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Meagre Subsets of ${}^{\omega}[0,1]$ and $\mathcal{B}(l^2)$

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We generalize Talagrand's characterization of meagre subsets of $\mathscr{P}(\omega)$ to certain meagre subsets of ${}^{\omega}X$, where X is a topological space like the real unit interval $[0,1]$, and to certain meagre subsets of $\mathcal{B}(l^2)_1$, the collection of bounded linear operators of norm at most 1 on the Hilbert space $l^2=\{f\in{}^{\omega}\mathbb{F}:\sum|f(n)|^2<\infty\}$ (where $\mathbb{F}=\mathbb{R}$ or $\mathbb{C}$), with the weak operator topology. We obtain a fairly complete characterization of meagre subsets of ${}^{\omega}X$ that are closed under finite (initial segment) changes and, for metric spaces X, meagre subsets of ${}^{\omega}X$ that are closed under limit 0 changes. To a lesser extent, we generalize this to characterize meagre subsets of $\mathcal{B}(l^2)_1$ (and the following subsets of $\mathcal{B}(l^2)_1$ - the norm ≤ 1 self-adjoint operators, non-negative operators and orthogonal projections) that are closed under finite rank and compact operator changes.

1 Introduction

Given an interval partition $(I_n)\subseteq\omega$ and a real $A\subseteq\omega$,

$$\begin{aligned}\mathcal{M}_{A,(I_n)} &= \{B\subseteq\omega:\forall^{\infty}n\in\omega(A\cap I_n\neq B\cap I_n)\}\\ &= \bigcup_{m\in\omega}\bigcap_{n\geq m}\{B\subseteq\omega:A\cap I_n\neq B\cap I_n\}\end{aligned}$$

is a countable union of closed nowhere dense sets, and hence meagre, when identifying $\mathscr{P}(\omega)$ with ${}^{\omega}2$ with the usual product topology. Conversely, every meagre subset of $\mathscr{P}(\omega)$ is contained in a set of this form ([1] 5.2). In particular if, for arbitrary (I_n), we let $\mathcal{M}_{(I_n)}=\mathcal{M}_{\omega,(I_n)}$, then $\mathcal{M}_{(I_n)}$ will be meagre and closed under taking almost subsets. Conversely, every meagre subset of $\mathscr{P}(\omega)$ closed under taking almost subsets (or even just subsets) will be contained in $\mathcal{M}_{(I_n)}$, for some interval partition (I_n) ([1] 6.27). We want to generalize this result in some way to functions from ω to the real unit interval $[0,1]$ and, in turn, to (orthogonal) projections on the Hilbert space l^2 (i.e. (linear) operators P on H that are idempotent ($P^2=P$) and self-adjoint ($P^*=P$)) with the weak operator topology (or the strong operator topology, as they agree when restricted to just the projections).

To see the motivation for this, let us first make some definitions. Let H be a fixed infinite dimensional separable Hilbert space (i.e. a Hilbert space isometrically isomorphic to l^2) with some fixed orthonormal basis (e_n). For any subspace V of H let P_V be the unique orthogonal projection onto V (i.e. $\mathcal{R}(P_V)=V$). For any $A\subseteq\omega$, let $P_A=P_{A,(e_n)}=P_{\mathrm{span}\{e_n:n\in A\}}$. For any $\mathcal{A}\subseteq\mathscr{P}(\omega)$ let $P_{\mathcal{A}}=\{P_A:A\in\mathcal{A}\}$. For any transitive relation R on a set S and any subset T of S let $\mathrm{cl}_R(T)=\{s\in S:\exists t\in T(sRt)\}$, the R-closure of T. Define transitive relations, for $A,B\subseteq\omega$ and $P,Q\in\mathcal{P}(\mathcal{B}(H))(=$ the projections on $H)$, as follows.

$$\begin{aligned}A\subseteq^* B &\Leftrightarrow |A\backslash B|<\infty.\\ A=^* B &\Leftrightarrow A\subseteq^* B\wedge B\subseteq^* A.\\ P\leq^* Q &\Leftrightarrow PQ-P\in\mathcal{K}(H)(=\text{ the compact operators on }H).\\ P=^* Q &\Leftrightarrow P\leq^* Q\wedge Q\leq^* P\Leftrightarrow P-Q\in\mathcal{K}(H).\end{aligned}$$

We would like to prove the following.

Conjecture 1.1 *For any $\mathcal{A} \subseteq \mathscr{P}(\omega)$, $\mathrm{cl}_{\subseteq^*}(\mathcal{A})$ is meagre if and only if $\mathrm{cl}_{\leq^*}(P_{\mathcal{A}})$ is meagre.*

This, in turn, is motivated by wanting to prove an inequality between certain cardinal invariants. In my presentation at RIMS 2009 I discussed a number of cardinal invariants defined from $\mathcal{P}(\mathcal{B}(H))/\mathcal{K}(H)$ in analogy with the classical cardinal invariants defined from $\mathscr{P}(\omega)/\mathrm{Fin}$. In particular, I showed how these new cardinal invariants could often be related to analogous cardinal invariants involving interval partitions on ω, essentially due to the fact that projections onto block subspaces are $\leq^*$-dense in $\mathcal{P}(\mathcal{B}(H))$. We need 1.1 to prove such a relation between the analogies of the groupwise density number $\mathfrak{g}$.

Specifically, recall that the (classical) groupwise density number $\mathfrak{g}$ is the minimum cardinality of a collection $\mathscr{A}$ of $\subseteq^*$-closed non-meagre subsets of $\mathscr{P}(\omega)$ whose intersection is empty. Equivalently, this is the minimum cardinality of a collection $\mathscr{A} \subseteq \mathscr{P}(\mathscr{P}(\omega))$ such that, for all $\mathcal{A} \in \mathscr{A}$, $\mathrm{cl}_{\subseteq^*}(\mathcal{A})$ is non-meagre and, for all $B \subseteq \omega$, there exists $\mathcal{A} \in \mathscr{A}$ such that, for all $A \in \mathcal{A}$, $B \nsubseteq^* A$.

Let us define a new cardinal invariant $\mathfrak{g}^{\mathrm{IP}}$ to be the minimum cardinality of a collection $\mathscr{A} \subseteq \mathscr{P}(\omega)$ such that, for all $\mathcal{A} \in \mathscr{A}$, $\mathrm{cl}_{\subseteq^*}(\mathcal{A})$ is non-meagre and, for all interval partitions (I_n) of ω, there exists $\mathcal{A} \in \mathscr{A}$ such that, for all $A \in \mathcal{A}$, I_n and A are disjoint for infinitely many $n \in \omega$. The first defining property of $\mathscr{A}$ in the definition of $\mathfrak{g}^{\mathrm{IP}}$ is the same as that in the definition $\mathfrak{g}$, but the second defining property is stronger and hence $\mathfrak{g} \leq \mathfrak{g}^{\mathrm{IP}}$. It can also be proved that $\mathfrak{g}^{\mathrm{IP}} \geq \mathfrak{b}$. Thus we can in fact have $\mathfrak{g} \neq \mathfrak{g}^{\mathrm{IP}}$, for example in the Hechler model where $\mathfrak{b} = \mathfrak{c} = \aleph_2$ and $\mathfrak{g} = \aleph_1$. Moreover, the proof that $\mathfrak{g} \leq \mathfrak{d}$ given in [1] 6.27 in fact shows that $\mathfrak{g}^{\mathrm{IP}} \leq \mathfrak{d}$, so $\mathfrak{g}^{\mathrm{IP}}$ is, at least, not always equal to $\mathfrak{c}$.

Let us define yet another cardinal invariant $\mathfrak{g}^{\perp}$ as the minimum cardinality of a collection $\mathscr{P} \subseteq \mathscr{P}(\mathcal{P}(\mathcal{B}(H)))$ such that, for all $\mathcal{P} \in \mathscr{P}$, $\mathrm{cl}_{\leq^*}(\mathcal{P})$ is non-meagre and, for all $Q \in \mathcal{P}(\mathcal{B}(H))$, there exists $\mathcal{P} \in \mathscr{P}$ such that, for all $P \in \mathcal{P}$, Q and $P^{\perp}(= 1 - P = P_{\mathcal{R}(P)^{\perp}})$ have a non-trivial (i.e. infinite rank projection) lower bound (w.r.t. $\leq^*$ which, in particular, implies that $Q \nleq^* P$). For any infinite rank $Q \in \mathcal{P}(\mathcal{B}(H))$ there exists a projection $Q' \leq^* Q$ such that $\mathcal{R}(Q')$ is a infinite dimensional block subspace, i.e. there exists a partition (I_n) of ω and an orthonormal basis (f_n) of $\mathcal{R}(Q')$ such that $f_n \in \mathrm{span}\{e_k : k \in I_n\}$ for all $n \in \omega$. If $\mathcal{A} \subseteq \mathscr{P}(\omega)$ is such that, for any $A \in \mathcal{A}$, there are infinitely many $n \in \omega$ such that I_n is disjoint from A then there exists infinitely many $n \in \omega$ such that $f_n \in \mathcal{R}(P_A)^{\perp} = \mathcal{R}(P_A^{\perp})$ and so the projection onto the closed linear span of all these f_n will be a non-trivial lower bound of $P_A^{\perp}$ and Q', and hence Q. Thus if we could prove 1.1, in particular if we could show that if $\mathcal{A} \subseteq \mathscr{P}(\omega)$ is such that $\mathrm{cl}_{\subseteq^*}(\mathcal{A})$ is non-meagre then $\mathrm{cl}_{\leq^*}(P_{\mathcal{A}})$ is also non-meagre, then it would follow that $\mathfrak{g}^{\perp} \leq \mathfrak{g}^{\mathrm{IP}}$.

