} & (I_{k,F(x_0)}^{(3)} \circ F)(x) \\ \text{Subject to} & x \in X \end{cases}$$

and if $x \in X$ then

$$I_{k,F(x_0)}^{(3)} F(x) = 0 \quad \text{if and only if} \quad F(x_0) \leq_C^{(3)} F(x) \quad \text{and} \quad F(x) \leq_C^{(3)} F(x_0).$$

Corollary 4.1. *Assume that F is $(-C)$ -closed, $(-C)$ -bounded valued on X and $x_0 \in X$. Let $k \in \text{int}C$. Then x_0 is a solution of (5-SVOP) if and only if x_0 is a solution of the following scalar optimization problem:*

$$(5 - \text{SOP}) \begin{cases} \text{Min} & (I_{k,F(x_0)}^{(5)} \circ F)(x) \\ \text{Subject to} & x \in X \end{cases}$$

and if $x \in X$ then

$$(I_{k,F(x_0)}^{(5)} \circ F)(x) = 0 \quad \text{if and only if} \quad F(x_0) \leq_C^{(5)} F(x) \quad \text{and} \quad F(x) \leq_C^{(5)} F(x_0).$$

Proof. We assume that x_0 is a solution of (5-SVOP). Then, for any $x \in X \setminus \{x_0\}$

$$F(x) \leq_C^{(5)} F(x_0) \quad \text{implies} \quad F(x_0) \leq_C^{(5)} F(x).$$

By Proposition 3.2 and 3.3, we obtain $(I_{k,F(x_0)}^{(5)} \circ F)(x_0) = 0$ and

$$F(x) \not\leq_C^{(5)} F(x_0) \quad \text{implies} \quad (I_{k,F(x_0)}^{(5)} \circ F)(x) \geq 0 = (I_{k,F(x_0)}^{(5)} \circ F)(x_0). \quad (4.1)$$

Moreover, by Proposition 3.2 (ii) we have

$$F(x_0) \leq_C^{(5)} F(x) \quad \text{implies} \quad (I_{k,F(x_0)}^{(5)} \circ F)(x_0) \leq (I_{k,F(x_0)}^{(5)} \circ F)(x). \quad (4.2)$$

Hence, by (4.1) and (4.2) we obtain $(I_{k,F(x_0)}^{(5)} \circ F)(x_0) \leq (I_{k,F(x_0)}^{(5)} \circ F)(x)$ for any $x \in X \setminus \{x_0\}$. Consequently, x_0 is a solution of (5-SOP).

Conversely, we assume that x_0 is a solution of (5-SOP). Then, $(I_{k,F(x_0)}^{(5)} \circ F)(x_0) \leq (I_{k,F(x_0)}^{(5)} \circ F)(x)$ for any $x \in X \setminus \{x_0\}$. Suppose that x_0 is not a solution of (5-SVOP). Then, there exist $\bar{x} \in X \setminus \{x_0\}$ such that

$$F(x) \leq_C^{(5)} F(x_0) \quad \text{and} \quad F(x_0) \not\leq_C^{(5)} F(x). \quad (4.3)$$

By Proposition 3.3, $(I_{k,F(x_0)}^{(5)} \circ F)(x_0) = 0$ and so we obtain

$$0 = (I_{k,F(x_0)}^{(5)} \circ F)(x_0) \leq (I_{k,F(x_0)}^{(5)} \circ F)(\bar{x}). \quad (4.4)$$

Moreover, by Proposition 3.2 (ii)

$$(I_{k,F(x_0)}^{(5)} \circ F)(\bar{x}) \leq (I_{k,F(x_0)}^{(5)} \circ F)(x_0) = 0. \quad (4.5)$$

Hence, by (4.4) and (4.5) we obtain $(I_{k,F(x_0)}^{(5)} \circ F)(\bar{x}) = (I_{k,F(x_0)}^{(5)} \circ F)(x_0) = 0$, and so we have

$$F(\bar{x}) \leq_C^{(5)} F(x_0) \quad \text{and} \quad F(x_0) \leq_C^{(5)} F(\bar{x}).$$

This contradicts (4.3). Consequently, x_0 is a solution of (5-SVOP). \square

Corollary 4.2. *Assume that F is $(-C)$ -closed, $(-C)$ -bounded valued on X and $x_0 \in X$. Let $k \in \text{int}C$. For each $j = 1, 4$, if x_0 is a solution of the following scalar optimization problem:*

$$(j\text{-SOP}) \begin{cases} \text{Min} & I_{k,F(x_0)}^{(j)} F(x) \\ \text{Subject to} & x \in X \end{cases}$$

and for any $x \in X$

$$I_{k,F(x_0)}^{(j)} F(x) = 0 \quad \text{if and only if} \quad F(x_0) \leq_C^{(j)} F(x) \quad \text{and} \quad F(x) \leq_C^{(j)} F(x_0),$$

then x_0 is a solution of (j -SVOP).

Proof. We can prove this corollary by a similar way in Corollary 4.2. \square

Corollary 4.3. *Assume that F is C -closed, C -bounded valued on X and $x_0 \in X$. Let $k \in \text{int}C$. If x_0 is a solution of the following scalar optimization problem:*

$$(2\text{-SOP}) \begin{cases} \text{Min} & I_{k,F(x_0)}^{(2)} F(x) \\ \text{Subject to} & x \in X \end{cases}$$

and for any $x \in X$ then

$$I_{k,F(x_0)}^{(2)}F(x) = 0 \quad \text{if and only if} \quad F(x_0) \leq_C^{(2)} F(x) \quad \text{and} \quad F(x) \leq_C^{(2)} F(x_0),$$

then x_0 is a solution of (2-SVOP).

