

# Ultrafilter limits of asymptotic density are not universally measurable

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Given a nonprincipal ultrafilter  $U$  on  $\omega$  and a sequence  $\bar{x} = \langle x_n : n \in \omega \rangle$  consisting of members of a compact Hausdorff space  $X$ , the  $U$ -limit of  $\bar{x}$  (written  $\lim_{n \rightarrow U} x_n$ ) is the unique  $y \in X$  such that for every open set  $O \subseteq X$  containing  $y$ ,  $\{n \mid x_n \in O\} \in U$ . Letting  $X$  be the unit interval  $[0, 1]$ , this operation defines a finitely additive measure  $\mu_U$  on  $\mathcal{P}(\omega)$  in terms of asymptotic density, letting  $\mu_U(A) = \lim_{n \rightarrow U} |A \cap n|/n$ , for each  $A \subseteq \omega$ . A *medial limit* is a finitely additive measure on  $\mathcal{P}(\omega)$ , giving singletons measure 0 and  $\omega$  itself measure 1, such that for each open set  $O \subseteq [0, 1]$ , the collection of  $A \subseteq \omega$  given measure in  $O$  is universally measurable, i.e., is measured by every complete finite Borel measure on  $\mathcal{P}(\omega)$  (see [1] for more on medial limits and universally measurable sets). If there could consistently be a nonprincipal ultrafilter  $U$  such that measure given by the  $U$ -limit of asymptotic density were universally measurable, this would give a relatively simple example of a medial limit. We show here, however, that this cannot be the case.

**Theorem 0.1.** *If  $U$  is a nonprincipal ultrafilter on  $\omega$ , then the function*

$$\mu_U: \mathcal{P}(\omega) \rightarrow [0, 1]$$

*defined by letting  $\mu_U(A) = \lim_{n \rightarrow U} |A \cap n|/n$  is not universally measurable.*

*Proof.* Let  $I_0 = \{0\}$ , and for each positive  $n \in \omega$  let

$$I_n = \{5^{n-1}, 5^{n-1} + 1, \dots, 5^n - 1\}.$$

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Either the union of the  $I_n$ 's for  $n$  even is in  $U$ , or the corresponding union for  $n$  odd is. In the first case, let  $J_0 = I_0 \cup I_1 \cup I_2$ , and for each positive  $n$ , let  $J_n = I_{2n+1} \cup I_{2n+2}$ . In the second case, let  $J_n = I_{2n} \cup I_{2n+1}$  for all  $n \in \omega$ . In either case, let  $S$  be the set of  $A \subseteq \omega$  such that  $A \cap J_n \in \{\emptyset, J_n\}$  for all  $n \in \omega$ . Then  $S$  is a perfect subset of  $\mathcal{P}(\omega)$ , and the mapping  $H: S \rightarrow \mathcal{P}(\omega)$  sending  $A \in S$  to  $\{n \mid A \cap J_n = J_n\}$  is a homeomorphism. Let  $F_0$  be the set of  $A \subseteq \omega$  such that  $\mu_U(A) \in [0, 1/4)$ , and let  $F_1$  be the set of  $A \subseteq \omega$  such that  $\mu_U(A) \in (3/4, 1]$  (so  $F_1$  is the set of complements of elements of  $F_0$ ). It will be enough to show that  $F_1 \cap S$  is not a universally measurable subset of  $S$ .

We claim for each  $A \in S$ , the set of  $n$  such that  $|A \cap n|/n \in [0, 1/4) \cup (3/4, 1]$  is in  $U$ . This follows from the fact that all (but possibly one) of the  $J_n$ 's are unions of two consecutive  $I_m$ 's, and that the union of the larger members of these pairs is in  $U$ . Each such consecutive pair (for  $n > 0$ ) has the form  $I_m = \{5^{m-1}, 5^{m-1}+1, \dots, 5^m-1\}$  and  $I_{m+1} = \{5^m, 5^m+1, \dots, 5^{m+1}-1\}$ , and if  $A \in S$ , then  $A$  either contains or is disjoint from  $I_m \cup I_{m+1}$ . If it contains both, then for each  $k \in I_{m+1}$ ,

$$|A \cap k|/k \geq (5^m - 5^{m-1})/5^m = 1 - 1/5 = 4/5 > 3/4,$$

and if it is disjoint from both then

$$|A \cap k|/k \leq 5^{m-1}/5^m = 1/5 < 1/4.$$

This establishes the claim. It follows that  $S \subseteq F_0 \cup F_1$ . Since  $\mu_U$  is a finitely additive measure, the intersection of two sets of  $\mu_U$ -measure greater than  $3/4$  cannot be less than  $1/4$ , so  $F_1 \cap S$  is closed under finite intersections. It follows that  $H$  maps  $F_1 \cap S$  homeomorphically to a nonprincipal ultrafilter, and thus that  $F_1 \cap S$  is not universally measurable.  $\square$

A version of the proof just given, in the special case  $\{5^n : n \in \omega\} \in U$ , led to the proof in [1] that consistently there are no medial limits.

## References

- [1] P.B. Larson, *The Filter Dichotomy and medial limits*, in preparation.

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