At first sight it might seem silly to be always taking $\subseteq^*$ and $\leq^*$ closures. Instead, one might simply deal only with $\mathcal{A}$ that are already $\subseteq^*$-closed. Indeed, if $A \subseteq^* B$ then $P_A \leq^* P_B$ so 1.1 is equivalent to saying that, for any $\subseteq^*$-closed $\mathcal{A} \subseteq \mathscr{P}(\omega)$, $\mathcal{A}$ is meagre if and only if $\mathrm{cl}_{\leq^*}(P_{\mathcal{A}})$ is meagre. However, we still can not replace $\mathrm{cl}_{\leq^*}(P_{\mathcal{A}})$ with $P_{\mathcal{A}}$ or even $\mathrm{cl}_{=^*}(P_{\mathcal{A}})$ because, for any infinite $A \subseteq \omega$, there will be many projections $P \leq^* P_A$ that are not of the form P_B for some $B \subseteq^* A$ or even $=^*$-equivalent to a projection of this form. Indeed, the conjecture will false if we do that replacement, as $\mathrm{cl}_{=^*}(P_{\mathscr{P}(\omega)})$ is meagre. In fact, $\mathrm{cl}_{=^*}(P_{\mathscr{P}(\omega)})$ is *good* meagre, as defined by Zamora-Aviles in [2] (and even *very* good meagre, as I show in 3.26).

Before reviewing and extending the Zamora-Aviles theory of good meagre sets, let us look at nowhere dense and meagre subsets of countable products of topological spaces (like $[0, 1]$) as these are interesting in their own right and provide a good basis for studying operators on Hilbert spaces. Furthermore, as we will see, we can obtain a more complete understanding in the case of countable products of topological spaces – unfortunately, some of the results in this case to not seem to easily generalize to the case of operators on Hilbert spaces.

2 Countable Products of Topological Spaces

For the rest of this section X is a topological space and ${}^{\omega}X$ is given the standard product topology, i.e. the sets $O_0 \times \ldots \times O_{n-1} \times {}^{\omega\setminus n}X = \{f \in {}^{\omega}X : \forall k \in n(f(k) \in O_k)\}$, for $n \in \omega$ and open subsets $O_0, \ldots, O_{n-1}$ of X, form a basis for the topology of ${}^{\omega}X$. For $\mathcal{F} \subseteq {}^{\omega}X$ and $A \subseteq \omega$ we define $\mathcal{F} \restriction A = \{f \restriction A : f \in \mathcal{F}\}$.

Definition 2.1 *A subset Y of X is* nowhere dense *in X if $X\backslash\overline{Y}$ is dense in X. Equivalently, Y is nowhere dense if, for every non-empty open subset O of X, there exists a non-empty open subset O' of O disjoint from Y. A subset Y of X is* meagre *in X if it is a countable union of nowhere dense sets.*

Proposition 2.2 *For any $\mathcal{F} \subseteq {}^\omega X$, the following are equivalent.*

(i) *For all $n \in \omega$, $\mathcal{F} \restriction \omega\backslash n$ is not dense in ${}^{\omega\backslash n}X$.*

(ii) *For all $n \in \omega$, $\mathcal{F} \restriction \omega\backslash n$ is nowhere dense in ${}^{\omega\backslash n}X$.*

(iii) *For all $n \in \omega$ there exists $m \geq n$ and non-empty open $O_n, \ldots, O_{m-1} \subseteq X$ such that*

$$\{f \in {}^\omega X : \forall k \in m\backslash n(f(k) \in O_k)\}$$

is disjoint from $\mathcal{F}$.

Proof:

(iii)⇒(ii) Assume (iii), fix $n \in \omega$ and take any basic open set $B = \{f \in {}^{\omega\backslash n}X : \forall k \in j\backslash n(f(k) \in O_k)\}$ of ${}^{\omega\backslash n}X$, where $j \in \omega\backslash n$ and $O_n, \ldots, O_{j-1}$ are non-empty open subsets of X. Take $m \geq j$ and open $O_j, \ldots, O_{m-1}$ such that $\{f \in {}^\omega X : \forall k \in m\backslash j(f(k) \in O_k)\}$ is disjoint from $\mathcal{F}$. Then $\{f \in {}^\omega X : \forall k \in m\backslash n(f(k) \in O_k)\}$ is also disjoint from $\mathcal{F}$ which is equivalent to saying $\{f \in {}^{\omega\backslash n}X : \forall k \in m\backslash n(f(k) \in O_k)\} \subseteq B$ is disjoint from $\mathcal{F} \restriction \omega\backslash n$.

(ii)⇒(i) Immediate.

(i)⇒(iii) Immediate. □

Definition 2.3 *Any $\mathcal{F} \subseteq {}^\omega X$ satisfying any of the equivalent conditions in the proposition above is said to be* good *nowhere dense. A countable union of good nowhere dense sets is said to be* good *meagre.*

Proposition 2.4 *Good nowhere dense sets are closed under subsets, finite unions and topological closures.*

Proof: Follows from characterization (ii) above and the corresponding closures for nowhere dense sets as $\mathcal{F} \subseteq \mathcal{G} \Rightarrow \mathcal{F} \restriction \omega\backslash n \subseteq \mathcal{G} \restriction \omega\backslash n$, $\mathcal{F} \cup \mathcal{G} \restriction \omega\backslash n = \mathcal{F} \restriction \omega\backslash n \cup \mathcal{G} \restriction \omega\backslash n$ and $\overline{\mathcal{F}} \restriction \omega\backslash n \subseteq \overline{\mathcal{F} \restriction \omega\backslash n}$ for all $\mathcal{F}, \mathcal{G} \subseteq {}^\omega X$ and $n \in \omega$. □

Lemma 2.5 *Any $\mathcal{F} \subseteq {}^{\omega\backslash n}X$ will be nowhere dense in ${}^{\omega\backslash n}X$ if and only if $\{f \in {}^\omega X : f \restriction \omega\backslash n \in \mathcal{F}\}$ is nowhere dense in ${}^\omega X$.*

Proposition 2.6 *If X is any finite set with the discrete topology then any nowhere dense subset $\mathcal{N}$ of ${}^\omega X$ is good nowhere dense.*

Proof: If $\mathcal{N}$ is not good nowhere dense then there exists $n \in \omega$ such that $\mathcal{N} \restriction \omega\backslash n$ is not nowhere dense. Wlog assume $X = |X| = \{0, \ldots, |X| - 1\}$ and, for each $t \in X^n$, define $f + t \in {}^\omega X$ by $(f+t)(k) = f(k) + t(k) \mod X$ for $k \in n$ and $(f+t)(k) = f(k)$ for $k \in \omega\backslash n$. For each $t \in X^n$, the map $f \mapsto f + t$ is a homeomorphism so if $\mathcal{N}$ is nowhere dense then so is $\mathcal{N} + t = \{f + t : f \in \mathcal{N}\}$. Then $\{f \in {}^\omega X : f \restriction \omega\backslash n \in \mathcal{N} \restriction \omega\backslash n\} = \bigcup_{t \in X^n} \mathcal{N} + t$ is also nowhere dense which, by the above lemma, means $\mathcal{N} \restriction \omega\backslash n$ is nowhere dense, a contradiction. □

The problem is, of course, that when X is not finite there will be other nowhere dense sets. Indeed, as long as X contains $\{x\}$ that is closed and not open (i.e. not isolated) then we see that $\{f \in {}^\omega X : f(0) = x\}$ is nowhere dense but not good nowhere dense. This gave me the idea that perhaps these are the only other kinds of nowhere dense sets. However, as the counterexample below shows, this is not the case.

Proposition 2.7 *For each $n \in \omega$, let $\mathcal{N}_n = \{f \in {}^\omega[0,1] : f(n) = nf(0)\}$. Then $\mathcal{N} = \bigcup \mathcal{N}_n$ is closed nowhere dense in ${}^\omega[0,1]$ but not good nowhere dense nor contained in $N \times {}^{\omega\setminus n}X$ for any $n \in \omega$ and nowhere dense subset N of ${}^n[0,1]$.*

Proof: If $f \in {}^\omega[0,1]\setminus\mathcal{N}$ then $f(0) \neq 0$, as $\mathcal{N}_0 = \{0\} \times {}^{\omega\setminus\{0\}}[0,1]$, so we may let

$$m = \max\{n \in \omega : nf(0) \leq 2\} \qquad \text{and} \qquad \epsilon = \min\{|f(n) - nf(0)|/(2n) : n \in m\setminus\{0\}\}.$$

Then $\{g \in {}^\omega[0,1] : |f(0) - g(0)| < \epsilon \wedge \forall n \in m|f(n) - g(n)| < |f(n) - nf(0)|/2\} \ni f$ is a basic open set disjoint from $\mathcal{N}$ which, as f was arbitrary, shows that $\mathcal{N}$ is closed. Furthermore, for any basic open set $B = \{f \in {}^\omega[0,1] : \forall k \in n(f(k) \in O_k)\}$, for non-empty open subsets $O_0, \ldots, O_{n-1}$ of $[0,1]$, we may pick $f \in {}^\omega[0,1]$ such that $f(0) \in O_0\setminus\{0\}$ and $f(k) \in O_k\setminus\{kf(0)\}$, for all $k \in n\setminus\{0\}$, and hence $f \in B\setminus\mathcal{N}$, i.e. $\mathcal{N}$ is closed nowhere dense. However, we have $\mathcal{N}_0 \restriction \omega\setminus\{0\} = {}^{\omega\setminus\{0\}}[0,1]$ so $\mathcal{N}$ is not good nowhere dense. Lastly, take any $n \in \omega$ and nowhere dense $N \subseteq {}^n[0,1]$. Then there exists open $O_0, \ldots, O_{n-1}$ such that $O_0 \times \ldots \times O_{n-1}$ is disjoint from N and $\sup O_n \leq 1/n$. Picking any $f \in {}^\omega[0,1]$ such that $f(k) \in O_k$ for all $k \in n$ and $f(n) = nf(0) \leq n \sup O_0 \leq 1$, we see that $f \in \mathcal{N}\setminus N \times {}^{\omega\setminus n}[0,1]$. □

Note, however, that, for every $n \in \omega$, $\mathcal{N}_n$ is indeed a subset of $N_n \times {}^{\omega\setminus n}[0,1]$ for a nowhere dense $N_n \subseteq [0,1]^n$. This begs the following question.

