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Forcing, Combinatorics and Definability

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In these lectures I will focus on the interaction between forcing, combinatorial methods and definability issues in set theory. The interplay between these topics has been remarkably fruitful, and I will illustrate it in the following three contexts: *Definable Wellorders*, *Cardinal Characteristics at Uncountable Cardinals* and *Models of the Proper Forcing Axiom*. The first of these topics is a classical one in set theory, although interesting results have recently been proved and interesting open problems remain. The second of these topics is quite new and provides fertile ground for the application of new forcing methods. The last topic is more specialised but does hold some surprises and has led to a new kind of forcing iteration.

Part 1: Definable Wellorders

In ZF, AC is equivalent to the statement:

$$H(\kappa^+) \text{ can be wellordered for every } \kappa.$$

A natural question is: When can we obtain a *definable* wellorder of $H(\kappa^+)$? By a Σ_n *definable wellorder of* $H(\kappa^+)$ we mean a wellorder of $H(\kappa^+)$ which is Σ_n definable over $H(\kappa^+)$ with κ as a parameter. We first note the following:

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Remarks. 1. If n is at least 3, then κ can be eliminated, as $\{\kappa\}$ is Π_2 definable.
2. If λ is a limit cardinal and $H(\kappa^+)$ has a definable wellorder for cofinally many $\kappa < \lambda$, then $H(\lambda)$ has a definable wellorder: We glue together the least definable wellorder of smaller $H(\kappa^+)$'s for those κ for which such a wellorder exists; if λ is a strong limit cardinal then the result is in fact a Σ_2 definable wellorder of $H(\lambda)$ (without parameters). For this reason we will focus on definable wellorders of $H(\kappa^+)$.

We also consider Σ_n definable wellorders of $H(\kappa^+)$ *with parameters*, i.e. wellorders of $H(\kappa^+)$ which are Σ_n definable over $H(\kappa^+)$ with *arbitrary* elements of $H(\kappa^+)$ as parameters

Definable wellorders and Large Cardinals

A Σ_n definable wellorder of $H(\omega_1)$ (with/without parameters) corresponds to a Σ^1_{n+1} definable wellorder of the reals (with/without real parameters). We have:

Theorem 1 *(Mansfield [25]) If there is a Σ^1_2 wellorder of the reals then every real belongs to L.*
(Martin-Steel [26]) A Σ^1_{n+2} wellorder of the reals is consistent with n Woodin cardinals but inconsistent with n Woodin cardinals and a measurable cardinal above them.

Do large cardinals impose a similar restriction on the existence of definable wellorders of $H(\omega_2)$? We say that a forcing is *small* iff it has size less than the least inaccessible cardinal. Small forcings preserve all large cardinals.

Theorem 2 *(Asperó-Friedman [2]) There is a small forcing which forces CH and a definable wellorder of $H(\omega_2)$.*

The above wellorder is not Σ_1. In fact:

Theorem 3 *(Woodin) If there is a measurable Woodin cardinal and CH holds then there is* no *wellorder of the reals which is Σ_1 over $H(\omega_2)$ in the parameter ω_1.*

However:

Theorem 4 *(Avraham-Shelah [3]) There is a small forcing which forces the negation of CH and a wellorder of the reals which is Σ_1 over $H(\omega_2)$ in the parameter ω_1.*

This leaves the following open problem:

Question 1. Is there a small forcing which forces a Σ_2 wellorder of $H(\omega_2)$?

I'll say something now about the proof of Theorem 2. It has two ingredients, *Canonical Function coding* and *Strongly type-guessing coding (Asperó).*

Canonical Function coding

For each $\alpha < \omega_2$ choose $f_\alpha : \omega_1 \longrightarrow \alpha$ onto and define $g_\alpha : \omega_1 \longrightarrow \omega_1$ by:

$$g_\alpha(\gamma) = \text{ordertype} \quad f_\alpha[\gamma].$$

g_α is a "canonical function" for α.

