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CONSTRUCTIONS OF HOLOMORPHIC VERTEX OPERATOR ALGEBRAS USING VIRASORO FRAMES

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1. INTRODUCTION

In this article, we shall report a construction of several holomorphic vertex operator algebras of central charge 24 using Virasoro frames. The Lie algebras associated to their weight one subspaces are of the types $A_1A_3^4$ and A_1D_5, $A_1^3A_7$, $A_1^2C_3D_5$, $A_2^2C_2A_5^2$,$A_3A_7C_3^2$, A_3C_7, $B_3A_9A_4$, $B_4C_6^2$ and B_6C_{10}. These vertex operator algebras correspond to number 7, 10, 18, 19, 26, 33, 35, 40, 48 and 56 in Schelleken's list [Sch]. We shall only report the main idea and the details will appear in another paper.

Holomorphic VOA and Schelleken's list. Let V be a vertex operator algebra (VOA). V is *rational* if all of its admissible modules are completely reducible [DLM1]. A rational vertex operator algebra V is said to be *holomorphic* if it has only one inequivalent irreducible module, namely V itself. It is well-known that a holomorphic VOA must have the central charge c divisible by 8.

Let $V=\oplus_{n\in\mathbb{Z}}V_n$ be a VOA of CFT-type. i.e.,

$$V_n=0 \quad \text{for } n<0 \quad \text{and } \dim V_0=1.$$

Then, the weight one subspace V_1 has a Lie algebra structure and possesses an invariant bilinear form [FLM]. When $c=8$ and 16, it is easy to determine $\dim V_1$ by using the theory of modular forms. The corresponding Lie algebra structure for V_1 can also be determined. When the central charge is 8 or 16, it is not difficult to classifiy all holomorphic VOAs.

The classification of holomorphic vertex operator algebras of central charge 24, on the other hand, is much more complicated. In 1993, Schelleken obtained a partially classification by determining the possible Lie algebra structures for the weight one subspaces. However, only 39 of the 71 cases in his list have been constructed explicitly. Besides, it is still an open question if the Lie algebra structure of V_1 will determine the VOA structure of V uniquely when $c=24$. The most difficult case is probably the case when $V_1=0$. Frenkel-Leposwky-Meurman [FLM] conjectured that such a VOA is isomorphic to the famous moonshine VOA $V^\natural$. This conjecture is one of the most difficult problem in VOA theory and has very little progress in the last 20 years.

Motivated by the work of Schelleken, Montague [Mon] proposed some constructions for 70 of the 71 theories by using the so-called $\mathbb{Z}_2$ and $\mathbb{Z}_3$ orbifold constructions. Unfortunately, the existences of such orbifold constructions have not been established rigorously (at least in mathematical sense).

In this article, we shall discuss an explicit construction of several holomorphic vertex operator algebras of central charge 24 by using Virasoro frames. Our main method is by successive $\mathbb{Z}_2$-orbifoldings on certain lattice type framed VOAs. In general, it is difficult to determine if the Fock space obtained from $\mathbb{Z}_2$-orbifold construction has a VOA structure; however, for a framed VOA, if an involution fixes the Virasoro frame pointwise, then the corresponding $\mathbb{Z}_2$-orbifold construction always gives a VOA (see [LY] and Section 2.2). By this method, we shall construct 10 vertex operator algebras (VOAs) in Schelleken's list [Sch]. They correspond to number 7, 10, 18, 19, 26, 33, 35, 40, 48 and 56 in his list and the Lie algebras associated to their weight one subspaces are of the types $A_1A_3^4$ and A_1D_5, $A_1^3A_7$, $A_1^2C_3D_5$, $A_2^2C_2A_5^2$,$A_3A_7C_3^2$, A_3C_7, $B_3A_9A_4$, $B_4C_6^2$ and B_6C_{10}. For simplicity, only the case $V_1 \cong A_1D_5$ will be given in this article. The other cases can be constructed using the similar method. We believe that these VOAs have not been constructed explicitly before in the literature.