Question 2.8 *For $X = [0,1]$ or some non-trivial class of topological spaces X, is every nowhere dense $\mathcal{M} \subseteq {}^\omega X$ either good nowhere dense or a subset of some $\mathcal{N} = \bigcup N_n \times {}^{\omega\setminus n}X$ where N_n is nowhere dense in ${}^nX$ for each $n \in \omega$? What if we further require $\mathcal{N}$ to be nowhere dense? Is every meagre $\mathcal{M} \subseteq {}^\omega X$ a subset of some $\bigcup \mathcal{N}_n \cup N_n \times {}^{\omega\setminus n}X$ where N_n is nowhere dense in ${}^nX$ and $\mathcal{N}_n$ is good nowhere dense for each $n \in \omega$?*

If the answer to the first question is yes then so is the answer to the last, but unfortunately I do not know more than that for general nowhere dense and meagre subsets of ${}^\omega X$. However, I can obtain some relations between good meagre sets and meagre sets that are closed under the following relations.

Definition 2.9 *For any set X and any $f, g \in {}^\omega X$ define*

$$f =^\bullet_X g \Leftrightarrow \forall^\infty n \in \omega(f(n) = g(n)).$$

If X is a metric space then define

$$f =^*_X g \Leftrightarrow d(f(n), g(n)) \to 0.$$

We drop the X subscript when it is clear from the context.

First we find a nice collection of sets which are cofinal in the ideal of good nowhere dense sets.

Lemma 2.10 *For any $g \in {}^\omega X$, interval partition (I_n) of ω and $m \in \omega$,*

$$\{f \in {}^\omega X : \exists n \in \omega\setminus m(f \restriction I_n = g \restriction I_n)\}$$

is dense in ${}^\omega X$.

Proposition 2.11 *For any sequence of non-empty open subsets (O_n) of X, any interval partition (I_n) of ω and $m \in \omega$,*

$$\mathcal{N}_{(O_n),(I_n),m} = \{f \in {}^\omega X : \forall n \in \omega\setminus m \exists k \in I_n(f(k) \notin O_k)\}$$

is closed good nowhere dense. Furthermore, any good nowhere dense $\mathcal{N} \subseteq {}^\omega X$ is contained in such a set, with $m = 0$ in fact.

Proof: We have $\mathcal{N}_{(O_n),(I_n),m} = \bigcap_{n\in\omega\setminus m}\bigcup_{k\in I_n}\{f \in {}^\omega X : f(k) \in X\setminus O_k\}$, which is closed. Good nowhere density is immediate from 2.2(iii). On the other hand, given a good nowhere dense $\mathcal{N} \subseteq {}^\omega X$ recursively choose an interval partition (I_n) and open subsets (O_n) of X such that $\min(I_n) = \max(I_{n-1}) + 1 (= 0 \text{ if } n = 0)$ and $\{f \in {}^\omega X : \forall k \in I_n(f(k) \in O_k)\}$ is disjoint from $\mathcal{N}$. Then $\mathcal{N} \subseteq \mathcal{N}_{(O_n),(I_n),0}$. □

Likewise, we have nice collection of sets which are cofinal in the σ-ideal of good meagre sets.

Proposition 2.12 *For any sequence of non-empty open subsets (O_n) of X and any interval partition (I_n) of ω,*

$$\mathcal{M}_{(O_n),(I_n)} = \bigcup_m \mathcal{N}_{(O_n),(I_n),m} = \{f \in {}^\omega X : \forall^\infty n \in \omega \exists k \in I_n(f(k) \notin O_k)\}$$

is $=^\bullet$-closed good meagre F_σ. Furthermore, any good meagre $\mathcal{M} \subseteq {}^\omega X$ is contained in such a set.

Proof: The first part is immediate. On the other hand, given a sequence $(\mathcal{N}_n)$ of nowhere dense subsets of X, recursively choose an interval partition (I_n) and open subsets (O_n) of X such that $\min(I_n) = \max(I_{n-1}) + 1 (= 0 \text{ if } n = 0)$ and $\{f \in {}^\omega X : \forall k \in I_n(f(k) \in O_k)\}$ is disjoint from $\bigcup_{k\in n}\mathcal{N}_k$, which is possible because a finite union of good nowhere dense sets is again good nowhere dense. Then the meagre set $\mathcal{M} = \bigcup \mathcal{N}_n$ is a subset of $\mathcal{M}_{(O_n),(I_n)}$. Alternatively, one may first find (O^n_m), and (I^m_n) such that $\mathcal{N}_m \subseteq \mathcal{N}_{(O^m_n),(I^m_n),0}$, for all $m \in \omega$, and then recursively define (I_n) and $(n_{j,k})_{j\in\omega,k\in j+1}$, such that for each $j \in \omega$, $(I^k_{n_{j,k}})_{k\in j+1}$ are disjoint subsets of I_j (and $\min(I_j) = \max(I_{j-1}) + 1 (= 0 \text{ if } j = 0)$) and for all $j \in \omega$, $k \in j + 1$ and $l \in I^k_{n_{j,k}}$, $O_l = O^k_{n_{j,k}}$. Then letting $O_n = X$ for those $n \in \omega\setminus\bigcup_{j\in\omega,k\in j+1} I^k_{n_{j,k}}$ (i.e. those $n \in \omega$ for which O_n has not yet been defined) we see that $\mathcal{M} = \bigcup \mathcal{N}_n \subseteq \mathcal{M}_{(O_n),(I_n)}$. □

The more interesting result is that we can obtain a converse of this for suitable X.

Theorem 2.13 *If X is compact and $\mathcal{F} \subseteq {}^\omega X$ is $=^\bullet$-closed F_σ then, for any $f \in {}^\omega X\setminus\mathcal{F}$ there exists an interval partition (I_n) of ω together with (O_n) such that O_n is an open subset of X containing $f(n)$, for each $n \in \omega$, and $\mathcal{F} \subseteq \mathcal{M}_{(O_n),(I_n)}$*

Proof: Let $\mathcal{F} = \bigcup \mathcal{N}_n$ for closed subsets $(\mathcal{N}_n)$ of X. Define (O_n) and (I_n) recursively as follows. Let $\min(I_n) = \max(I_{n-1})+1 (= 0 \text{ if } n = 0)$ and, for all $t \in {}^{\min(I_n)}X$ set $f_t(k) = t(k)$ for $k \in \min(I_n)$ and $f_t(k) = f(k)$ for $k \in \omega\setminus\min(I_n)$. As $\mathcal{F}$ is $=^\bullet$-closed, $f_t \notin \mathcal{F}$ and we can find $n_t \in \omega\setminus\min(I_n)$ and open $O^t_k \ni f_t(k)$, for all $k \in n_t$, such that $O^t_0 \times \ldots \times O^t_{n_t-1} \times {}^{\omega\setminus n_t}X$ is disjoint from the closed set $\bigcup_{k\in n}\mathcal{N}_k (\subseteq \mathcal{F})$. As ${}^{\min(I_n)}X$ is compact, there exists $m_n \in \omega$ and $t_0, \ldots, t_{m_n-1}$ with ${}^{\min(I_n)}X \subseteq \bigcup_{j\in m_n}\prod_{k\in\min(I_n)} O^{t_j}_k$. Let $\max(I_n) = \max\{n_{t_k} : k \in m_n\}$ and, for all $k \in I_{n+1}$, let $O_k = \bigcap_{j\in m_n} O^{t_j}_k \ni f_{t_j}(k) = f(k)$ (for $k \geq n_{t_j}$ set $O^{t_j}_k = X$ in this intersection). This completes the recursion and now note that if $n \in \omega$ then $g \restriction \min(I_n) \in \prod_{k\in\min(I_n)} O^{t_j}_k$, for some $j \in m_n$, so if $g(k) \in O_k \subseteq O^{t_j}_k$ for all $k \in I_n$ then $g \notin \bigcup_{k\in n}\mathcal{N}_k$. Thus if there are infinitely many $n \in \omega$ such that $g(k) \in O_k \subseteq O^{t_j}_k$ for all $k \in I_n$ then $g \notin \mathcal{F}$, i.e. $\mathcal{F} \subseteq \mathcal{M}_{(O_n),(I_n)}$. □

Note that this theorem holds, with exactly the same proof, if we replace ${}^\omega X$ with a closed subset $\mathcal{X}$ of ${}^\omega X$.

Corollary 2.14 *If X is compact and $\mathcal{F} \subseteq {}^\omega X$ is $=^\bullet$-closed meagre F_σ then $\mathcal{F}$ is good meagre.*

Proof: As X is compact, ${}^\omega X$ is also compact, by Tychonoff's Theorem, and hence a Baire space, by the Baire Category Theorem. So there exists $f \in {}^\omega X\setminus\mathcal{F}$ and the result now follows from 2.13 and (the first part of) 2.12. □

Corollary 2.15 *If X is compact then $\mathcal{F} \subseteq {}^\omega X$ is good meagre if and only if $\mathcal{F}$ is contained in a $=^\bullet$-closed meagre F_σ.*

Proof: Follows from 2.14 and (the second part of) 2.12. □

Question 2.16 *For compact X, is every $=^\bullet$-closed meagre $\mathcal{M} \subseteq {}^\omega X$ contained in a $=^\bullet$-closed meagre F_σ subset? Can every $=^\bullet$-closed meagre $\mathcal{M} \subseteq {}^\omega X$ be written as $\mathcal{M} = \bigcup \mathcal{N}_n$ where, for all $n \in \omega$, $\mathcal{N}_n$ is nowhere dense and $\mathrm{cl}_{=^\bullet}(\overline{\mathcal{N}_n})$ is meagre?*

Note that a positive answer to the second question would imply a positive answer to the first. For then we could find nowhere dense $(\mathcal{N}_t)_{t\in\omega^{<\omega}}$ such that $\mathrm{cl}_{=^\bullet}(\overline{\mathcal{N}_{n_0,\ldots,n_k}}) = \bigcup_m \mathcal{N}_{n_0,\ldots,n_k,m}$ for all $k, n_0, \ldots, n_k \in \omega$ and $\mathcal{F} = \bigcup \mathcal{N}_n \subseteq \bigcup_{t\in\omega^{<\omega}} \overline{\mathcal{N}_t}$, where this last set is $=^\bullet$-closed meagre F_σ. Also note the answer to the second question would be negative if the last 'meagre' were to be replaced by 'nowhere dense', indeed, every non-empty $=^\bullet$-closed set is dense and hence not nowhere dense.

However, the answer to the first question above is positive if X is a metric space and the first (and even the second) $=^\bullet$ is replaced by $=^*$, as shown below in 2.21.