Now code $A \subseteq \omega_2$ by $B \subseteq \omega_1$ as follows:

$$\alpha \in A \text{ iff } g_\alpha(\gamma) \in B \text{ for a club of } \gamma.$$

Assuming GCH, the forcing to do this is ω-strategically closed and ω_2-cc.

Asperó coding

A *club-sequence in ω_1 of height τ* is a sequence $\vec{C} = (C_\delta \mid \delta \in S)$ where $S \subseteq \omega_1$ is stationary and each C_δ is club in δ of ordertype τ. $\vec{C}$ is *strongly type-guessing* iff for every club $C \subseteq \omega_1$ there is a club $D \subseteq \omega_1$ such that for all δ in $D \cap S$, ordertype $(C \cap C_\delta^+) = \tau$, where C_δ^+ denotes the set of successor elements of C_δ. An ordinal γ is *perfect* iff $\omega^\gamma = \gamma$.

Lemma 5 *(Asperó) Assume GCH. Let $B \subseteq \omega_1$. Then there is an ω-strategically closed, ω_2-cc forcing that forces: $\gamma \in B$ iff the γ-th perfect ordinal is the height of a strongly type-guessing club sequence.*

Now to prove Theorem 2, proceed as follows: Assume GCH. Write $H(\omega_2)$ as $L_{\omega_2}[A]$, $A \subseteq \omega_2$. Use Canonical Function coding to code A by $B \subseteq \omega_1$. And use Asperó coding to code B definably over $H(\omega_2)$. There is a problem however: B only codes $H(\omega_2)$ of the ground model, not $H(\omega_2)$ of the extension!

The solution is to perform both codings "simultaneously". The forcing is a hybrid forcing, situated halfway between iteration and product.

Now we look beyond ω_2.

Theorem 6 *(Asperó-Friedman [1]) There is a class forcing which forces GCH, preserves all supercompact cardinals (as well as a proper class of n-huge cardinals for each n) and adds a definable wellorder of $H(\kappa^+)$ for all regular $\kappa \geq \omega_1$.*

Using the Remark 2 from the start our discussion of definable wellorders, we then have:

Corollary 7 *There is a class forcing which forces GCH, preserves all supercompact cardinals (as well as a proper class of n-huge cardinals for each n) and adds a (parameter-free) definable wellorder of $H(\delta)$ for all cardinals $\delta \geq \omega_2$ which are not successors of singulars.*

What about successors of singulars? Here there is an obstruction:

Theorem 8 *(Asperó-Friedman [1]) Suppose that there is a $j : L(H(\lambda^+)) \longrightarrow L(H(\lambda^+))$ fixing λ, with critical point less than λ. Then there is no definable wellorder of $H(\lambda^+)$ with parameters.*

The reason for this is that by Kunen's argument refuting the existence of a nontrivial elementary embedding of V into V, the hypotheses of Theorem 8 imply that there can in fact be no wellorder of $H(\lambda^+)$ inside the model $L(H(\lambda^+))$. But the following remains open:

Question 2. Is there a small forcing that adds a definable wellorder of $H(\aleph_{\omega+1})$ with parameters?

Regarding Σ_1 definable wellorders:

Theorem 9 *(Friedman-Holy [14]) There is a class forcing which forces GCH, preserves all supercompact cardinals (as well as a proper class of n-huge cardinals for each n) and adds a Σ_1 definable wellorder of $H(\kappa^+)$ with parameters for all regular $\kappa \geq \omega_1$.*

This result is based on a stronger result, which shows that one can in fact force a strong form of condensation for a predicate on A on κ^+ which codes $H(\kappa^+)$, assuming that κ is regular and uncountable. However if one is not allowed arbitrary parameters, then the following is still open:

Question 3. Is there a small forcing that adds a Σ_1 definable wellorder of $H(\omega_3)$?

