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Fodor-type Reflection Principle and Balogh's reflection theorems †

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Abstract

In this note, we show that the theorems in Z. Balogh [2] proved there under Axiom R are already provable under Fodor-type Reflection Principle (FRP) introduced in [9] or under a slight extension of FRP still much weaker than Axiom R.

1 Introduction

The purpose of this note is to show that the theorems in [2] proved there under Axiom R are already provable under Fodor-type Reflection Principle (FRP) introduced in [9] or a slight extension of it still much weaker than Axiom R.

In Section 2, we begin with checking the proof of a slight extension of Dow's theorem mentioned in [2]. This is used in Section 3 to show that Balogh's

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`http://kurt.scitec.kobe-u.ac.jp/~fuchino/papers/balogh-x.pdf`

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theorem on reflection of metrizability (Theorem 2.2 in [2]) is a consequence of the reflection theorem on metrizability proved under FRP by Fuchino, Juhász, Soukup, Szentmiklóssy and Usuba (Theorem 4.3 in [9]).

In Section 4, we prove that Balogh's reflection theorem on paracompactness (Theorem 1.6 in [2]) holds under FRP.

In Section 5, we consider another reflection theorem on paracompactness by Balogh (Theorem 1.4 in [2]) for which we need a slight strengthening of FRP which is provable from Axiom R. The status of the axiom we use here is still largely unknown (see Problems 2, 3) except that it is still much weaker than Axiom R.

In the following, we consider the topology of a space X as given either by an open base τ of X or by the family $\mathcal{O}$ of all open sets of X. We write $X = (X, \tau)$ or $X = (X, \mathcal{O})$. If $\mathcal{O}$ is generated from the open base τ we write $\mathcal{O} = \mathcal{O}_\tau$.

The approach $X = (X, \tau)$ with an open base τ is convenient in connection with the method of elementary submodels. This is because, for an open basis τ of a topological space X, $\tau \cap M$ is also an open basis of $X \cap M$ for an elementary submodel M of $\mathcal{H}(\theta)$ for a sufficiently large cardinal θ with $(X, \tau) \in M$ while $\mathcal{O} \cap M$ for such M does not build in general the set of all open sets of a topology on $X \cap M$.

Here, we call a cardinal θ *sufficiently large* if it is regular and $2^{|X|}$, $2^{2^{|X|}}$, $\cdots < \theta$ for all (small) sets X relevant in the context following the declaration of θ being "sufficiently large".

A set M of cardinality $\aleph_1$ is *internally approachable* if M is the union of a continuously increasing chain $\langle M_\alpha : \alpha < \omega_1 \rangle$ of countable subsets of M such that $M_\alpha \in M_{\alpha+1}$ for all $\alpha < \omega_1$. If we consider M as an $\in$-structure, we assume also that each M_α is an elementary submodel of $M = \langle M, \in \rangle$. For an internally approachable M, the sequence $\langle M_\alpha : \alpha < \omega_1 \rangle$ as above is called *internally approachable filtration* of M.

A set M is *ω-bounding* if $[M]^{\aleph_0} \cap M$ is cofinal in $[M]^{\aleph_0}$ with respect to $\subseteq$. For a regular uncountable θ any internally approachable $M \prec \mathcal{H}(\theta)$ is ω-bounding. It follows that there are cofinally may ω-bounding $M \prec \mathcal{H}(\theta)$ of cardinality $\aleph_1$.

A space is said to be *(countably) compact* here if it is Hausdorff and satisfies the usual (countably) compactness condition. So a compact space is normal. Note also that

(1.1) a first countable and countably compact space is regular.

Following the definition in Engelking [6], a Lindelöf space is a regular topological space X with the Lindelöf property:

every open cover of X has a countable subcover.

Similarly to the case of compact spaces, Lindelöf spaces are normal ([6, Theorem 3.8.2]).

For a property P of a topological space and a cardinal κ, we say that a given topological space X is $\leq \kappa$-P ($< \kappa$-P, respectively) if every subspace Y of X of cardinality $\leq \kappa$ ($< \kappa$, respectively) has the property P. In this notation, we shall always put '$\leq$' or '$<$' to the cardinal κ since very often "κ P" or "κ-P" is already used for some other notions (this is e.g. the case with "$\aleph_1$ meta-Lindelöf"). X is said to be *almost* P if X is $< | X |$- P, that is, if every subspace of X of cardinality $< | X |$ has the property P.

The following notation and the lemma have been introduced in [9].

For a family $\mathcal{F}$ of sets, let $\sim_{\mathcal{F}}$ be the intersection relation on $\mathcal{F}$, i.e. let $F \sim_{\mathcal{F}} G$ if and only if $F \cap G \neq \emptyset$ for F, $G \in \mathcal{F}$, and let $\approx_{\mathcal{F}}$ be the transitive closure of $\sim_{\mathcal{F}}$. An argument in elementary cardinal arithmetic shows the following:

Lemma 1.1. *Let μ be an uncountable regular cardinal and $\mathcal{F}$ a family of sets such that, for all $F \in \mathcal{F}$, we have $| \{G \in \mathcal{F} : F \sim_{\mathcal{F}} G\} | < \mu$. Then every equivalence class of $\approx_{\mathcal{F}}$ has cardinality $< \mu$.* ❑

2 Dow's theorem

A. Dow [4] proved (in ZFC) that every countably compact $\leq \aleph_1$-metrizable space is metrizable. Z. Balogh [1] noted that practically the same proof of Dow's theorem as stated in [4] shows that every countably compact $\leq \aleph_1$-P space is metrizable where P here is the property: *there exists a point countable base.* In this section we will check the details of the proof of this assertion (Theorem 2.8).

Dow gives an elegant proof of the following Proposition as an application of the method of elementary submodels (see [4, Proposition 3.2]).

Proposition 2.1 (Juhász [11]). *For any space X if every subspace of X of cardinality $\leq \aleph_1$ has countable weight then X itself has countable weight.* ❑

Bing metrization theorem implies the following.

Lemma 2.2. *A countably compact space X is metrizable if and only if X has countable weight.* ❑

The following is trivial.

Lemma 2.3. *For a space X and $Y \subseteq X$, $w(X) \leq \kappa$ for a cardinal κ implies $w(Y) \leq \kappa$.* □

Lemma 2.4. *Suppose that $X = (X, \tau)$, $Y \subseteq X$ and $x \in Y$. If X is regular at x and $\mathcal{B}$ is a neighborhood base for x in (the subspace topology of) Y, then $\mathcal{B}$ is a neighborhood base for x in $\overline{Y}$ as well. Thus, for such x, we have $\chi(x, Y) = \chi(x, \overline{Y})$.*

Proof. Suppose that $O \in \mathcal{O}_\tau$ with $y \in O$. We have to show that there is $U \in \mathcal{B}$ such that $U \cap \overline{Y} \subseteq O \cap \overline{Y}$.