For the rest of this section, X is not just a topological space but a metric space.

Definition 2.17 *A subset $\mathcal{F} \subseteq {}^\omega X$ is* very good *nowhere dense if there exists $\epsilon > 0$ such that, for all $n \in \omega$, there exists $m \geq n$ and $t \in {}^{m\setminus n}X$ such that*

$$\{f \in {}^\omega X : \forall k \in m\setminus n |f(k) - t(k)| < \epsilon\}$$

is disjoint from $\mathcal{F}$. A countable union of very good nowhere dense sets is very good *meagre.*

Proposition 2.18 *The very good nowhere dense sets are closed under subsets, finite unions and topological closures.*

Like before, we have a nice collection of sets which are cofinal in the ideal of very good meagre sets.

Proposition 2.19 *For any $\epsilon > 0$, $g \in {}^\omega X$, interval partition (I_n) of ω and $m \in \omega$,*

$$\mathcal{N}_{\epsilon,g,(I_n),m} = \{f \in {}^\omega X : \forall n \in \omega\setminus m \exists k \in I_n |f(k) - g(k)| \geq \epsilon\}$$

is closed very good nowhere dense. Furthermore, any very good nowhere dense $\mathcal{N} \subseteq {}^\omega X$ is contained in such a set, with $m = 0$ in fact.

Proof: Proved analogously to 2.11. □

Likewise, we have nice collection of sets which are cofinal in the σ-ideal of very good meagre sets.

Proposition 2.20 *For any $g \in {}^\omega X$ and any interval partition (I_n) of ω,*

$$\mathcal{M}_{g,(I_n)} = \bigcup_m \mathcal{N}_{1/m,g,(I_n),m} = \{f \in {}^\omega X : \liminf_n \max_{k\in I_n} |f(k) - g(k)| > 0\}$$

is $=^*$-closed very good meagre F_σ. Furthermore, any very good meagre $\mathcal{M} \subseteq {}^\omega X$ is contained in such a set.

Proof: Proved analogously to 2.12. □

We again have a converse of this for suitable X.

Theorem 2.21 *If X is compact and $\mathcal{F} \subseteq {}^\omega X$ is $=^*$-closed meagre then there exists $g \in {}^\omega X$ and an interval partition (I_n) of ω such that $\mathcal{F} \subseteq \mathcal{M}_{g,(I_n)}$.*

Proof: Let $\mathcal{F} = \bigcup \mathcal{N}_n$ for nowhere dense $(\mathcal{N}_n)$. Recursively define a sequence $(n_m) \subseteq \omega$, an interval partition (I_m) of ω, open subsets $(O_{m,k} \subseteq {}^{I_m}X)_{m\in\omega,k\in n_m}$ and, for each $m \in \omega$, open subsets $(B_t \subseteq {}^{\min(I_m)}X)_{t\in n_0\times\ldots\times n_{m-1}}$ as follows. First set $B_0 = 0$, take arbitrary I_0 and open $O_{0,0} \subseteq {}^{I_0}X$ of diameter < 1 in each coordinate. Once n_{m-1}, if $m > 0$, $(B_t)_{t\in n_0\times\ldots\times n_{m-1}}$, I_m and $O_{m,0}$ have been defined take n_m and open $(O_{m,k})_{k\in n_m\setminus\{0\}}$ of ${}^{I_m}X$, each of diameter $< 1/(m+1)$ in each coordinate, such that $\bigcup_{k\in n_m} O_{m,k} = {}^{I_m}X$. Let $(t_k)_{k\in n_0 n_1\ldots n_m}$ enumerate $t \in n_0 \times \ldots \times n_m$. Recursively define $(B_t \subseteq {}^{\min(I_{m+1})}X)_{t\in n_0\times\ldots\times n_m}$ and $(Y_t \subseteq {}^{\omega\setminus\min(I_{m+1})}X)_{t\in n_0\times\ldots\times n_m}$ by letting $B_{t_j} \times Y_{t_j}$ be a non-empty open subset disjoint from $\bigcup_{k<m+1} \mathcal{N}_k$ such that

$$\overline{B_{t_j} \times Y_{t_j}} \subseteq B_{t_j\restriction m} \times O_{m,t_j(m)} \times Y_{t_{j-1}} \quad \text{(with } Y_{t_{-1}} = {}^{\omega\setminus\min(I_{m+1})}X \text{ by convention).}$$

Then take $\max(I_{m+1}) + 1 = \min(I_{m+2})$ large enough that we can find $O_{m+1,0} \subseteq {}^{I_{m+1}}X$, of diameter less than $1/(m+2)$ in each coordinate, such that $O_{m+1,0} \times {}^{\omega\setminus\min(I_{m+2})}X \subseteq Y_{t_{n_0 n_1\ldots n_m}}$. This completes the recursion.

Let $g \in {}^\omega X$ be such that $g \restriction I_m \in O_{m,0}$ for all $m \in \omega$. Take any $f \in {}^\omega X\setminus\mathcal{M}_{g,(I_n)}$ and let $A \in [\omega]^\omega$ be such that $\lim_{n\to\infty} \max_{k\in I_{a_n}} |f(k) - g(k)| = 0$, where (a_n) is the increasing enumeration of A. For each $m \in A$, let $h(m) = 0$ and, for each $m \in \omega\setminus A$, let $h(m) \in n_m$ be such that $f \restriction I_m \in O_{m,h(m)}$. Choose $f' \in \bigcap_k B_{h\restriction k} \times {}^{\omega\setminus\min(I_k)}X = \bigcap_k \overline{B_{h\restriction k}} \times {}^{\omega\setminus\min(I_k)}X$, noting that this last expression is non-empty because it is an intersection of non-empty decreasing compact sets. For each $m \in A$ we have

$$f' \in B_{h\restriction m} \times O_{m,0} \times {}^{\omega\setminus\min(I_{m+1})}X \subseteq B_{h\restriction m} \times Y_{h\restriction m} \subseteq {}^\omega X\setminus \bigcup_{k<m} \mathcal{N}_k$$

which, as A is infinite, implies $f' \notin \bigcup \mathcal{N}_k$. Furthermore, by our choice of h, $|f(n) - f'(n)| \to 0$ and hence $f \notin \bigcup \mathcal{N}_k$, as $\mathcal{F}$ is $=^*$-closed. As f was arbitrary we have $\mathcal{F} \subseteq \mathcal{M}_{g,(I_n)}$. □

Corollary 2.22 *If X is compact and $\mathcal{F}$ is $=^*$-closed meagre then $\mathcal{F}$ is very good meagre.*

Corollary 2.23 *If X is compact then $\mathcal{F}$ is very good meagre if and only if $\mathcal{F}$ is contained in a $=^*$-closed meagre set.*

3 Operators on a Hilbert Space

In this section we will be dealing with the collection of bounded linear operators $\mathcal{B}(H)$ on a separable infinite dimensional Hilbert space H (i.e. a Hilbert space H isometrically isomorphic to l^2) with the weak operator topology, i.e. the weakest topology in which, for each $x, y \in H$, the functions $T \mapsto \langle Tx, y\rangle$ are continuous. If we are dealing with a uniformly bounded subcollection $\mathcal{B}$ of such operators and we fix a basis (e_n) of H then sets of the form $\{T \in \mathcal{B} : ||P_n(S-T)P_n|| < \epsilon\}$, for $\epsilon > 0$, $n \in \omega$ and $S \in \mathcal{B}$, form a basis for this topology. We will also have occasion to mention the strong operator topology, i.e. the weakest topology in which, for each $x \in H$, the functions $T \mapsto Tx$ are continuous. Again, if we are dealing with a uniformly bounded subcollection $\mathcal{B}$ of such operators and we fix a basis (e_n) of H then sets of the form $\{T \in \mathcal{B} : ||(S-T)P_n|| < \epsilon\}$, for $\epsilon > 0$, $n \in \omega$ and $S \in \mathcal{B}$, form a basis for this topology. It follows that both of these topologies are second countable when restricted to a uniformly bounded subcollection and hence, being (completely) regular (as, indeed, are all Hausdorff topological vector spaces) also metrizable, by Urysohn's metrization theorem, which, in particular, implies that all closed subsets are G_δ. Also note that both the weak and strong operator topologies are coarser than the norm topology.

Definition 3.1 *Let $\mathcal{B}(H)^{+/-}$, $\mathcal{B}(H)^+$ and $\mathcal{P}(\mathcal{B}(H))$ be the collection of self-adjoint operators, non-negative operators and projections on H respectively. If $\mathcal{B} \subseteq \mathcal{B}(H)$ and $r \geq 0$ then let $\mathcal{B}_r$ be* *operators in $\mathcal{B}$ of norm at most r.*

$$\begin{aligned}
\mathcal{B}(H)^{+/-} &= \{T \in \mathcal{B}(H) : T = T^*\}. \\
\mathcal{B}(H)^{+} &= \{T \in \mathcal{B}(H) : \forall x \in H(\langle Tx, x\rangle \geq 0)\} = \{T^*T : T \in \mathcal{B}(H)\}. \\
\mathcal{P}(\mathcal{B}(H)) &= \{T \in \mathcal{B}(H) : T^2 = T \wedge T = T^*\}. \\
\mathcal{B}_r &= \{T \in \mathcal{B} : ||T|| \leq r\}.
\end{aligned}$$

For what follows we need a couple of technical lemmas about projections.