Definable wellorders and Forcing Axioms

I'll next look at definable wellorders in the presence of generalisations of Martin's axiom, such as BPFA (the Bounded Proper Forcing Axiom) and BMM (the Bounded Martin's Maximum). For the definition of these axioms please look ahead to the start of Part 3 of these notes.

Theorem 10 *(Friedman [10]) MA is consistent with a Σ^1_3 wellorder of the reals (assuming only Con ZFC).*
(Caicedo-Friedman [6]) BPFA + $\omega_1 = \omega_1^L$ (which is consistent relative to a reflecting cardinal) implies that there is a Σ^1_3 wellorder of the reals.

Theorem 11 *(Hjorth [23]) Assuming not CH and every real has a #, there is no Σ^1_3 wellorder of the reals.*

Question 4. Does BPFA + "$0^\#$ does *not* exist" imply that there is a Σ^1_3 wellorder of the reals?
Question 5. Is BMM consistent with a projective wellorder of the reals? (PFA is not.)

Surprisingly, the following appears to be unresolved (but should not be difficult):

Question 6. Is MA consistent with the *nonexistence* of a projective wellorder of the reals (relative to Con ZFC)?

For $H(\omega_2)$:

Theorem 12 *(Caicedo-Velickovic [7]) BPFA + $\omega_1 = \omega_1^L$ implies that there is a Σ_1 definable wellorder of $H(\omega_2)$.*

Theorem 13 *(Larson [24]) Relative to a supercompact limit of supercompacts, there is a model of MM with a definable wellorder of $H(\omega_2)$.*

The forcing axioms mentioned have little effect at cardinals past ω_2. Indeed the above results easily imply the following for larger $H(\kappa)$:

Theorem 14 *MA is consistent with a definable wellorder of $H(\kappa^+)$ for all κ.*
(Reflecting cardinal) BSPFA is consistent with a definable wellorder of $H(\kappa^+)$ for all κ.
(A supercompact limit of supercompacts) MM is consistent with a definable wellorder of $H(\kappa^+)$ for all regular $\kappa \geq \omega_1$.

Definable Wellorders and Cardinal Characteristics

This adds to large cardinals and forcing axioms a new context for the study of definable wellorders. The general question is: To what extent are the known results about cardinal characteristics on ω consistent with the existence of a projective wellorder of the reals? A general method for attacking such questions is provided by:

The Template iteration $\mathbb{T}$ [12]: This is a countable support, ω_2-cc iteration which adds a Σ^1_3 wellorder of the reals (and a Σ_1 wellorder of $H(\omega_2)$). The iteration is not proper, but is S-proper for certain stationary $S \subseteq \omega_1$.

This leads to the broad project of mixing the template iteration with a variety of iterations for controlling cardinal characteristics. Results obtained in this way are the following:

Theorem 15 *(Fischer-Friedman [9]) Each of the following is consistent with a Σ^1_3 wellorder of the reals: $\mathfrak{d} < \mathfrak{c}$, $\mathfrak{b} < \mathfrak{a} = \mathfrak{s}$, $\mathfrak{b} < \mathfrak{g}$.*

Here, $\mathfrak{b}$ = the bounding number, $\mathfrak{a}$ = the almost disjointness number, $\mathfrak{s}$ = the splitting number and $\mathfrak{g}$ = the groupwise density number. What allows these results to go through is that the template iteration $\mathbb{T}$ is ω^ω bounding and can be mixed with any countable support proper iteration of posets of size ω_1.

One can also ask for projective *witnesses* to cardinal characteristics. A sample result:

Theorem 16 *(Friedman-Zdomskyy [21]) It is consistent that* $\mathfrak{a} = \omega_2$ *and there is an infinite* Π^1_2 *maximal almost disjoint family.*

Question 7. Is it consistent with $\mathfrak{a} = \omega_2$ that there is an infinite Σ^1_2 maximal almost disjoint family?