Now, since X is regular at x, there is $O' \in \mathcal{O}_\tau$ such that $y \in O'$ and $\overline{O'} \subseteq O$. Let $U \in \mathcal{B}$ be such that $U \cap Y \subseteq O' \cap Y$. Then we have

$$U \cap \overline{Y} \subseteq \overline{U} \cap \overline{Y} = \overline{U \cap Y} \subseteq \overline{O' \cap Y} = \overline{O'} \cap \overline{Y} \subseteq O \cap \overline{Y}.$$

This shows that $\mathcal{B}$ is also a neighborhood base of x in $\overline{Y}$. $\chi(x, Y) = \chi(x, \overline{Y})$ follows from this by Lemma 2.3. □ (Lemma 2.4)

Lemma 2.5 (Proposition 2.3 in [4]). *If a space $X = (X, \tau)$ has a point countable base then, for a sufficiently large θ and $M \prec \mathcal{H}(\theta)$ with $\langle X, \tau \rangle \in M$, $\tau \cap M$ is a base for (each point of) $\overline{X \cap M}$.*

Proof. Suppose that $X = (X, \tau)$, θ and M are as above. By elementarity, there is a point countable base $\mathcal{B}$ of X with $\mathcal{B} \in M$.

Suppose that

$$(2.1) \quad x \in \overline{X \cap M}$$

and $B_0 \in \mathcal{B}$ is a neighborhood of x. Let $O_0 \in \tau$ and $C_0 \in \mathcal{B}$ be such that $x \in C_0 \subseteq O_0 \subseteq B_0$. By (2.1), there is $y \in C_0 \cap (X \cap M) = C_0 \cap M$. Since there are only countably many $B \in \mathcal{B}$ with $y \in B$, all such B's are in M. In particular, we have C_0, $B_0 \in M$.

Again by elementarity, we have $M \models \exists O \in \tau(C_0 \subseteq O \subseteq B_0)$. Hence there is an $O_1 \in \tau \cap M$ such that $x \in C_0 \subseteq O_1 \subseteq B_0$. This shows that $\tau \cap M$ is a local base for x. □ (Lemma 2.5)

Lemma 2.6 (Proposition 2.4 in [4]). *Suppose that $X = (X, \tau)$ is a countably compact space. If $M \prec \mathcal{H}(\theta)$ is countable with $\langle X, \tau \rangle \in M$ and $\tau \cap M$ is not a base for (X, τ) then there is $z \in \overline{X \cap M}$ such that $\tau \cap M$ is not a base at z.*

Proof. If $\overline{X \cap M} = X$ then the assertion is just trivial. So assume that there is $x \in X \setminus \overline{X \cap M}$. Suppose, toward a contradiction, that $\tau \cap M$ is a base at each $z \in \overline{X \cap M}$. Then we can choose $O_z \in \tau \cap M$ such that $z \in O_z$ and

(2.2) $x \notin O_z$

for each $z \in \overline{X \cap M}$. Since $\overline{X \cap M}$ is countably compact and $\{O_z : z \in \overline{X \cap M}\} \subseteq \tau \cap M$ is a countable open covering of $\overline{X \cap M}$, there are $z_1, \ldots, z_n \in \overline{X \cap M}$ for some $n \in \omega$ such that $\overline{X \cap M} \subseteq O_{z_1} \cup \cdots \cup O_{z_n}$. It follows that $M \models$ "$O_{z_1}, \ldots, O_{z_n}$ covers X". By elementarity it follows that $O_{z_1}, \ldots, O_{z_n}$ really covers X. But this is a contradiction to (2.2). ❑ (Lemma 2.6)

Using the lemmas above, we can prove the following theorem of Miščenko:

Theorem 2.7 (Miščenko). *A countably compact space with a point countable base has a countable base (i.e. it is metrizable).* ❑

We can even prove the following. Note that a countably compact space with a point countable base is regular as noted before. Thus the following Theorem 2.8 indeed generalizes Miščenko's Theorem.

Theorem 2.8 (A variant of Theorem 3.1 in Dow [4]. See also [2]). *If X is a regular countably compact space such that every subspace of X of cardinality $\leq \aleph_1$ has a point countable base, then X is metrizable.*

Proof. Suppose, for contradiction, that $X = (X, \tau)$ is a countably compact space such that every subspace of X of cardinality $\leq \aleph_1$ has a point countable base but X is not metrizable.

Let θ be sufficiently large and let M be an internally approachable elementary submodel of $\mathcal{H}(\theta)$ and $\langle X, \tau \rangle \in M$.

Since $w(X) > \aleph_0$ (by Lemma 2.2), there is a $Z \in [X]^{\aleph_1}$ such that $w(Z) > \aleph_0$ by Proposition 2.1. By elementarity, there is such a $Z \in M$.

We have $w(\overline{Z}) > \aleph_0$ by Lemma 2.3. Since $\overline{Z}$ is countably compact, $\overline{Z}$ is non metrizable by Lemma 2.2. Thus we may assume without loss of generality $X = \overline{Z}$. For each $x \in X \cap M$, $Z \cup \{x\}$ has cardinality $\aleph_1$ and hence it has a point countable base. In particular $\chi(x, Z \cup \{x\}) = \aleph_0$ by Lemma 2.4. It follows that $\tau \cap M$ is a base of $(X \cap M, \tau)$. Thus

(2.3) $(X \cap M, \tau \cap M)$ has a point countable base.

Let $\langle M_\alpha : \alpha < \omega_1 \rangle$ be an internally approachable filtration of M such that Z, $\langle X, \tau \rangle \in M_0$.

Since $w(X) > \aleph_0$ and M_α is countable $\tau \cap M_\alpha$ is not a base of (X, τ) for any $\alpha < \omega_1$. Thus, by Lemma 2.6, there is $z \in \overline{X \cap M_\alpha}$ such that$\tau \cap M_\alpha$ is not a base at z. Since $M_\alpha \in M_{\alpha+1}$, there is such z in $M_{\alpha+1}$ by elementarity.

Let N be a countable elementary submodel of $\mathcal{H}(\theta)$ such that

(2.4) $X, Z, M, \langle M_\alpha : \alpha < \omega_1 \rangle \in N$.