Definition 3.2 *Let P_L and P_R be the projections $H \oplus H \to H$ onto the first and second coordinate respectively and let I_L and I_R be the injections $H \to H \oplus H$ into the first and second coordinate respectively. Let $P'_L = I_L P_L$ and $P'_R = I_R P_R$*

Proposition 3.3 ([2] 3.1.8.) *For any $T \in \mathcal{B}(H)^+_1$ there exists a projection P on $H \oplus H$ such that $T = P_L P I_L$. (Equivalently, for any $T \in \mathcal{B}(H \oplus H)^+_1$ such that $P'_L T P'_L = T$ there exists a projection P on $H \oplus H$ such that $T = P'_L P P'_L$.)*

Proof: As T is non-negative and of norm at most 1, $T - T^2$ is also non-negative and hence has a non-negative square root $\sqrt{T - T^2}$. Thus we may let P be the operator whose matrix representation is

$$P = \begin{bmatrix} T & \sqrt{T - T^2} \\ \sqrt{T - T^2} & 1 - T \end{bmatrix},$$

i.e. $P = (TP_L + \sqrt{T - T^2}P_R) \oplus (\sqrt{T - T^2}P_L + (1 - T)P_R)$, which is immediately verified to satisfy $P^2 = P$ and $P^* = P$. □

For the next lemma, note that if $(T_n), (S_n) \subseteq \mathcal{B}(H)$ are such that (T_n) is uniformly bounded, $T_n \xrightarrow{s} T$ and $S_n \xrightarrow{s} S$ then $T_n S_n \xrightarrow{s} TS$. Also note that, for any $S \in \mathcal{B}(H)^+$ and $x \in H$, $||\sqrt{S}x||^2 = \langle Sx, x\rangle \leq ||Sx||||x||$ so if $S_n \xrightarrow{s} 0$ then $\sqrt{S_n} \xrightarrow{s} 0$.

Lemma 3.4 *For any $Q \in \mathcal{P}(\mathcal{B}(H))$, $n \in \omega$ and $\epsilon > 0$, there exists $m \in \omega\backslash n$ and $P \in \mathcal{P}(\mathcal{B}(H))$ such that $P_{2m} P P_{2m} = P$, $P_m P P_m = P_m Q P_m$ and $||(P - Q)P_n|| < \epsilon$.*

Proof: We have $P_m Q P_m \xrightarrow{s} Q$ which, as $Q^2 = Q$, gives $\sqrt{P_m Q P_m - (P_m Q P_m)^2} \xrightarrow{s} 0$. Let $m \geq n$ be large enough that $||\sqrt{P_m Q P_m - (P_m Q P_m)^2} P_n|| < \epsilon/2$ and $||(1 - P_m)QP_n|| < \epsilon/2$ and let P be the projection such that $P_{2m} P P_{2m} = P$ and $P_m P P_m = P_m Q P_m = T$ defined in 3.3. So $PP_m = T + S_m\sqrt{T - T^2}$, where S_m is the operator that shifts any $x \in H$ m places to the right (i.e. such that $S_m e_k = e_{k+m}$ for all $k \in \omega$) and then

$$PP_n = PP_m P_n = TP_n + S_m\sqrt{T - T^2}P_n = P_m Q P_n + S_m\sqrt{P_m Q P_m - (P_m Q P_m)^2}P_n$$

and hence $||(P - Q)P_n|| \leq ||(P_m - 1)QP_n|| + ||\sqrt{P_m Q P_m - (P_m Q P_m)^2}P_n|| < \epsilon$. □

Proposition 3.5 *A bounded linear operator T on H is a projection if and only if $T^*T = T$.*

Proof: If $T^*T = T$ then $T^* = (T^*T)^* = T^*T^{**} = T^*T = T$, i.e. T is self-adjoint, and hence also $T^2 = T^*T = T$, i.e. T is idempotent. On the other hand if T is a projection then $T^*T = T^2 = T$. □

In fact, we can do even better and not even assume T is bounded.

Proposition 3.6 *A linear operator T on H is a projection if and only if $\langle Tx, Ty\rangle = \langle Tx, y\rangle$ for all $x, y \in H$.*

Proof: If T is a projection then $\langle Tx, Ty\rangle = \langle T^*Tx, y\rangle = \langle Tx, y\rangle$, for all $x, y \in H$. On the other hand, if $\langle Tx, Ty\rangle = \langle Tx, y\rangle$ for all $x, y \in H$ then

$$\langle Tx, y\rangle = \langle Tx, Ty\rangle = \overline{\langle Ty, Tx\rangle} = \overline{\langle Ty, x\rangle} = \langle x, Ty\rangle,$$

for all $x, y \in H$ and hence $\langle Tx, y\rangle = \langle Tx, Ty\rangle = \langle T^2x, y\rangle$ for all $x, y \in H$. So T is self-adjoint (which, being defined everywhere, implies it is bounded, by the Hellinger-Toeplitz theorem) and idempotent. □

Proposition 3.7 *If scalars are complex (i.e. if $\mathbb{F} = \mathbb{C}$) then a linear operator T on H is a projection if and only if $\langle Tx, T^{\perp}x\rangle = 0$ for all $x \in H$.*

Proof: If T is a projection then $\langle Tx, T^{\perp}x\rangle = \langle T^{\perp *}Tx, x\rangle = 0$, for all $x \in H$, as we have $T^{\perp *}T = T - T^*T = 0$. On the other hand if $\langle Tx, Tx\rangle = \langle Tx, x\rangle$ for all $x \in H$ then, for all $x, y \in H$,

$$\begin{aligned}
\langle T(x+y), T(x+y)\rangle &= \langle T(x+y), x+y\rangle \\
\langle Tx, Tx\rangle + \langle Tx, Ty\rangle + \langle Ty, Tx\rangle + \langle Ty, Ty\rangle &= \langle Tx, x\rangle + \langle Tx, y\rangle + \langle Ty, x\rangle + \langle Ty, y\rangle \\
\langle Tx, Ty\rangle + \langle Ty, Tx\rangle &= \langle Tx, y\rangle + \langle Ty, x\rangle.
\end{aligned}$$

But also

$$\begin{aligned}
\langle T(x-iy), T(x-iy)\rangle &= \langle T(x-iy), x-iy\rangle \\
\langle Tx, Tx\rangle + i\langle Tx, Ty\rangle - i\langle Ty, Tx\rangle + \langle Ty, Ty\rangle &= \langle Tx, x\rangle + i\langle Tx, y\rangle - i\langle Ty, x\rangle + \langle Ty, y\rangle \\
\langle Tx, Ty\rangle - \langle Ty, Tx\rangle &= \langle Tx, y\rangle - \langle Ty, x\rangle.
\end{aligned}$$

Adding these two equations together and dividing by 2 gives $\langle Tx, Ty\rangle = \langle Tx, y\rangle$ for all $x, y \in H$. □

Proposition 3.8 *If scalars are real (i.e. if $\mathbb{F} = \mathbb{R}$) then T is a projection if and only if T is self-adjoint and $\langle Tx, T^{\perp}x\rangle = 0$ for all $x \in H$.*

Proof: For all $x, y \in H$ we have $\langle Tx, Tx\rangle = \langle Tx, x\rangle$ and $\langle Ty, Ty\rangle = \langle Ty, y\rangle$ and hence

$$\begin{aligned}
\langle T(x+y), T(x+y)\rangle &= \langle T(x+y), x+y\rangle \\
\langle Tx, Tx\rangle + \langle Tx, Ty\rangle + \langle Ty, Tx\rangle + \langle Ty, Ty\rangle &= \langle Tx, x\rangle + \langle Tx, y\rangle + \langle Ty, x\rangle + \langle Ty, y\rangle \\
\langle Tx, Ty\rangle + \langle Ty, Tx\rangle &= \langle Tx, y\rangle + \langle Ty, x\rangle. \\
\langle Tx, Ty\rangle + \langle Tx, Ty\rangle &= \langle Tx, y\rangle + \langle x, Ty\rangle. \\
\langle Tx, Ty\rangle + \langle Tx, Ty\rangle &= \langle Tx, y\rangle + \langle Tx, y\rangle. \\
\langle Tx, Ty\rangle &= \langle Tx, y\rangle. \ \square
\end{aligned}$$

Proposition 3.9 *For any $r \in \mathbb{R}$, $B_r = \{x \in H : ||x|| \leq r\}$ is closed w.r.t. the weak topology.*

Proof: If $||y|| > r$ then the weakly open set $\{x \in H : \langle x, y\rangle > r||y||\}$ contains y and is disjoint from B_r (because if $||x|| \leq r$ then $\langle x, y\rangle \leq ||x||||y|| \leq r||y||$). As y was arbitrary, B_r is closed. □

Proposition 3.10 *In the weak operator topology, $\mathcal{P}(\mathcal{B}(H))$ is a dense G_δ in $\mathcal{B}(H)^+_1$.*

Proof: Density follows from 3.3. Next note that, for any $x \in H$,

$$||Tx - \frac{1}{2}x||^2 = \langle Tx, Tx\rangle - \frac{1}{2}\langle Tx, x\rangle - \frac{1}{2}\langle x, Tx\rangle + \frac{1}{4}||x||^2$$

so

$$\langle Tx, T^{\perp}x\rangle = 0 \Leftrightarrow \langle x, Tx\rangle = \langle Tx, Tx\rangle = \langle Tx, x\rangle \Leftrightarrow ||Tx - \frac{1}{2}x|| = \frac{1}{2}||x||.$$

Hence, defining $f_x(T) = Tx - \frac{1}{2}x$, which is a continuous map from $\mathcal{B}(H)_1$ with the weak operator topology to H with the weak topology, we have

$$\{T \in \mathcal{B}(H)_1 : \langle Tx, T^{\perp}x\rangle = 0\} = f_x^{-1}(B_{\frac{1}{2}||x||}) \cap \bigcap_n (\mathcal{B}(H)_1 \backslash f_x^{-1}(B_{\frac{1}{2}||x||-\frac{1}{n}}))$$

which is the intersection of a closed and hence G_δ set with another G_δ set and thus itself a G_δ set. If D is a countable dense subset of H then

$$\{T \in \mathcal{B}(H)_1 : \forall x \in H \langle Tx, T^{\perp}x\rangle = 0\} = \bigcap_{x\in D} \{T \in \mathcal{B}(H)_1 : \langle Tx, T^{\perp}x\rangle = 0\}$$

is also a G_δ set. If we are dealing with complex scalars then this is already precisely the collection of projections, while if we are dealing with real scalars then we must intersect this with the collection of self-adjoint operators $\{T \in \mathcal{B}(H)_1 : \forall x, y \in H \langle Tx, y\rangle = \langle x, Ty\rangle\}$, which is closed and hence G_δ, to get the collection of projections, which again shows it is a G_δ set. □

Proposition 3.11 *If Y is a comeagre subset of a Baire space X and $Z \subseteq X\backslash Y$ then, for any $A \subseteq Y$, A is comeagre in Y if and only if $A \cup Z$ is comeagre in X.*

Proof: Let $Y \supseteq \bigcap O_n^Y$ for open dense (O_n^Y) in X. If $A \cup (X\backslash Y) \supseteq A \cup Z \supseteq \bigcap O_n$, for open dense (O_n) in X, then $O_n \cap \bigcap_k O_k^Y$ is comeagre and hence dense in X and hence in Y, for each $n \in \omega$. Therefore $A \supseteq \bigcap_n (O_n \cap \bigcap_k O_k^Y)$ is comeagre in Y.