Definable Wellorders can be consider in many other contexts as well. Here are some sample open problems:

Questions.
8. Is it consistent that for all infinite regular κ, GCH fails at κ and there is a definable wellorder of $H(\kappa^+)$?
9. Is the tree property at ω_2 consistent with a projective wellorder of the reals?
10. Is it consistent that the nonstationary ideal on ω_1 is saturated and there is a Σ^1_4 wellorder of the reals?
11. [16] shows that it is consistent for GCH to fail at a measurable cardinal κ while there is a definable wellorder of $H(\kappa^+)$. Is this consistent if one requires 2^κ to be greater than κ^{++}?

Part 2: Cardinal Characteristics at κ

Cardinal characteristics on ω is a vast subject. Some examples of such characteristics appear in Blass' survey article for the Handbook of Set Theory:

$$\mathfrak{a}, \mathfrak{b}, \mathfrak{d}, \mathfrak{e}, \mathfrak{g}, \mathfrak{h}, \mathfrak{i}, \mathfrak{m}, \mathfrak{p}, \mathfrak{r}, \mathfrak{s}, \mathfrak{t}, \mathfrak{u}$$

These are all at most $\mathfrak{c}$, the cardinality of the continuum.

Now suppose that κ is regular and uncountable. We consider analogues of some of the above for κ

$$\mathfrak{a}(\kappa), \mathfrak{b}(\kappa), \mathfrak{d}(\kappa) \ldots$$

Why is this worth doing? Here are four reasons:

1. Higher iterated forcing: Cardinal Characteristics at ω are to countable support iterations what Cardinal Characteristics at κ are to κ-supported iterations. The theory of iterations with uncountable support is an interesting and developing area in the theory of iterated forcing.

2. The large cardinal context: Cardinal Characteristics at a measurable cardinal present interesting new challenges, mixing iterated forcing techniques with elementary embedding techniques.

3. Global behaviour as κ varies, internal consistency: There are sometimes interesting interactions between cardinal characteristics at different cardinals. Also, one can ask for inner models exhibiting global behaviours of cardinal characteristics ("internal consistency"), which as in the large cardinal context demand a mix of iterated forcing and elementary embedding techniques.

4. Solve problems at κ that are unsolved at ω: In at least one case, the natural analogue of an unsolved problem at ω can be solved at an uncountable regular κ, which at least gives us a feeling of progress.

I'll illustrate these themes with some examples. First let's consider the most fundamental of all cardinal characteristics, the cardinal characteristic 2^κ.

Global behaviour:

Theorem 17 *(Corollary to Easton's Theorem) It is consistent that* $2^\alpha = \alpha^{++}$ *for all regular* α.

To prove this, Easton used an *Easton product* of the forcings Cohen(α, α^{++}) (this adds α^{++} many α-Cohen subsets of α, for each regular α).

Internal consistency:

Theorem 18 *(Friedman-Ondrejović [17]) Assuming that $0^{\#}$ exists, there is an* inner model *in which $2^\alpha = \alpha^{++}$ for all regular α.*

The Easton product does not work for this result, as it can be shown that the existence of $0^{\#}$ rules out the existence of generics over L for α-Cohen forcing when α has uncountable cofinality in V. Instead, one uses a *Reverse Easton iteration* of the Cohen(α, α^{++}) forcings.

Large cardinal context:

Theorem 19 *(Woodin) Assume that κ is hypermeasurable. Then in a forcing extension, κ is measurable and $2^\kappa = \kappa^{++}$.*

By "hypermeasurable" I mean $H(\kappa^{++})$-hypermeasurable, i.e., there is an elementary embedding $j : V \to M$ with critical point κ such that $H(\kappa^{++})$ belongs to M. Woodin used a Reverse Easton iteration of the Cohen(α, α^{++}) forcings for $\alpha \leq \kappa$, α inaccessible, followed by Cohen(κ^+, κ^{++}). But we'll see later that the proof is much easier if we replace Cohen by Sacks. This was discovered by considering our next cardinal characteristic, the dominating number $\mathfrak{d}(\kappa)$:

Global Behaviour:

Theorem 20 *(Cummings-Shelah [8]) It is consistent that $\mathfrak{d}(\alpha) = \alpha^+ < 2^\alpha$ for all regular α.*

The forcing used is a Reverse Easton iteration of a product of α-Cohen forcings followed by an α^+-iteration of α-Hechler forcings.