Let $\alpha^* = \omega_1 \cap N$. By the remark above there is $z^* \in M_{\alpha^*+1}$ such that

(2.5) $z^* \in \overline{X \cap M_{\alpha^*}}$ and $\tau \cap M_{\alpha^*}$ is not a neighborhood base at z^*.

On the other hand, by (2.4), we have

$$(\tau \cap M) \cap N = \bigcup\{\tau \cap M_\beta : \beta < \alpha^*\} = \tau \cap M_{\alpha^*}.$$

Hence by (2.3) and by Lemma 2.5, $\tau \cap M_{\alpha^*}$ is a neighborhood base for any $z \in \overline{X \cap M_{\alpha^*}}$. This is a contradiction. □ (Theorem 2.8)

3 Balogh's metrization theorem under FRP

The following two theorems were proved in S. Fuchino, I. Juhasz, L. Soukup, Z. Szentmiklóssy and T. Usuba [9].

Theorem 3.1 (Fuchino, Juhász, Soukup, Szentmiklóssy and Usuba, [9, Theorem 4.2]). *Suppose that X is a locally countably compact and meta-Lindelöf space. If X is $\leq \aleph_1$-metrizable then it is actually metrizable.* □

Theorem 3.2 (Fuchino, Juhász, Soukup, Szentmiklóssy and Usuba [9, Theorem 4.3]). (1) *Assume that* FRP(κ) *holds for every regular cardinal κ with $\omega_1 < \kappa \leq \lambda$ and X is a locally separable, countably tight space with $L(X) \leq \lambda$. If X is $\leq \aleph_1$-meta-Lindelöf then X is actually meta-Lindelöf.*

(2) *Under* FRP *every locally separable, countably tight and $\leq \aleph_1$-meta-Lindelöf space is meta-Lindelöf.* □

Here, for a regular cardinal $\kappa \geq \omega_1$, FRP(κ) (*The Fodor-type Reflection Principle for κ*) is the following statement:

FRP(κ): For any stationary $S \subseteq E^\kappa_\omega = \{\alpha < \kappa : \mathrm{cf}(\alpha) = \omega\}$ and mapping $g : S \to [\kappa]^{\leq\aleph_0}$ there is $I \in [\kappa]^{\aleph_1}$ such that

(3.1) $\mathrm{cf}(I) = \omega_1$;

(3.2) $g(\alpha) \subseteq I$ for all $\alpha \in I \cap S$;

(3.3) for any regressive $f : S \cap I \to \kappa$ such that $f(\alpha) \in g(\alpha)$ for all $\alpha \in S \cap I$, there is $\xi^* < \kappa$ such that $f^{-1}{}''\{\xi^*\}$ is stationary in $sup(I)$.

FRP is the axiom which asserts that FRP(κ) holds for all regular cardinal $\kappa \geq \aleph_2$. Note that we can only demand FRP(κ) for a regular κ since FRP(κ) for a singular κ is easily shown to be inconsistent (see Lemma 2.2 in [9]).

In [9], it is shown that FRP(κ) for a regular cardinal κ follows from RP(κ) which is a weakening of of Axiom R for κ. Thus FRP is a consequence of Axiom R. On the other hand, it is also proved in [9] that FRP(κ) is preserved by c.c.c.-extension of the universe. Thus FRP is strictly weaker than Axiom R.

Here, the Reflection Principle RP(κ) and Axiom R for κ (Notation: AR(κ)) are defined as follows:

RP(κ): For any stationary $S \subseteq [\kappa]^{\aleph_0}$, there is an $I \in [\kappa]^{\aleph_1}$ such that

(3.4) $\omega_1 \subseteq I$;

(3.5) $\mathrm{cf}(I) = \omega_1$;

(3.6) $S \cap [I]^{\aleph_0}$ is stationary in $[I]^{\aleph_0}$.

AR(κ): For any stationary $S \subseteq [\kappa]^{\aleph_0}$ and ω_1-club $\mathcal{T} \subseteq [\kappa]^{\aleph_1}$, there is $I \in \mathcal{T}$ such that $S \cap [I]^{\aleph_0}$ is stationary in $[I]^{\aleph_0}$

where $\mathcal{T} \subseteq [X]^{\aleph_1}$ for an uncountable set X is said to be ω_1-*club* (or *tight and unbounded* in Fleissner's terminology in [7]) if

(3.7) $\mathcal{T}$ is cofinal in $[X]^{\aleph_1}$ with respect to $\subseteq$ and

(3.8) for any increasing chain $\langle I_\alpha : \alpha < \omega_1 \rangle$ in $\mathcal{T}$ of length ω_1, we have $\bigcup_{\alpha<\omega_1} I_\alpha \in \mathcal{T}$.

Axiom R is the assertion that AR(κ) holds for all cardinals $\kappa \geq \aleph_2$ and RP is the assertion that RP(κ) holds for all cardinals κ with $\kappa \geq \aleph_2$.

It is easy to see that AR(κ) implies RP(κ). R.E. Beaudoin [3] proved that Axiom R follows from MA^+(σ-closed).

By the theorems above and by Theorem 2.8, we can prove the following improvement of Theorem 2.2 in Z. Balogh [2] where the assertion (2) of the following theorem was proved under Axiom R.

Theorem 3.3. (1) *Let λ be a cardinal such that for each regular cardinal κ with $\omega_1 < \kappa \leq \lambda$ we have* FRP(κ). *If X is a regular locally countably compact space with $L(X) \leq \lambda$ and*

(3.9) *every subspace of X of cardinality $\leq \aleph_1$ has a point countable base,* *then* X *is metrizable.*

(2) *Assume* FRP. *If* X *is a regular locally countably compact space satisfying* (3.9), *then* X *is metrizable.*

Proof. We prove only (1) since (2) clearly follows from (1).