On the other hand, if A is comeagre in Y then $A \supseteq \bigcap O_n$ for open dense (O_n) in Y. But then there are open (O_n') in X such that, for all $n \in \omega$, $O_n = O_n' \cap Y$. As O_n is dense in Y and Y is dense in X, $O_n' \supseteq O_n$ is also dense in X, for all $n \in \omega$. Thus $A \cup Z \supseteq A \supseteq \bigcap O_n' \cap O_n^Y$ is comeagre in X. □

Corollary 3.12 *If Y is a comeagre subset of a Baire space X and $Z \subseteq X\backslash Y$ then, for any $A \subseteq Y$, A is meagre in Y if and only if $A \cup Z$ is meagre in X.*

Proof:

A meagre in $Y \Leftrightarrow Y\backslash A$ comeagre in $Y \Leftrightarrow (Y\backslash A)\cup(X\backslash Y)\backslash Z$ comeagre in $X \Leftrightarrow A\cup Z$ meagre in X.□

Corollary 3.13 *If $\mathcal{Z} \subseteq \mathcal{B}(H)_1^+ \backslash \mathcal{P}(\mathcal{B}(H))$ then, for any $\mathcal{F} \subseteq \mathcal{P}(\mathcal{B}(H))$, $\mathcal{F}$ is meagre in $\mathcal{P}(\mathcal{B}(H))$ w.r.t. the weak operator topology if and only if $\mathcal{F} \cup \mathcal{Z}$ is meagre in $\mathcal{B}(H)_1^+$ w.r.t. the weak operator topology.*

From now until 3.27 let $\mathcal{B}$ consistently be $\mathcal{B}(H)_1$, $\mathcal{B}(H)_1^{+/-}$, $\mathcal{B}(H)_1^+$ or $\mathcal{P}(\mathcal{B}(H))$.

Proposition 3.14 *For any $\mathcal{F} \subseteq \mathcal{B}$, the following are equivalent*

(i) For all $n \in \omega$, $P_{\omega\backslash n}\mathcal{F}P_{\omega\backslash n}$ is not dense in $P_{\omega\backslash n}\mathcal{B}P_{\omega\backslash n}$ w.r.t. to the weak operator topology.

(ii) For all $n \in \omega$, $P_{\omega\backslash n}\mathcal{F}P_{\omega\backslash n}$ is nowhere dense in $P_{\omega\backslash n}\mathcal{B}P_{\omega\backslash n}$ w.r.t. to the weak operator topology.

(iii) For all $n \in \omega$ there exists $m \geq n$, $\epsilon > 0$ and $S \in \mathcal{B}$ such that $P_{m\backslash n}SP_{m\backslash n} = S$ and $\{T \in \mathcal{B} : ||S - P_{m\backslash n}TP_{m\backslash n}|| < \epsilon\}$ is disjoint from $\mathcal{F}$

Proof:

(iii)⇒(ii) Assume (iii), fix $n \in \omega$ and take any basic open set $\mathcal{O}$ of $P_{\omega\setminus n}\mathcal{B}P_{\omega\setminus n}$ of the form

$$\mathcal{O} = \{T \in P_{\omega\setminus n}\mathcal{B}P_{\omega\setminus n} : ||P_m(S-T)P_m|| < \epsilon\}$$

for some $S \in P_{\omega\setminus n}\mathcal{B}P_{\omega\setminus n}$, $\epsilon > 0$ and $m \geq n$. Take $\delta > 0$, $l \geq m$ and $R \in \mathcal{B}$ such that $P_{l\setminus m}RP_{l\setminus m} = R$ and

$$\mathcal{L} = \{T \in \mathcal{B} : ||R - P_{l\setminus m}TP_{l\setminus m}|| < \delta\}$$

is disjoint from $\mathcal{F}$. If $T = P_{\omega\setminus n}LP_{\omega\setminus n} = P_{\omega\setminus n}FP_{\omega\setminus n}$ for some $L \in \mathcal{L}$ and $F \in \mathcal{F}$ then $||R - P_{l\setminus m}FP_{l\setminus m}|| = ||R - P_{l\setminus m}TP_{l\setminus m}|| = ||R - P_{l\setminus m}LP_{l\setminus m}|| < \delta$, i.e.

$$\begin{aligned} P_{\omega\setminus n}\mathcal{L}P_{\omega\setminus n} \cap P_{\omega\setminus n}\mathcal{F}P_{\omega\setminus n} &= P_{\omega\setminus n}\{F \in \mathcal{F} : ||R - P_{l\setminus m}FP_{l\setminus m}|| < \delta\}P_{\omega\setminus n} \\ &= P_{\omega\setminus n}(\mathcal{L} \cap \mathcal{F})P_{\omega\setminus n}, \end{aligned}$$

so $P_{\omega\setminus n}\mathcal{L}P_{\omega\setminus n}$ is disjoint from $P_{\omega\setminus n}\mathcal{F}P_{\omega\setminus n}$ too. Let

$$\begin{aligned} \mathcal{K} &= \{T \in P_{\omega\setminus n}\mathcal{B}P_{\omega\setminus n} : ||R + P_mSP_m - P_lTP_l|| < \min(\epsilon,\delta)\} \\ &\subseteq \{T \in P_{\omega\setminus n}\mathcal{B}P_{\omega\setminus n} : ||R - P_{l\setminus m}TP_{l\setminus m}|| < \delta\} \cap \{T \in P_{\omega\setminus n}\mathcal{B}P_{\omega\setminus n} : ||P_m(S-T)P_m|| < \epsilon\} \\ &= P_{\omega\setminus n}\{T \in \mathcal{B} : ||R - P_{l\setminus m}TP_{l\setminus m}|| < \delta\}P_{\omega\setminus n} \cap \mathcal{O} \\ &= P_{\omega\setminus n}\mathcal{L}P_{\omega\setminus n} \cap \mathcal{O}. \end{aligned}$$

Note that $\mathcal{K}$ is non-empty as it contains $R + P_mSP_m$, if $\mathcal{B}$ is not $\mathcal{P}(\mathcal{B}(H))$, while it contains $P_{\omega\setminus n}PP_{\omega\setminus n}$ for any $P \in \mathcal{P}(\mathcal{B}(H))$ such that $P_lPP_l = R + P_mSP_m$, which exists by 3.3. Thus $\mathcal{K}$ is a non-empty open subset of $\mathcal{O}$ disjoint from $P_{\omega\setminus n}\mathcal{F}P_{\omega\setminus n}$ which, as n and $\mathcal{O}$ were arbitrary, shows that (ii) holds.

(ii)⇒(i) Immediate.

(i)⇒(iii) Assume that (i) holds so, for any $n \in \omega$, there exists a basic open set $\mathcal{O}$ of $P_{\omega\setminus n}\mathcal{B}P_{\omega\setminus n}$ disjoint from $P_{\omega\setminus n}\mathcal{F}P_{\omega\setminus n}$ of the form

$$\mathcal{O} = \{T \in P_{\omega\setminus n}\mathcal{B}P_{\omega\setminus n} : ||P_m(R-T)P_m|| < \epsilon\}$$

for some $R \in P_{\omega\setminus n}\mathcal{B}P_{\omega\setminus n}$, $\epsilon > 0$ and $m \geq n$. Setting $S = P_mRP_m$ gives $P_{m\setminus n}SP_{m\setminus n} = S$ and

$$\mathcal{O} = \{T \in P_{\omega\setminus n}\mathcal{B}P_{\omega\setminus n} : ||S - P_mTP_m|| < \epsilon\} = P_{\omega\setminus n}\{T \in \mathcal{B} : ||S - P_{m\setminus n}TP_{m\setminus n}|| < \epsilon\}P_{\omega\setminus n}$$

which implies $\{T \in \mathcal{B} : ||S - P_{m\setminus n}TP_{m\setminus n}|| < \epsilon\}$ is disjoint from $\mathcal{F}$. □

Definition 3.15 ([2] 3.1.4) *Any $\mathcal{F} \subseteq \mathcal{B}$ satisfying any of the equivalent conditions in the proposition above is said to be* good nowhere dense *(w.r.t. the basis (e_n)). A countable union of good nowhere dense sets is said to be* good meagre *(w.r.t. the basis (e_n)).*

Proposition 3.16 *Good nowhere dense sets are closed under subsets, finite unions and topological closures.*

Proof: Follows from characterization (ii) above and the corresponding closures for nowhere dense sets. □

Lemma 3.17 *For any $S \in \mathcal{B}$, $n \in \omega$ and $\epsilon > 0$, there exists $m \in \omega\setminus n$ and $T \in \mathcal{B}$ such that $P_mTP_m = T$ and $||(S-T)P_n|| < \epsilon$.*

Proof: If $\mathcal{B}$ is $\mathcal{P}(\mathcal{B}(H))$ then this follows from 3.4. Otherwise simply let $m \geq n$ be large enough that $||(1 - P_m)SP_n|| < \epsilon$, let $T = P_mSP_m$ and note that then $P_mTP_m = T$, $||(S - T)P_n|| < \epsilon$ and $||T|| \leq ||S||$ and if S is self-adjoint or non-negative then the same applies to T. □

Proposition 3.18 ([2] 3.1.5.) *Take a partition (I_n) of ω and a sequence $(T_n) \subseteq \mathcal{B}$ such that $P_{I_n}T_nP_{I_n} = T_n$, for all $n \in \omega$. Then, for all $m \in \omega$,*

$$\mathcal{D}_m = \{T \in \mathcal{B} : \exists n \geq m(T_n = P_{I_n}TP_{I_n})\}$$

is dense in $\mathcal{B}$ w.r.t. the strong operator topolgy.