Large cardinal context:

Theorem 21 *(Friedman-Thompson [18]) Assume that κ is hypermeasurable. Then in a generic extension, κ is measurable and $\mathfrak{d}(\kappa) = \kappa^+ < 2^\kappa$.*

The forcing used is a Reverse Easton iteration of α-Sacks products, $\alpha \leq \kappa$, α inaccessible. Two interesting points are:

i. If you try this with κ-Cohen and κ-Hechler instead of κ-Sacks then you need some supercompactness; hypermeasurability is insufficient.

ii. The proof is *easier* than Woodin's proof, which only gives $\kappa^+ < 2^\kappa$ and not $\mathfrak{d}(\kappa) < 2^\kappa$.

Global Behaviour in the Large Cardinal Context

Theorem 22 *(Friedman-Thompson [19]) Assume that κ is hypermeasurable. Then in a generic extension, κ is measurable and $\mathfrak{d}(\alpha) = \alpha^+ < 2^\alpha$ for all regular α.*

Here one needs (a Reverse Easton iteration of) an α-Sacks product at inaccessible $\alpha \leq \kappa$, an α-Cohen product followed by α-Hechler iteration at successors of non-inaccessibles, and something new at α^+, α inacessible (an α^+-Cohen product followed by a mixture of an α-Sacks product and α^+-Hechler iteration).

Remark. Friedman-Honzik [15] works out Easton's Theorem in the large cardinal setting. A sample result is: Suppose that F is an Easton function of the form $F(\kappa) =$ the least λ such that $H(\lambda^+) \models \varphi(\kappa)$ for some formula φ. Then there is a cofinality-preserving forcing that realises the Easton function F at all regulars and preserves the measurability of κ whenever κ is $F(\kappa)$-hypermeasurable.

Question 12. What Global Behaviours for $\mathfrak{d}(\alpha)$ are possible when there is a measurable cardinal?

We now look at the cardinal characteristic CofSym(α). Let Sym(α) denote the group of permutations of α under composition. Then CofSym(α)

denotes the least λ such that Sym(α) is the union of a strictly increasing λ-chain of subgroups.

Sharp and Thomas [27]: CofSym(α) can be anything reasonable.

But its Global Behaviour is nontrivial!

Theorem 23 *(Sharp-Thomas [27]) (a) Suppose that $\alpha < \beta$ are regular and GCH holds. Then in a cofinality-preserving forcing extension, CofSym(α) = β.*
(b) If CofSym(α) $> \alpha^+$ then CofSym(α^+) $\leq$ CofSym(α).

Question 13. Is it consistent that CofSym(ω) = CofSym(ω_1) = ω_3?

CofSym has been studied in the Large Cardinal setting:

Theorem 24 *(Friedman-Zdomskyy [20]) Suppose that κ is hypermeasurable. Then in a forcing extension, κ is measurable and CofSym(κ) = κ^{++}.*

As in earlier cases, the obvious approach does not work: The forcing used in [27] adds a dominating real and therefore supercompactness would be needed to adapt it to the context of measurable cardinals. Instead, we use an iteration of Miller(κ) (with continuous club-splitting) and a generalisation of Sacks(κ), based on a different proof of the Sharp-Thomas result that is found in [28]. The proof also introduces yet another cardinal characteristic, $\mathfrak{g}_{\mathrm{cl}}(\kappa)$, the groupwise density number for *continuous* partitions.