Let X be as in (1). Then every point of X has a countably compact neighborhood, and this neighborhood is compact metrizable by Theorem 2.8. By Lemma 2.2, it follows that X is both locally separable and countably tight. Also X is $\leq \aleph_1$-meta-Lindelöf by (3.9). Hence X is meta-Lindelöf by Theorem 3.2 (1). By Theorem 3.1, it follows that X is metrizable. □ (Theorem 3.3)

Theorem 3.3 implies the following theorem which can be also derived directly form Theorem 3.2:

Theorem 3.4 (Fuchino, Juhász, Soukup, Szentmiklóssy and Usuba [9]). (1) *Let* λ *be a cardinal such that for each regular cardinal* κ *with* $\omega_1 < \kappa \leq \lambda$ *we have* FRP(κ). *If* X *is a locally countably compact and* $\aleph_1$*-metrizable space with* $L(X) \leq \lambda$ *then* X *is metrizable.*

(2) *Assume* FRP. *Then every locally countably compact and* $\aleph_1$*-metrizable space is metrizable.* □

In S. Fuchino, H. Sakai, L. Soukup and T. Usuba [10], it is proved that the assertion of Theorem 3.2, (1) as well as Theorem 3.4, (1) are equivalent to:

FRP($\leq \lambda$): FRP(κ) holds for each regular cardinal κ with $\omega_1 < \kappa \leq \lambda$

over ZFC. Thus also we obtain the following:

Theorem 3.5. *The assertion of* Theorem 3.3, (1) *is equivalent to* FRP($\leq \lambda$) *over* ZFC. □

4 Reflection of paracompactness in countably tight locally Lindelöf spaces

In this section we prove that Theorem 1.6 in Balogh [2] is already provable under FRP (Theorem 4.5).

Recall that a space X is *locally Lindelöf* if every point x of X has an open neighborhood O such that $\overline{O}$ is a Lindelöf subspace of X.

Lemma 4.1. *For a topological space* $X = (X, \mathcal{O})$, *if* $\mathcal{F} \subset \mathcal{P}(X)$ *is locally finite, then we have* $\bigcup\{\overline{Y} : Y \in \mathcal{F}\} = \overline{\bigcup \mathcal{F}}$.

Proof. The inclusion "$\subseteq$" is clear. To show the other inclusion "$\supseteq$", suppose $x \in \overline{\bigcup\mathcal{F}}$. Let $O \in \mathcal{O}$ be such that $x \in O$ and $\mathcal{F}_0 = \{Y \in \mathcal{F} : O \cap Y \neq \emptyset\}$ is finite. Then we have $x \in \overline{\bigcup\mathcal{F}_0} = \bigcup\{\overline{Y} : Y \in \mathcal{F}_0\}$. Thus $x \in \bigcup\{\overline{Y} : Y \in \mathcal{F}\}$.

❑ (Lemma 4.1)

Lemma 4.2. *For a topological space $X = (X, \mathcal{O})$, if $\mathcal{F} \subseteq \mathcal{P}(X)$ is locally finite, then $\overline{\mathcal{F}} = \{\overline{Y} : Y \in \mathcal{F}\}$ is also locally finite.*

Proof. For $x \in X$, let $O \in \mathcal{O}$ be such that $x \in O$ and $\mathcal{F}_0 = \{Y \in \mathcal{F} : O \cap Y \neq \emptyset\}$ is finite. For any $y \in O$ if $y \in \overline{Y}$ for some $Y \in \mathcal{F}$ then $O \cap Y \neq \emptyset$, i.e. $Y \in \mathcal{F}_0$. So we have $\{Y \in \mathcal{F} : O \cap \overline{Y} \neq \emptyset\} = \mathcal{F}_0$. ❑ (Lemma 4.2)

The following characterization of paracompactness of locally Lindelöf spaces was already mentioned in [2]. In the proof of Theorem 4.5 we actually only use the trivial direction of this characterization. Nevertheless the characterization explains the need to look at open partitions of a given locally Lindelöf space to prove the paracompactness of the space.

Lemma 4.3. *A regular locally Lindelöf space X is paracompact if and only if it is partitioned into clopen Lindelöf subspaces.*

Proof. Suppose first that X is partitioned into clopen Lindelöf subspaces. By Morita's theorem each subspace in the partition is paracompact. Hence it follows that X itself is also paracompact.

Suppose now that X is a locally Lindelöf paracompact space. We show that there is a partition of X into clopen Lindelöf subspaces. Let $\mathcal{A} \subseteq \mathcal{O}$ be an open covering of X such that $\overline{Y}$ is Lindelöf for all $Y \in \mathcal{A}$. Let $\mathcal{B}$ be a locally finite open refinement of $\mathcal{A}$. Then elements of $\mathcal{B}' = \{\overline{Y} : Y \in \mathcal{B}\}$ are Lindelöf and $\mathcal{B}'$ is still locally finite by Lemma 4.2.

Claim 4.3.1. *For any $Y \in \mathcal{B}'$, $\{Z \in \mathcal{B}' : Y \cap Z \neq \emptyset\}$ is countable.*

$\vdash$ Suppose $Y \in \mathcal{B}'$. Let $S = \{Z \in \mathcal{B}' : Y \cap Z \neq \emptyset\}$. For each $y \in Y$, let $O_y \in \mathcal{O}$ be such that $y \in O_y$ and $\{Z \in \mathcal{B}' : O_y \cap Z \neq \emptyset\}$ is finite. Note that we can find such O_y since $\mathcal{B}'$ is locally finite. Since Y is Lindelöf, there is a countable $Y_0 \subseteq Y$ such that $\{O_y : y \in Y_0\}$ is a cover of Y. Then we have $S \subseteq \{Z \in \mathcal{B}' : O_y \cap Z \neq \emptyset$ for some $y \in Y_0\}$ and the right side of the inclusion is easily seen to be countable. $\dashv$ (Claim 4.3.1)

Let $\sim_{\mathcal{B}'}$ be the intersection relation on $\mathcal{B}'$ and $\approx_{\mathcal{B}'}$ be its transitive closure. Let $\mathbb{E}$ be the set of all equivalence classes of $\approx_{\mathcal{B}'}$. By the claim above and by Lemma 1.1, each $e \in \mathbb{E}$ is countable. It follows that $\bigcup e$ is Lindelöf and $\bigcup e$

is closed by Lemma 4.1. Thus $\{\bigcup e : e \in \mathbb{E}\}$ is a partition of X as desired.