Proof. We can prove this corollary by a similar way in Corollary 4.2. □

Remark 4.1. In the case of $j = 1, 2, 4$, even if $x_0 \in X$ is a solution of (j -SVOP), x_0 is not necessary a solution of each scalar optimization problem (j -SOP) in Corollary 4.2 and 4.3. We consider the case of $j = 1$. Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $C = \mathbb{R}_+$,

$$A := \left\{ \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \mid (a_1 - 1)^2 + (a_2 - 1)^2 = 1 \right\},$$

$$B := \left\{ \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \mid 0 \leq b_1 \leq 1, b_2 = -b_1 + 1 \right\},$$

and $k = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Then, we consider a set-valued map $F : X \rightarrow 2^Y$

$$F(x) := \begin{cases} A & (x \geq 0), \\ B & (x < 0). \end{cases}$$

Since $A \not\leq_C^{(1)} B$ and $B \not\leq_C^{(1)} A$ we obtain (1)-Min $F(X)=X$. Let $x_1 = 1$, $x_2 = -1$ and we consider $I_{k,F(x_i)}^{(1)}$. Then, $(I_{k,F(1)}^{(1)} \circ F)(1) = 2$ and $(I_{k,F(-1)}^{(1)} \circ F)(-1) = 1$. Hence, x_1 is not a solution of (1-SOP) although x_1 is a solution of (1-SVOP). This example is the counter example of the other cases, too.

Theorem 4.2. Assume that F is C -closed, C -bounded valued on X , $V' \in 2^Y \setminus \{\emptyset\}$ is C -bounded, and $x_0 \in X$. Let $k \in \text{int}C$. For each $j = 1, \dots, 3$, x_0 is a solution of the following scalar optimization problem then x_0 is a weak solution of (j -SVOP):

$$(j\text{-SOP}) \begin{cases} \text{Min} & I_{k,V'}^{(j)}F(x) \\ \text{Subject to} & x \in X. \end{cases}$$

Proof. We assume that x_0 is a solution of (j -SOP). Then, for any $x \in X \setminus \{x_0\}$

$$(I_{k,V'}^{(j)} \circ F)(x_0) \leq (I_{k,V'}^{(j)} \circ F)(x). \tag{4.6}$$

Suppose that x_0 is not a weak solution of (j -SVOP). Then, there exists $\bar{x} \in X \setminus \{x_0\}$ such that

$$F(\bar{x}) \leq_{\text{int}C}^{(j)} F(x_0) \quad \text{and} \quad F(x_0) \not\leq_{\text{int}C}^{(j)} F(\bar{x}).$$

By Proposition 3.6 (i), $(I_{k,V'}^{(j)} \circ F)(\bar{x}) < (I_{k,V'}^{(j)} \circ F)(x_0)$. This contradicts (4.6). Consequently, x_0 is a weak solution of (j -SVOP). □

Corollary 4.4. Assume that F is $(-C)$ -closed, $(-C)$ -bounded valued on X , $V' \in 2^Y \setminus \{\emptyset\}$ is $(-C)$ -bounded and $x_0 \in X$. Let $k \in \text{int}C$. For each $j = 4, 5$, if x_0 is a solution of the following scalar optimization problem then x_0 is a weak solution of (j -SVOP):

$$(j\text{-SOP}) \begin{cases} \text{Min} & I_{k,V'}^{(j)} F(x) \\ \text{Subject to} & x \in X. \end{cases}$$

Proof. We can prove this corollary by a similar way in Theorem 4.2. \square

Theorem 4.3. Assume that F is C -closed, C -bounded valued on X , $V' \in 2^Y \setminus \{\emptyset\}$ is C -bounded and $x_0 \in X$. Let $k \in \text{int}C$. For each $j = 1, \dots, 3$, x_0 is a unique solution of the following scalar optimization problem then x_0 is a solution of (j -SVOP):

$$(j\text{-SOP}) \begin{cases} \text{Min} & I_{k,V'}^{(j)} F(x) \\ \text{Subject to} & x \in X. \end{cases}$$

Proof. We assume that x_0 is a unique solution of (j -SOP). Then, for any $x \in X \setminus \{x_0\}$

$$(I_{k,V'}^{(j)} \circ F)(x_0) \leq (I_{k,V'}^{(j)} \circ F)(x).$$

Suppose that x_0 is not a solution of (j -SVOP). Then, there exists $\bar{x} \in X \setminus \{x_0\}$ such that

$$F(\bar{x}) \leq_C^{(j)} F(x_0) \quad \text{and} \quad F(x_0) \not\leq_C^{(j)} F(\bar{x}).$$

By Proposition 3.2 (ii), $(I_{k,V'}^{(j)} \circ F)(\bar{x}) \leq (I_{k,V'}^{(j)} \circ F)(x_0)$. Hence \bar{x} is a solution of (j -SOP). This is a contradiction to the uniqueness of x_0 . Consequently, x_0 is a solution of (j -SVOP). \square

Corollary 4.5. Assume that F is $(-C)$ -closed, $(-C)$ -bounded valued on X , $V' \in 2^Y \setminus \{\emptyset\}$ is $(-C)$ -bounded and $x_0 \in X$. Let $k \in \text{int}C$. For each $j = 4, 5$, if x_0 is a unique solution of the following scalar optimization problem then x_0 is a solution of (j -SVOP):

$$(j\text{-SOP}) \begin{cases} \text{Min} & I_{k,V'}^{(j)} F(x) \\ \text{Subject to} & x \in X. \end{cases}$$

Proof. We can prove this corollary by a similar way in Theorem 4.3. \square

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