I turn now to the relationship between $\mathfrak{a}(\kappa)$ and $\mathfrak{d}(\kappa)$. The former is the minimum size of a (size at least κ) maximal almost disjoint family of subsets of κ An old open problem is the following:

Question 14. Does $\mathfrak{d}(\omega) = \omega_1$ imply $\mathfrak{a}(\omega) = \omega_1$?

Suprisingly, this has been solved at uncountable cardinals!

Theorem 25 *(Blass-Hyttinen-Zhang [5]) For uncountable α, $\mathfrak{d}(\alpha) = \alpha^+$ implies $\mathfrak{a}(\alpha) = \alpha^+$*

Are there other open questions for ω which can be solved for uncountable cardinals? Here are some sample problems:

Question 15. Can $\mathfrak{p}(\kappa)$ be less than $\mathfrak{t}(\kappa)$? Maybe it will help to assume that κ is a large cardinal.
Question 16. Can $\mathfrak{s}(\kappa)$ be singular?

More open Questions.
17. (Without large cardinals) Is $\mathfrak{b}(\kappa) < \mathfrak{a}(\kappa)$ consistent for an uncountable κ?
18. Which Global Behaviours for $\mathfrak{b}(\alpha), \mathfrak{d}(\alpha)$ are internally consistent? Cummings-Shelah answered this for ordinary consistency.
19. (Without supercompactness) Can $\mathfrak{s}(\kappa)$ be greater than κ^+? Zapletal showed that one (almost) needs a hypermeasurable.
20. Is it consistent that $\mathrm{CofSym}(\kappa) = \kappa^{+++}$ for a measurable κ?

Part 3: Some models of PFA and BPFA

Let $\mathcal{C}$ be a class of forcings.

FA($\mathcal{C}$) is the Forcing Axiom for $\mathcal{C}$: For P in $\mathcal{C}$, there is a filter on P which hits ω_1-many predense sets in P.

BFA($\mathcal{C}$) is the Bounded Forcing Axiom for $\mathcal{C}$: For P in $\mathcal{C}$, there is a filter which hits ω_1-many predense sets of size $\leq \omega_1$ in P.

PFA = FA(Proper) = the Proper Forcing Axiom,
BPFA = BFA(Proper) = the Bounded Proper Forcing Axiom.

Useful Fact. (Bagaria, Stavi-Väänänen) BPFA is equivalent to the Σ_1 elementarity of $H(\omega_2)^V$ in $H(\omega_2)^{V[G]}$ for all proper P and P-generic G.

Theorem 26 *(a) (Baumgartner [4]) If there is a supercompact then PFA holds in a proper forcing extension.*
(b) (Goldstern-Shelah [22]) If there is a reflecting cardinal (i.e., a regular κ such that $H(\kappa) \prec_{\Sigma_2} V$) then BPFA holds in a proper forcing extension.

Cardinal Minimality

V is *cardinal minimal* iff whenever M is an inner model with the correct cardinals (i.e., $\mathrm{Card}^M = \mathrm{Card}^V$) then $M = V$.

A "local" version: Fix a cardinal κ. V is *κ-minimal* iff whenever M is an inner model with the correct cardinals $\leq \kappa$ then $H(\kappa)^M = H(\kappa)$.

Examples
L is trivially cardinal minimal.
Let x be κ-Sacks, κ-Miller or κ-Laver over L. Then $L[x]$ is *not* cardinal minimal, because L and $L[x]$ have the same cardinals.
Let $f : \kappa \to \kappa^+$ be a minimal collapse of κ^+ to κ over L. Then $L[f]$ is cardinal minimal.

More interesting examples are given by the core models.

Theorem 27 *Let K be the core model for a measurable, hypermeasurable, strong or Woodin cardinal. Then K is cardinal minimal. In fact, K is κ-minimal for all $\kappa \geq \omega_2$.*

ω_1-minimality fails for core models, and in fact whenever $0^\#$ exists:

Theorem 28 *Suppose that $0^\#$ exists. Then V is not ω_1-minimal. In fact, there is an inner model M with the correct ω_1 which is a forcing extension of L.*

Other sources of cardinal minimality are models of forcing axioms.