❑ (Lemma 4.3)

Lemma 4.4 (Proposition 1.1 in Balogh [2]). *If a topological space $X = (X, \mathcal{O})$ is locally Lindelöf, then $\mathcal{B} = \{V \subseteq X : V$ is an open Lindelöf subspace of $X\}$ forms a base of X.*

Proof. Note that a closed subspace of a Lindelöf space is also Lindelöf. Hence, for $x \in X$ and $x \in O \in \mathcal{O}$, there is a $U \in \mathcal{O}$ such that $x \in U \subseteq O$ and $\overline{U}$ is Lindelöf. Since $\overline{U}$ is a Lindelöf space and thus normal, we can construct a sequence $\langle O_i : i \in \omega\rangle$ of open sets such that

$$(4.1) \quad x \in O_0 \subseteq \overline{O_0} \subseteq O_1 \subseteq \overline{O_1} \subseteq \cdots \subseteq U.$$

Let $O^* = \bigcup_{i\in\omega} O_i$. Then O^* is open neighborhood of x and $O^* \subseteq O$. O^* is Lindelöf since we can also represent O^* as the countable union of Lindelöf spaces, namely as $O^* = \bigcup_{i\in\omega} \overline{O_i}$. ❑ (Lemma 4.4)

S. Balogh [2] proved the following theorem under Axiom R.

Theorem 4.5 (FRP). *Suppose that X is locally Lindelöf and countably tight. If every open subspace Y of X with $L(Y) \leq \aleph_1$ is paracompact then X itself is paracompact.*

Proof. A variation of the proof of Theorem 4.3 in S. Fuchino, I. Juhasz, L. Soukup, Z. Szentmiklóssy and T. Usuba[9] will do.

It is enough to prove that the following $(4.2)_\kappa$ holds for all cardinal κ by induction on κ:

$(4.2)_\kappa$ For any countably tight and locally Lindelöf space X with $L(X) \leq \kappa$, if every open subspace of X of Lindelöf degree $\leq \aleph_1$ is paracompact then X itself is also paracompact.

For $\kappa \leq \aleph_1$, $(4.2)_\kappa$ trivially holds. So assume that $\kappa > \aleph_1$ and that $(4.2)_\lambda$ holds for all $\lambda < \kappa$. Let X be as in $(4.2)_\kappa$. We have to show that X is paracompact.

Case 1. κ is regular.

Let $\{L_\alpha : \alpha < \kappa\}$ be a cover of X consisting of Lindelöf subspaces of X. By Lemma 4.4, we may assume that each L_α is open. For $\beta < \kappa$, let $X_\beta = \bigcup\{L_\alpha : \alpha < \beta\}$. By $L(X) = \kappa$, we have $X \neq X_\beta$ for every $\beta < \kappa$. We may also assume that the continuously increasing sequence $\langle X_\beta : \beta < \kappa\rangle$ of open set in X is strictly increasing.

Let $S = \{\alpha < \kappa : X_\alpha \neq \overline{X_\alpha}\}$.

Claim 4.5.1. *S is non-stationary in κ.*

$\vdash$ We prove first the following weakening of the claim:

Subclaim 4.5.1.1. *$S \cap E^\kappa_\omega$ is non-stationary in κ.*

$\vdash$ For a contradiction, suppose that $S \cap E^\kappa_\omega$ were stationary. For each $\alpha \in S \cap E^\kappa_\omega$, let $p_\alpha \in \overline{X_\alpha} \setminus X_\alpha$ and let $h(\alpha) \in \kappa$ be such that $p_\alpha \in L_{h(\alpha)}$. Since X is countably tight, there is $c_\alpha \in [\alpha]^{\aleph_0}$ such that $p_\alpha \in \overline{\bigcup_{\beta \in c_\alpha} L_\beta}$.

Now, by FRP, there is $I \in [\kappa]^{\aleph_1}$ such that

(4.3) $\mathrm{cf}(I) = \omega_1$;

(4.4) $h(\alpha) \in I$ for all $\alpha \in S \cap E^\kappa_\omega \cap I$;

(4.5) $c_\alpha \subseteq I$ for all $\alpha \in S \cap E^\kappa_\omega \cap I$;

(4.6) if $f : S \cap E^\kappa_\omega \cap I \to \kappa$ is such that $f(\alpha) \in c_\alpha$ for all $\alpha \in S \cap E^\kappa_\omega \cap I$, then there is $\xi^* \in I$ with $\sup(f^{-1}(\{\xi^*\})) = \sup(I)$.

Let $Y = \bigcup_{\beta \in I} L_\beta$. Note that, by (4.4), $p_\alpha \in Y$ for all $\alpha \in S \cap E^\kappa_\omega \cap I$.

By $|\, I \,| = \aleph_1$ and since each L_β is open Lindelöf subspace of X, it follows that Y is open and $L(Y) \leq \aleph_1$. Hence, by the assumption on X, Y is a paracompact subspace of X. Thus the open cover $\mathcal{L} = \{L_\beta : \beta \in I\}$ of Y has a locally finite open refinement $\mathcal{E}$. Since each L_β ($\beta \in I$) is Lindelöf, it follows that, for each $\beta \in I$,

(4.7) $\{E \in \mathcal{E} : E \cap L_\beta \neq \emptyset\}$ is countable.

Now, for each $\alpha \in S \cap E^\kappa_\omega \cap I$, let $E_\alpha \in \mathcal{E}$ be such that $p_\alpha \in E_\alpha$. Since $p_\alpha \in \overline{\bigcup\{L_\beta : \beta \in c_\alpha\}}$, there is $f(\alpha) \in c_\alpha$ such that $E_\alpha \cap L_{f(\alpha)} \neq \emptyset$. Thus, by (4.6), there is a $\xi^* \in I$ such that $\sup(f^{-1}{}''\{\xi^*\}) = \sup(I)$. By (4.7), we have $E \subseteq X_\eta$ for all $E \in \mathcal{E}$ such that $E \cap L_{\xi^*} \neq \emptyset$ for some large enough $\eta \in S \cap E^\kappa_\omega \cap I$ with $f(\eta) = \xi^*$. But, since $\emptyset \neq E_\eta \cap L_{f(\eta)} = E_\eta \cap L_{\xi^*}$ we have $p_\eta \in E_\eta \subseteq X_\eta$. This is a contradiction to the choice of p_η. $\dashv$ (Subclaim 4.5.1.1)

Let C be a club subset of κ consisting of limit ordinals such that $S \cap E^\kappa_\omega \cap C = \emptyset$ and let

(4.8) $D = \{\alpha \in C : \alpha \setminus S \text{ is cofinal in } \alpha\}$.

Clearly D is also a club subset of κ. So the following subclaim proves the claim.

Subclaim 4.5.1.2. *$S \cap D = \emptyset$.*

$\vdash$ For $\alpha \in D \cap E^\kappa_\omega$, we have $\alpha \notin S$ by $D \subseteq C$.

For $\alpha \in D \cap E^\kappa_{>\omega}$, suppose $p \in \overline{X_\alpha}$. By the countable tightness of X there is $\beta < \alpha$ such that $p \in \overline{X_\beta}$. By (4.8), we may assume that $\beta \in E^\kappa_\omega \setminus S$. Thus we have $p \in \overline{X_\beta} = X_\beta \subseteq X_\alpha$. This shows that $X_\alpha = \overline{X_\alpha}$ and hence $\alpha \notin S$.