Proof: Take any basic open set $\mathcal{O} = \{T \in \mathcal{B} : ||(S - T)P_k|| < \epsilon\}$, for some $k \in \omega$, $S \in \mathcal{B}$ and $\epsilon > 0$. Let T and m be as in the above lemma (with n replaced by k), let n be large enough that $m \leq \min(I_n)$ and note that then $T + T_n \in \mathcal{O} \cap \mathcal{D}_m$. □

Proposition 3.19 *Take a partition (I_n) of ω and sequences $(T_n) \subseteq \mathcal{B}$ and $(\epsilon_n) \subseteq \mathbb{R}$ such that $P_{I_n}T_nP_{I_n} = T_n$ and $\epsilon_n > 0$ for all $n \in \omega$. Then, for all $m \in \omega$,*

$$\mathcal{N}_{(\epsilon_n),(T_n),(I_n),m} = \{T \in \mathcal{B} : \forall n \geq m(||T_n - P_{I_n}TP_{I_n}|| \geq \epsilon_n)\} = \bigcap_{n \geq m} \{T \in \mathcal{B} : ||T_n - P_{I_n}TP_{I_n}|| \geq \epsilon_n\}$$

is a closed nowhere dense subset of $\mathcal{B}$ w.r.t. both the weak and strong operator topology. Furthermore every good nowhere dense $\mathcal{N} \subseteq \mathcal{B}$ is contained in such a set, with $m = 0$ in fact.

Definition 3.20

$$T =^{\bullet}_{(e_n)} S \Leftrightarrow \exists n \in \omega(P_{\omega \setminus n}TP_{\omega \setminus n} = P_{\omega \setminus n}SP_{\omega \setminus n}).$$

Proposition 3.21 ([2] 3.1.5.) *Take a partition (I_n) of ω and sequences $(T_n) \subseteq \mathcal{B}$ and $(\epsilon_n) \subseteq \mathbb{R}$ such that $P_{I_n}T_nP_{I_n} = T_n$ and $\epsilon_n > 0$ for all $n \in \omega$. Then*

$$\mathcal{M}_{(\epsilon_n),(T_n),(I_n)} = \bigcup_m \mathcal{N}_{(\epsilon_n),(T_n),(I_n),m} = \{T \in \mathcal{B} : \forall^\infty n(||T_n - P_{I_n}TP_{I_n}|| \geq \epsilon_n)\}$$

is a $=^{\bullet}_{(e_n)}$-closed meagre F_σ subset of $\mathcal{B}$ w.r.t. both the weak and strong operator topology. Furthermore every good meagre $\mathcal{M} \subseteq \mathcal{B}$ is contained in such a set.

Definition 3.22 *A subset $\mathcal{F} \subseteq \mathcal{B}$ is* very good *nowhere dense (w.r.t. the basis (e_n)) if there exists $\epsilon > 0$ such that, for all $n \in \omega$, there exists $m \geq n$ and $S \in \mathcal{B}$ such that $P_{m \setminus n}SP_{m \setminus n} = S$*

$$\{T \in \mathcal{B} : ||S - P_{m \setminus n}TP_{m \setminus n}|| < \epsilon\}$$

is disjoint from $\mathcal{F}$. A countable union of very good nowhere dense sets is sadi to be very good *meagre (w.r.t. the basis (e_n)).*

Proposition 3.23 *The very good nowhere dense sets are closed under subsets, finite unions and topological closures.*

Proposition 3.24 *Take $\epsilon > 0$, a partition (I_n) of ω and $(T_n) \subseteq \mathcal{B}$ such that $P_{I_n}T_nP_{I_n} = T_n$ for all $n \in \omega$. Then, for all $m \in \omega$,*

$$\mathcal{N}_{\epsilon,(T_n),(I_n),m} = \{T \in \mathcal{B} : \forall n \geq m(||T_n - P_{I_n}TP_{I_n}|| \geq \epsilon)\} = \bigcap_{n \geq m} \{T \in \mathcal{B} : ||T_n - P_{I_n}TP_{I_n}|| \geq \epsilon\}$$

is a closed nowhere dense subset of $\mathcal{B}$ w.r.t. both the weak and strong operator topology. Furthermore every very good nowhere dense $\mathcal{N} \subseteq \mathcal{B}$ is contained in such a set, with $m = 0$ in fact.

Proposition 3.25 *Take a partition* (I_n) *of* ω *and* $(T_n) \subseteq \mathcal{B}$ *such that* $P_{I_n} T_n P_{I_n} = T_n$ *for all* $n \in \omega$. *Then*

$$\mathcal{M}_{(T_n),(I_n)} = \bigcup_m \mathcal{N}_{1/m,(T_n),(I_n),m} = \{T \in \mathcal{B} : \liminf_n ||T_n - P_{I_n} T P_{I_n}|| > 0\}$$

is a $=^*$*-closed meagre* F_σ *subset of* $\mathcal{B}$ *w.r.t. both the weak and strong operator topology. Furthermore every very good meagre* $\mathcal{M} \subseteq \mathcal{B}$ *is contained in such a set.*

Proposition 3.26 $\mathrm{cl}_{=^*}(P_{\mathscr{P}(\omega)})$ *is very good meagre.*

Proof: For each $n \in \omega$ let $I_n = \{2n, 2n+1\}$ and let P_n be the projection onto the one dimensional subspace $\mathrm{span}\{e_{2n} + e_{2n+1}\}$. For any $A \in \omega$, we have $||P_n - P_{I_n} P_A P_{I_n}|| \geq 1/\sqrt{2}$ and hence $\liminf ||P_n - P_{I_n} P P_{I_n}|| \geq 1/\sqrt{2}$ for all $P =^* P_A$. Thus $\mathrm{cl}_{=^*}(P_{\mathscr{P}(\omega)}) \subseteq \mathcal{M}_{(P_n),(I_n)}$. □

Next we prove the analog of 2.13. Note however that, w.r.t. the weak operator topology, $\mathcal{B}(H)_1$, $\mathcal{B}(H)_1^{+/-}$ and $\mathcal{B}(H)_1^+$ are closed subsets of $\mathcal{B}(H)$, however $\mathcal{P}(\mathcal{B}(H))$ is not and hence the proof of 3.27 below does not work in this case. Indeed, it follows from 3.3 that $\mathcal{P}(\mathcal{B}(H))$ is dense in $\mathcal{B}(H)_1^+$.

For the rest of this section let $\mathcal{B}$ consistently be $\mathcal{B}(H)_1$, $\mathcal{B}(H)_1^{+/-}$ or $\mathcal{B}(H)_1^+$ (but not $\mathcal{P}(\mathcal{B}(H))$).

Theorem 3.27 *If* $\mathcal{F} \subseteq \mathcal{B}$ *is* $=^\bullet_{(e_n)}$*-closed* F_σ *w.r.t. the weak operator topology and* $T \in \mathcal{B} \setminus \mathcal{F}$ *then there exists an interval partition* (I_n) *of* ω *and* $(\epsilon_n) > 0$ *such that* $\mathcal{F} \subseteq \mathcal{M}_{(\epsilon_n),(P_{I_n} T P_{I_n}),(I_n)}$

Proof: Let $\mathcal{F} = \bigcup \mathcal{N}_n$ for closed subsets $(\mathcal{N}_n)$ of X. Define (ϵ_n) and (I_n) recursively as follows. Let $\min(I_n) = \max(I_{n-1}) + 1 (= 0$ if $n = 0)$ and, for all $S \in \mathcal{B}(H)_3$ such that

$$P_{\omega \setminus \min(I_n)}(S - T) P_{\omega \setminus \min(I_n)} = 0$$

and hence $S \notin \mathcal{F}$ (and quite possibly $S \notin \mathcal{B}$), as $\mathcal{F}$ is $=^\bullet_{(e_n)}$-closed, there exists $n_S \in \omega \setminus \min(I_n)$ and $\epsilon_S > 0$ such that the set

$$\mathcal{R}_{S,n} = \mathcal{R}_S = \{R \in \mathcal{B}(H) : \forall j,k \in n_S(\langle (R - S)e_j, e_k \rangle < \epsilon_S)\}$$

is disjoint from the closed set $\bigcup_{k \in n} \mathcal{N}_k (\subseteq \mathcal{F})$. As

$$\begin{aligned} \mathcal{S} &= \{S \in \mathcal{B}(H)_3 : P_{\omega \setminus \min(I_n)}(S - T) P_{\omega \setminus \min(I_n)} = 0\} \\ &= \{S \in \mathcal{B}(H)_3 : \forall j,k \in \omega \setminus \min(I_n)(\langle S e_j, e_k \rangle = \langle T e_j, e_k \rangle)\} \end{aligned}$$

is closed and hence compact there exist $S_0, \ldots, S_{m_n - 1}$ such that $\mathcal{R}_{S_0}, \ldots, \mathcal{R}_{S_{m_n-1}}$ cover $\mathcal{S}$. Let $\max(I_{n+1}) = \max\{n_{S_k} : k \in m_n\}$ and $\epsilon_n = \min\{\epsilon_{S_k} : k \in m_n\}$. This completes the recursion.

Now note that if $n \in \omega$, $R \in \mathcal{B}$ and $||P_{I_n}(R - T)P_{I_n}|| < \epsilon_n$ then

$$R' = R + P_{\omega \setminus \min(I_n)}(T - R) P_{\omega \setminus \min(I_n)}$$

has norm at most 3 and $P_{\omega \setminus \min(I_n)}(R' - T) P_{\omega \setminus \min(I_n)} = 0$ so $R' \in \mathcal{R}_{S_m,n}$ for some $m \in m_n$. For $j,k \in n_{S_m} \setminus \min(I_n) \subseteq I_n$ we have $\langle (R - S_m)e_j, e_k \rangle = \langle (R - T)e_j, e_k \rangle < \epsilon_n \leq \epsilon_{S_m}$, while for $j,k \in n_{S_m}$ with either j or k less than $\min(I_n)$ we have $\langle (R - S_m)e_j, e_k \rangle = \langle (R' - S_m)e_j, e_k \rangle < \epsilon_{S_m}$. So we in fact have $R \in \mathcal{R}_{S_m,n}$ too and hence $R \notin \bigcup_{k \in n} \mathcal{N}_k$. Thus if $||P_{I_n}(R - T)P_{I_n}|| < \epsilon_n$ for infinitely many $n \in \omega$ then $R \notin \mathcal{F}$, i.e. $\mathcal{F} \subseteq \mathcal{M}_{(\epsilon_n),(P_{I_n} T P_{I_n}),(I_n)}$. □

Corollary 3.28 *If* $\mathcal{F} \subseteq \mathcal{B}$ *is* $=^\bullet_{(e_n)}$*-closed meagre* F_σ *w.r.t. the weak operator topology then* $\mathcal{F}$ *is good meagre w.r.t.* (e_n).