SPFA = FA(Semiproper) = the Semiproper Forcing Axiom
BSPFA = BFA(Semiproper) = the Bounded Semiproper Forcing Axiom

Theorem 29 *(Velickovic [29]) Suppose that SPFA holds. Then V is ω_2-minimal.*

There is a related result for BPFA:

Theorem 30 *(Caicedo-Velickovic [7]) Suppose that BPFA holds. Then V is ω_2-minimal* with respect to inner models satisfying BPFA*: If M is an inner model* satisfying BPFA *with the correct ω_2 then $H(\omega_2)^M = H(\omega_2)$.*

The above results are optimal in the following sense:

Theorem 31 *(Friedman [13]) (a) Suppose that there is a supercompact. Then in some forcing extension, PFA holds and the universe is not ω_2-minimal.*
(b) Suppose that there is a reflecting cardinal. Then in some forcing extension, BPFA holds and the universe is not ω_2-minimal.

The proofs are based on:

Lemma 32 *(Collapsing to ω_2 with "finite conditions" [11]) Assume GCH. Suppose that κ is inaccessible and $\mathcal{S}$ denotes $[\kappa]^\omega$ of V. Then there is a forcing P such that:*
(a) P forces $\kappa = \omega_2$.
(b) P is $\mathcal{S}$-proper, and hence preserves ω_1, in any extension of V in which $\mathcal{S}$ remains stationary.

We sketch the proof of Theorem 31(a). Suppose that κ is supercompact. Collapse κ to ω_2 with finite conditions, producing $V[F]$. Now perform Baumgartner's PFA iteration, but at stage $\alpha < \omega_2$, choose a forcing *in the model* $V[F \restriction \alpha, G_\alpha]$ which is $\mathcal{S}$-proper there; argue that it is also $\mathcal{S}$-proper in $V[F, G_\alpha]$. *Important:* Only use names from $V[F \restriction \alpha, G_\alpha]$, to keep the forcing small! (This is the method of *"Diagonal Iteration".*) Then verify that PFA (indeed FA($\mathcal{S}$ − Proper)) holds in $V[F, G]$. As $\kappa = \omega_2$ holds both in $V[F]$ and in $V[F, G]$, this shows that $V[F, G]$ is not ω_2-minimal.

How do we collapse an inaccessible κ to ω_2 with finite conditions? Here is a brief sketch (for more details see [11]).

Let $\# : [\kappa]^\omega \longrightarrow \kappa$ be injective. P consists of all pairs $p = (A, S)$ such that:

1. A is a finite set of disjoint closed intervals $[\alpha, \beta]$, $\alpha \le \beta < \kappa$, cof $(\alpha) \le \omega_1$.
2. S is a finite subset of $[\kappa]^\omega$ ("side conditions").
3. Technical (see [11]).
4. Let F be the set of uncountable cofinality α for $[\alpha, \beta]$ in A, together with κ. The *height* of $x \in S$ is the least element of F greater than $\sup x$. Then:

i. (Closure under truncation) x in S, α in F implies $x \cap \alpha$ in S.
ii. (Almost an $\in$-chain) If $x, y \in S$ have the same height then $\#(x) \in y$, $\#(y) \in x$ or $x = y$.

The forcing is κ-cc and adds a club in κ consisting only of ordinals of cofinality $\leq \omega_1$. So κ becomes ω_2.

Questions.
21. Suppose that BSPFA holds. Then is V ω_2-minimal with respect to inner models satisfying BSPFA?
22. Is there a forcing which collapses an inaccessible to ω_3 "with finite conditions"?

In conclusion I would like to say that it was a pleasure for me to deliver these lectures to the stimulating Japanese set theory community, and I look forward to further collaborations between the Gödel Center and Japan!

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