$\dashv$ (Subclaim 4.5.1.2)

$\dashv$ (Claim 4.5.1)

Now let D be a club subset of κ such that $D \cap S = \emptyset$ and let $\langle \xi_\alpha : \alpha < \kappa \rangle$ be an increasing enumeration of $D \cup \{0\}$. Let $Y_\alpha = X_{\xi_{\alpha+1}} \setminus X_{\xi_\alpha}$ for $\alpha < \kappa$. Then $\{Y_\alpha : \alpha < \kappa\}$ is a partition of X into clopen subspaces. Since each Y_α is the union of $< \kappa$ many Lindelöf spaces, namely $L_\delta \setminus X_{\xi_\alpha}$, $\xi_\alpha \le \delta < \xi_{\alpha+1}$, we have $L(Y_\alpha) < \kappa$. It follows from the induction hypothesis that each Y_α is paracompact. Hence X itself is also paracompact.

Case 2. κ is singular.

Similarly to Case 1., let $\{L_\alpha : \alpha < \kappa\}$ be a cover of X consisting of open Lindelöf subspaces of X. Let $\langle \kappa_i : i < \mathrm{cf}(\kappa) \rangle$ be a continuously and strictly increasing sequence of cardinals cofinal in κ. For $i < \mathrm{cf}(\kappa)$, let $X_i = \bigcup\{L_\alpha : \alpha < \kappa_i\}$. By the induction hypothesis, there is a locally finite open refinement $\mathcal{C}_i$ of the open cover $\{L_\alpha : \alpha < \kappa_i\}$ of X_i for each $i < \mathrm{cf}(\kappa)$. Let $\mathcal{C} = \bigcup_{i<\mathrm{cf}(\kappa)} \mathcal{C}_i$.

Let $\sim_\mathcal{C}$ be the intersection relation on $\mathcal{C}$ and $\approx_\mathcal{C}$ be its transitive closure. Since each $\mathcal{C}_i$ is locally finite and each $C \in \mathcal{C}_i$ is Lindelöf, we have $|\{C' \in \mathcal{C} : C \approx_\mathcal{C} C'\}| \le \mathrm{cf}(\kappa) < \kappa$ for all $C \in \mathcal{C}$.

Let $\mathbb{E}$ be the set of all equivalence classes of $\approx_\mathcal{C}$. Then, by Lemma 1.1, each $e \in \mathbb{E}$ has cardinality $\le \mathrm{cf}(\kappa)$.

$\mathcal{P} = \{\bigcup e : e \in \mathbb{E}\}$ is a partition of X into clopen subspaces. Since each $Y \in \mathcal{P}$ is the union of $\le \mathrm{cf}(\kappa)$ many Lindelöf subspaces, we have $L(Y) \le \mathrm{cf}(\kappa) < \kappa$. It follows that each $Y \in \mathcal{P}$ is paracompact by the induction hypothesis and hence X is also paracompact. □ (Theorem 4.5)

In contrast to reflection theorem in the last section, the following is still open:

Problem 1. *Is the assertion of* Theorem 4.5 *equivalent to* FRP *?*

5 Axiom R-like extension of FRP and a stronger reflection property of paracompactness

Similarly to the extension of RP to Axiom R, FRP(κ) for a regular cardinal $\kappa \ge \aleph_2$ can be enhanced with the additional requirement that the reflection point I be an element of a given ω_1-club family $\subseteq [\kappa]^{\aleph_1}$:

FRP$^R(\kappa)$: For any ω_1-club $\mathcal{T} \subseteq [\kappa]^{\aleph_1}$, stationary $S \subseteq E^\kappa_\omega$ and mapping $g : S \rightarrow [\kappa]^{\leq\aleph_0}$ there is $I \in \mathcal{T}$ such that

(5.1) for any regressive $f : S \cap I \rightarrow \kappa$ such that $f(\alpha) \in g(\alpha)$ for all $\alpha \in S \cap I$, there is $\xi^* < \kappa$ such that $f^{-1}{''}\{\xi^*\}$ is stationary in $sup(I)$.

Similarly to FRP, let FRP^R be the axiom asserting that FRP$^R(\kappa)$ holds for all regular $\kappa \geq \aleph_2$.

Note that we can put the constraints (3.1) and (3.2) on I by thinning out the ω_1-club family $\mathcal{C}$. Thus FRP$^R(\kappa)$ implies FRP(κ) for all regular $\kappa \geq \aleph_2$. The proof of the implication "RP$(\kappa) \Rightarrow$ FRP(κ)" in [9] can be slightly modified to show the implication "AR$(\kappa) \Rightarrow$ FRP$^R(\kappa)$".

A straight forward modification of Theorem 3.4 in [9] shows also that FRP$^R(\kappa)$ is preserved in generic extensions by c.c.c. forcing.

Shelah proved that SCH follows from a weakening of RP ([16]). Since RP also implies $2^{\aleph_0} \leq \aleph_2$ (Todorcevic, see [12]), it follows that, under RP, we have $\mathrm{cf}([\kappa]^{\aleph_0}, \subseteq) = \kappa^+$ for all cardinal κ with $cf(\kappa) = \omega$. Thus the assumption FRP^R + (5.2) of Theorem 5.1 below is a consequence of Axiom R. This assumption is also still much weaker than Axiom R, since it is easy to see that this is still preserved in extensions by c.c.c. forcing.

Balogh proved the following theorem under Axiom R (Theorem 1.4 in [2]).

Theorem 5.1. *Assume* FRP^R *and*

(5.2) $\{\kappa < \lambda : \mathrm{cf}([\kappa]^{\aleph_0}) = \kappa\}$ *is cofinal in* λ *for any singular cardinal* λ.

Suppose that X *is a countably tight locally Lindelöf space such that*

(5.3) *for all open subspaces* Y *of* X *with* $L(Y) \leq \aleph_1$, *we have* $L(\overline{Y}) \leq \aleph_1$ *and*

(5.4) *every clopen subspace* Y *of* X *with* $L(Y) \leq \aleph_1$ *is paracompact.*

Then X *itself is paracompact.*

Proof of Theorem 5.1: The proof is a modification of the proof of Theorem 4.5.

It is enough to prove that the following $(5.5)_\kappa$ holds for all cardinal κ by induction on κ:

$(5.5)_\kappa$ For any countably tight and locally Lindelöf space X with $L(X) = \kappa$, if X satisfies (5.3) and (5.4), then X is paracompact.