Proof: As $\mathcal{B}$ is compact it is a Baire space, by the Baire Category Theorem, so there exists $T \in \mathcal{B}\backslash\mathcal{F}$ and the result now follows from 3.27 and (the first part of) 3.21. □

Corollary 3.29 *A subset $\mathcal{F}$ of $\mathcal{B}$ is good meagre w.r.t. (e_n) if and only if $\mathcal{F}$ is contained in a $=^\bullet_{(e_n)}$-closed meagre F_σ w.r.t. the weak operator topology.*

Proof: Follows from 3.28 and (the second part of) 3.21. □

I should point out that in [2] page 47 (after 3.1.4) Zamora-Aviles makes the comment that if $\mathcal{F}$ is closed under finite-rank changes (and $\mathcal{B} = \mathcal{B}(H)^+_1$) then good meagreness does not depend on the basis. While this does indeed follow from the above corollary if $\mathcal{F}$ is meagre F_σ (because if $\mathcal{F}$ is closed under finite rank changes then it will be $=^\bullet_{(e_n)}$-closed for any basis (e_n)) and might indeed be true even without the F_σ assumption, I do not see how it follows directly from what is written in [2].

Unfortunately, the proof of 2.21, where we get rid of the F_σ assumption, does not, as far as I can tell, generalize to $\mathcal{B}$, essentially because it does not generalize to closed subsets of ${}^\omega X$ (unlike the proof of 2.13). The best I can do is the following.

Theorem 3.30 *If $\mathcal{F} \subseteq \mathcal{B}$ is $=^*$-closed F_σ w.r.t. the weak operator topology and $T \in \mathcal{B}\backslash\mathcal{F}$, there exists and interval partition (I_n) of ω such that $\mathcal{F} \subseteq \mathcal{M}_{(P_{I_n}TP_{I_n}),(I_n)}$.*

Proof: By 3.27 there exists and interval partition (I_n) of ω together with $(\epsilon_n) > 0$ such that $\mathcal{F} \subseteq \mathcal{M}_{(\epsilon_n),(P_{I_n}TP_{I_n}),(I_n)}$. We claim that, in fact, $\mathcal{F} \subseteq \mathcal{M}_{(P_{I_n}TP_{I_n}),(I_n)}$. If not then there exists $S \in \mathcal{F}\backslash\mathcal{M}_{(P_{I_n}TP_{I_n}),(I_n)}$ which means we have $A \in [\omega]^\omega$ such that $||P_{I_{a_n}}(S-T)P_{I_{a_n}}|| \rightarrow 0$ where (a_n) is the increasing enumeration of A. So if we let $K = \sum_{a\in A} P_{I_a}(S-T)P_{I_a} \in \mathcal{K}(H)$ then $S - K \in \mathcal{F}$ but $P_{I_a}(S-K)P_{I_a} = P_{I_a}TP_{I_a}$ for all $a \in A$ and hence $S - K \notin \mathcal{M}_{(\epsilon_n),(P_{I_n}TP_{I_n}),(I_n)}$, a contradiction. □

Corollary 3.31 *If $\mathcal{F} \subseteq \mathcal{B}$ is $=^*$-closed meagre F_σ w.r.t. the weak operator topology then $\mathcal{F}$ is very good meagre.*

Corollary 3.32 *A subset $\mathcal{F}$ of $\mathcal{B}$ is very good meagre if and only if $\mathcal{F}$ is contained in a $=^*$-closed meagre F_σ w.r.t. the weak operator topology.*

Proposition 3.33 *If (I_n) is an interval partition of ω and $(T_n) \in \mathcal{B}(H)^+$ is such that we have $P_nT_nP_n = T_n$, for all $n \in \omega$, then there exists an interval partition (J_n) of ω and $(P_n) \subseteq \mathcal{P}(\mathcal{B}(H))$ such that $P_{J_n}P_nP_{J_n} = P_n$, for all $n \in \omega$, and $\mathcal{M}_{(P_n),(J_n)} \subseteq \mathcal{M}_{(P_{I_n}TP_{I_n}),(I_n)}$ (where these $\mathcal{M}$'s are defined within $\mathcal{B}(H)^+_1$ or $\mathcal{P}(\mathcal{B}(H))$, or even $\mathcal{B}(H)_1$).*

Proof: Let (J_n) be such that, for each $n \in \omega$, J_n contains some I_{m_n} and is at least twice the size of I_{m_n}. By 3.3 we can find $(P_n) \subseteq \mathcal{P}(\mathcal{B}(H))$ such that $P_{J_n}P_nP_{J_n} = P_n$ and $P_{I_{m_n}}P_nP_{I_{m_n}} = T_{m_n}$, for all $n \in \omega$, from which $\mathcal{M}_{(T_n),(I_n)} \subseteq \mathcal{M}_{(P_n),(J_n)}$ follows. □

Finally let us summarize we what we can now say about meagre subsets and their closures.

Definition 3.34 *For any $f, g \in {}^\omega[0,1]$ define*

$$f \leq^*_{[0,1]} g \Leftrightarrow \limsup(f(n) - g(n)) \leq 0.$$

Definition 3.35 *For any $A \subseteq \omega$ let $\mathbf{1}_A : \omega \rightarrow 2$ be its characteristic function, i.e. $\mathbf{1}_A(n) = 1$ if $n \in A$ and $\mathbf{1}_A(n) = 0$ otherwise. If $\mathcal{A} \subseteq \mathscr{P}(\omega)$ then let $\mathbf{1}_\mathcal{A} = \{\mathbf{1}_A : A \in \mathcal{A}\}$.*

Proposition 3.36 *For $\mathcal{A} \subseteq \mathscr{P}(\omega)$, the following are equivalent.*

(i) There exists an interval partition (I_n) of ω such that $\forall A \in \mathcal{A} \forall^\infty n \exists m \in I_n \backslash A$.

(ii) $\mathrm{cl}_{\leq_2^*}(\mathbf{1}_{\mathcal{A}}) = \mathrm{cl}_{\leq_2^\bullet}(\mathbf{1}_{\mathcal{A}}) = \mathbf{1}_{\mathrm{cl}_{\subseteq^*}(\mathcal{A})}$ *is meagre.*

(iii) $\mathrm{cl}_{\leq^*_{[0,1]}}(\mathbf{1}_{\mathcal{A}})$ *is meagre.*

Furthermore, they imply

(iv) $\mathrm{cl}_{\leq^*}(P_{\mathcal{A}})$ *is meagre.*

Proof:

(i)⇒(ii),(iii),(iv) Simply note that (i) implies that

$$\begin{aligned} \mathrm{cl}_{\leq_2^*}(\mathbf{1}_{\mathcal{A}}) &\subseteq \mathcal{M}_{\mathbf{1}_\omega,(I_n)}(\subseteq {}^\omega 2), \\ \mathrm{cl}_{\leq^*_{[0,1]}}(\mathbf{1}_{\mathcal{A}}) &\subseteq \mathcal{M}_{\mathbf{1}_\omega,(I_n)}(\subseteq {}^\omega[0,1]) \quad \text{and} \\ \mathrm{cl}_{\leq^*}(P_{\mathcal{A}}) &\subseteq \mathcal{M}_{P_{I_n},(I_n)}(\subseteq \mathcal{P}(\mathcal{B}(H))) \end{aligned}$$

(iii)⇒(i) Take arbitrary $g \in {}^\omega[0,1]$ and an interval partition (I_n) of ω. If (i) fails then there exists $C \in \mathcal{A}$ and $A \in [\omega]^\omega$ such that $I_a \subseteq C$ for all $a \in A$. If we let $C' = \bigcup_{a \in \omega \backslash A} I_a$ and $f = g \upharpoonright C' \cup \mathbf{1}_C \upharpoonright \omega \backslash C'$ then $f \leq \mathbf{1}_C$ (everywhere) and $f \upharpoonright I_a = g \upharpoonright I_a$ for all $a \in A$ and hence $f \in \mathrm{cl}_{\leq^*_{[0,1]}}(\mathbf{1}_{\mathcal{A}}) \nsubseteq \mathcal{M}_{g,(I_n)}$. Thus, as g and (I_n) were arbitrary, $\mathrm{cl}_{\leq^*_{[0,1]}}(\mathbf{1}_{\mathcal{A}})$ is not very good meagre, by 2.20. But $\mathrm{cl}_{\leq^*_{[0,1]}}(\mathbf{1}_{\mathcal{A}})$ is $=^*$-closed meagre and hence very good meagre by 2.21, a contradiction.

(ii)⇒(i) Just like (iii)⇒(i), instead taking arbitrary $g \in {}^\omega 2$. □

What we wanted, of course, was to prove that (iv) implies (ii), which would follow if we could prove 3.31 without the F_σ assumption because then we could also prove it for $\mathcal{P}(\mathcal{B}(H))$ – given any $=^*$-closed meagre subset $\mathcal{F}$ of $\mathcal{P}(\mathcal{B}(H))$, the $=^*$-closure of $\mathcal{F}$ in $\mathcal{B}(H)_1^+$ would be meagre by 3.13, hence very good meagre in $\mathcal{B}(H)_1^+$ (by 3.31 without the F_σ assumption) and thus $\mathcal{F}$ would be very good meagre in $\mathcal{P}(\mathcal{B}(H))$ by 3.33. Then the proof of (iv)⇒(i) would go like the proof of (iii)⇒(i) above.

References

[1] Andreas Blass, *Combinatorial Cardinal Characteristics of the Continuum*, Handbook of Set Theory November 24, 2003 preprint.

[2] Beatriz Zamora-Aviles, *The Structure of Order Ideals and Gaps in the Calkin Algebra*, PhD Thesis, October 2009, York University.