For $\kappa \leq \aleph_1$, $(5.5)_\kappa$ trivially holds. So assume that $\kappa > \aleph_1$ and that $(5.5)_\lambda$ holds for all $\lambda < \kappa$. Let X be a countably tight and locally Lindelöf space with $L(X) = \kappa$ such that X satisfies (5.3) and (5.4). We have to show that X is paracompact. Let $\{L_\alpha : \alpha < \kappa\}$ be a cover of X consisting of Lindelöf subspaces of X. By Lemma 4.4, we may assume that each L_α is open. Let

$$\mathcal{T} = \{I \in [\kappa]^{\aleph_1} : \bigcup_{\alpha \in I} L_\alpha \text{ is a clopen subspace of } X\}.$$

By (5.3) and since X is countably tight, it is easy to see that $\mathcal{T}$ is ω_1-club.

Case 1. κ is regular.

For $\beta < \kappa$, let $X_\beta = \bigcup\{L_\alpha : \alpha < \beta\}$. By induction hypothesis we may also assume that $X \neq X_\beta$ for every $\beta < \kappa$ and that the sequence $\langle X_\beta : \beta < \kappa \rangle$ is strictly increasing.

Let $S = \{\alpha < \kappa : X_\alpha \neq \overline{X_\alpha}\}$.

Claim 5.1.1. *S is non-stationary in κ.*

$\vdash$ We prove first the following weakening of the claim:

Subclaim 5.1.1.1. *$S \cap E^\kappa_\omega$ is non-stationary in κ.*

$\vdash$ For a contradiction, suppose that $S \cap E^\kappa_\omega$ were stationary. For each $\alpha \in S \cap E^\kappa_\omega$, let $p_\alpha \in \overline{X_\alpha} \setminus X_\alpha$ and let $h(\alpha) \in \kappa$ be such that $p_\alpha \in L_{h(\alpha)}$. Since X is countably tight, there is $c_\alpha \in [\alpha]^{\aleph_0}$ such that $p_\alpha \in \overline{\bigcup_{\beta \in c_\alpha} L_\beta}$.

Now, by FRP^R, there is $I \in \mathcal{T}$ such that

(5.6) $\mathrm{cf}(I) = \omega_1$;

(5.7) $h(\alpha) \in I$ for all $\alpha \in S \cap E^\kappa_\omega \cap I$;

(5.8) $c_\alpha \subseteq I$ for all $\alpha \in S \cap E^\kappa_\omega \cap I$;

(5.9) if $f : S \cap E^\kappa_\omega \cap I \to \kappa$ is such that $f(\alpha) \in c_\alpha$ for all $\alpha \in S \cap E^\kappa_\omega \cap I$, then there is $\xi^* \in I$ with $\sup(f^{-1}{}''\{\xi^*\}) = \sup(I)$.

Let $Y = \bigcup_{\beta \in I} L_\beta$. Note that, by (5.7), $p_\alpha \in Y$ for all $\alpha \in S \cap E^\kappa_\omega \cap I$.

By $I \in \mathcal{T}$ and since each L_β is open Lindelöf subspace of X, it follows that Y is clopen and $L(Y) \leq \aleph_1$. Hence, by (5.4), Y is a paracompact subspace of X. The rest of this case can be treated exactly as the Case 1 in the proof of Theorem 4.5.

Case 2. κ is singular.

Let θ be a sufficiently large cardinal. Let $\mathcal{L} = \{L_\alpha : \alpha < \kappa\}$. The singularity of κ is not yet necessary in the following claim:

Claim 5.1.2. *If $M \prec \mathcal{H}(\theta)$ is such that*

(5.10) $\omega_1 \subseteq M$;

(5.11) $X, \mathcal{L} \in M$;

(5.12) *M is ω-bounding,*

then $Z = \bigcup(\mathcal{L} \cap M)$ is a clopen subspace of X.

$\vdash$ Z is an open subspace of X as the union of open subspaces $\mathcal{L} \cap M$. Thus it is enough to show that X is closed. Suppose $x \in \overline{Z}$. By the countable tightness of X, there is $c \in [\mathcal{L} \cap M]^{\aleph_0}$ such that $x \in \overline{\bigcup c}$. By (5.12), there is $c' \in [\mathcal{L} \cap M]^{\aleph_0} \cap M$ such that $c \subseteq c'$. By (5.3) and by the elementarity of M, we have

$$M \models \exists d \in [\mathcal{L}]^{\aleph_1} (\overline{\bigcup c'} \subseteq \bigcup d).$$

Let $d \in [\mathcal{L}]^{\aleph_1} \cap M$ be such that $\overline{\bigcup c'} \subseteq \bigcup d$. By (5.10), we have $d \subseteq M$. Thus there is an $L^* \in d = d \cap M$ such that $x \in L^* \subseteq \bigcup d \subseteq \bigcup(\mathcal{L} \cap M)$.

$\dashv$ (Claim 5.1.2)

Let $\langle M_i : i < \mathrm{cf}(\kappa) \rangle$ be an increasing sequence of elementary submodels of $\mathcal{H}(\theta)$ such that, for $i < \mathrm{cf}(\kappa)$,

(5.13) $|\, M_i \,| < \kappa$;

(5.14) $\omega_1 \subseteq M_i$;

(5.15) $X, \mathcal{L} \in M_i$;

(5.16) M_i is ω-bounding and

(5.17) $\kappa \subseteq \bigcup_{i<\mathrm{cf}(\kappa)} M_i$.

We can construct such a sequence in particular with the property (5.16) by the assumption on the cardinal arithmetic.

Let $X_i = \bigcup(\mathcal{L} \cap M_i)$ for $i < \mathrm{cf}(\kappa)$. By Claim 5.1.2, each X_i is a clopen subspace of X. Since $L(X_i) \leq |\, M_i \,| < \kappa$, each X_i is paracompact by induction hypothesis . Note that we need here the closedness of X_i so that (5.3) holds for X_i.

$\mathcal{L} \cap M_i$ has a locally finite open refinement $\mathcal{C}_i$ for each $i < \mathrm{cf}(\kappa)$. Let $\mathcal{C} = \bigcup_{i<\mathrm{cf}(\kappa)} \mathcal{C}_i$.

Let $\sim_{\mathcal{C}}$ be the intersection relation on $\mathcal{C}$ and $\approx_{\mathcal{C}}$ be its transitive closure. Since each $\mathcal{C}_i$ is locally finite and each $C \in \mathcal{C}_i$ is Lindelöf, $|\, \{C' \in \mathcal{C}_i : C' \approx_{\mathcal{C}_i} C\} \,| \leq \aleph_0$ for all $i < \mathrm{cf}(\kappa)$. Hence $|\, \{C' \in \mathcal{C} : C \approx_{\mathcal{C}} C'\} \,| \leq \mathrm{cf}(\kappa) < \kappa$ for all $C \in \mathcal{C}$.

Let $\mathbb{E}$ be the set of all equivalence classes of $\approx_C$. Then, by Lemma 1.1, each $e \in \mathbb{E}$ has cardinality $\leq \mathrm{cf}(\kappa)$.

$\mathcal{P} = \{\bigcup e : e \in \mathbb{E}\}$ is a partition of X into clopen subspaces. Since each $Y \in \mathcal{P}$ is the union of $\leq \mathrm{cf}(\kappa)$ many Lindelöf subspaces, we have $L(Y) \leq \mathrm{cf}(\kappa) < \kappa$. It follows that each $Y \in \mathcal{P}$ is paracompact by the induction hypothesis and hence X is also paracompact. □ (Theorem 5.1)

Though we presently do not know if $\mathrm{FRP}^R(\kappa)$ is equivalent to $\mathrm{FRP}(\kappa)$ for all regular κ, it *is* the case for many instances of κ:

Theorem 5.2. *Suppose that κ is regular and*

(5.18) $cf([\lambda]^{\aleph_0}, \subseteq) < \kappa$ *for all* $\lambda < \kappa$.

Then we have $\mathrm{FRP}^R(\kappa) \Leftrightarrow \mathrm{FRP}(\kappa)$.

Proof. It is enough to show the direction "$\Leftarrow$".

Assume that κ is a regular cardinal $> \aleph_1$ with (5.18) and $\mathrm{FRP}(\kappa)$ holds. Let $S \subseteq E^\kappa_\omega$ be stationary, $g : S \to [\kappa]^{\aleph_0}$ and $\mathcal{T} \subseteq [\kappa]^{\aleph_1}$ be ω_1-club. We want to show that there is $I \in \mathcal{T}$ such that I satisfies (5.1).

Let θ be sufficiently large and let $\mathcal{M}^* = \langle \mathcal{H}(\theta), S, g, \mathcal{T}, \ldots, \trianglelefteq, \in \rangle$ and let $\mathcal{M} \prec \mathcal{M}^*$ be the union of the continuously increasing chain $\langle M_\alpha : \alpha < \kappa \rangle$ of elementary submodels of $\mathcal{M}^*$ such that

(5.19) $|\, M_\alpha \,| < \kappa$ for all $\alpha < \kappa$;

(5.20) $M_{\alpha+1}$ is ω-bounding for all $\alpha < \kappa$;

(5.21) $M_\alpha \in M_{\alpha+1}$ for all $\alpha < \kappa$ and

(5.22) $\kappa \subseteq \mathcal{M}$.

Note that (5.20) is possible by (5.18). Let $C = \{\alpha \in \kappa : \kappa \cap M_\alpha = \alpha\}$. Since C is club in κ, $S_0 = S \cap C$ is stationary. Applying $\mathrm{FRP}(\kappa)$ to S_0 and $g \restriction S_0$ we obtain $I_0 \in [\lambda]^{\aleph_0}$ such that, letting $\alpha_0 = \sup(I_0)$,

(5.23) $\mathrm{cf}(\alpha_0) = \omega_1$;

(5.24) $g(\alpha) \subseteq I_0$ for all $\alpha \in I \cap S_0$;

(5.25) for any regressive $f : S_0 \cap I \to \kappa$ such that $f(\alpha) \in g(\alpha)$ for all $\alpha \in S_0 \cap I$, there is $\xi^* < \kappa$ such that $f^{-1}{}''\{\xi^*\}$ is stationary in $sup(I_0)$.

Since $S_0 \cap \alpha_0$ is cofinal in α_0 by (5.26), we have $\alpha_0 \in C$. By (5.23) and (5.20) it follows that

(5.26) M_{α_0} is ω-bounding.

Let $\langle N_\alpha : \alpha < \omega_1 \rangle$ be a continuously increasing sequence of elementary submodels of M_{α_0} such that

(5.27) $|N_\alpha| = \aleph_0$ for every $\alpha < \omega_1$;

(5.28) there is a countable set $x_\alpha \in N_{\alpha+1}$ such that $N_\alpha \subseteq x_\alpha$ for every $\alpha < \omega_1$ and

(5.29) $I_0 \subseteq \bigcup_{\alpha<\omega_1} N_\alpha$.

The condition (5.28) is realizable by (5.26). Let $N = \bigcup_{\alpha<\omega_1} N_\alpha$ and $I = \kappa \cap N$. Then $I_0 \subseteq I$ by (5.29). So $|I| = \aleph_1$ by (5.27). Since $N \subseteq M_{\alpha_0}$, we have $\sup(I) = \alpha_0$.

Thus the following claim implies that this I is as in the definition of $\mathrm{FRP}^R(\kappa)$ for S, g and $\mathcal{T}$.

Claim 5.2.1. *$I \in \mathcal{T}$.*

$\vdash$ For $\alpha < \omega_1$ there is $A_\alpha \in \mathcal{T} \cap N_{\alpha+1}$ such that

(5.30) $\bigcup(\mathcal{T} \cap N_\alpha) \subseteq A_\alpha$

by (5.28) and elementarity. $\langle A_\alpha : \alpha < \omega_1 \rangle$ is then an increasing sequence in $\mathcal{T}$. Let $A = \bigcup_{\alpha<\omega_1} A_\alpha$. By the ω_1-clubness of $\mathcal{T}$, we have $A \in \mathcal{T}$. By (5.30) and (5.28), we have $I \cap N_\alpha \subseteq A_\alpha \subseteq I$ for all $\alpha < \omega_1$. By (5.29), it follows that $A = I$. $\dashv$ (Claim 5.2.1)

$\square$ (Theorem 5.2)

By the theorem above we have $\mathrm{FRP}^R(\aleph_n) \Leftrightarrow \mathrm{FRP}(\aleph_n)$ for all $n \in \omega \setminus 1$. Thus the test question in this connection would be the following:

Problem 2. *Is* $\mathrm{FRP}^R(\aleph_{\omega+1})$ *equivalent to* $\mathrm{FRP}(\aleph_{\omega+1})$ *?*

The following problem is also still open:

Problem 3. *Does* (5.2) *follow from* FRP *or* FRP^R *?*

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