TABLE 1. **Exceptional Framed VOAs**

Number in Schelleken's list	dimension of 1/16-code	T.E. Code	dim V_1	Lie algebra
10	9	[10]	48	$A_{1,2}D_{5,8}$
7	8	[5, 4]	48	$A_{1,2}A_{3,4}^3$
18	8	[8]	72	$A_{1,1}^3A_{7,4}$
7	7	[3, 3, 3]	48	$A_{1,2}A_{3,4}^3$
19	7	[5, 3]	72	$A_{1,1}^2C_{3,2}D_{5,4}$
26	7	[6, 2]	96	$A_{2,1}^2A_{5,2}^2C_{2,1}$
35	7	[7]	120	$A_{3,1}C_{7,2}$
48	6	[6]	192	$B_{4,1}C_{6,1}^2$
40	6	[5, 2]	144	$A_{4,1}A_{9,2}B_{3,1}$
33	6	[4, 3]	120	$A_{3,1}A_{7,2}C_{3,1}^2$
19	6	[3, 3, 2]	96	$A_{2,1}^2A_{5,2}^2C_{2,1}$
56	5	[5]	288	$B_{6,1}C_{10,1}$
48	5	[3, 3]	192	$B_{4,1}C_{6,1}^2$

2. FRAMED VERTEX OPERATOR ALGEBRAS

In this section, we review the notion of framed VOAs from [DGH, M3]. For the details of VOAs, see [FHL, FLM].

Definition 2.1. Let $V = \bigoplus_{n=0}^{\infty} V_n$ be a VOA. An element $e \in V_2$ is called an *Ising vector* or a *Virasoro element of central charge* 1/2 if the subalgebra $\mathrm{Vir}(e)$ generated by e is isomorphic to the simple Virasoro VOA $L(1/2, 0)$. Two Ising vectors $u, v \in V$ are said to be *orthogonal* if $[Y(u, z_1), Y(v, z_2)] = 0$.

Remark 2.2. It is well-known that $L(1/2,0)$ is rational, i.e., all $L(1/2,0)$-modules are completely reducible, and has only three inequivalent irreducible modules $L(1/2,0)$, $L(1/2,1/2)$ and $L(1/2,1/16)$. The fusion rules of $L(1/2,0)$-modules are computed in [DMZ]:

$$
\begin{aligned}
&L(1/2,1/2)\boxtimes L(1/2,1/2)=L(1/2,0), \quad L(1/2,1/2)\boxtimes L(1/2,1/16)=L(1/2,1/16),\\
&L(1/2,1/16)\boxtimes L(1/2,1/16)=L(1/2,0)\oplus L(1/2,1/2).
\end{aligned} \tag{2.1}
$$

Definition 2.3. ([DGH]) A simple VOA (V,ω) is said to be *framed* if there exists a set $\{e^1,\dots,e^n\}$ of mutually orthogonal Ising vectors of V such that $\omega=e^1+\cdots+e^n$. The sub VOA T_n generated by $e^1,\dots,e^n$ is thus isomorphic to $L(1/2,0)^{\otimes n}$ and is called a *Virasoro frame* or simply a *frame* of V. By abuse of notation, we sometimes call the set of Ising vectors $\{e^1,\dots,e^n\}$ a *frame*, also.

2.1. **Structure codes.** Given a framed VOA V with a frame T_n, one can associate two binary codes C and D of length n to V and T_n as follows:

Since $T_n=L(1/2,0)^{\otimes n}$ is rational, V is a completely reducible T_n-module. That is,

$$
V\cong\bigoplus_{h_i\in\{0,\frac12,\frac1{16}\}} m_{h_1,\dots,h_n}L(1/2,h_1)\otimes\cdots\otimes L(1/2,h_n),
$$

where the nonnegative integer $m_{h_1,\dots,h_n}$ is the multiplicity of $L(1/2,h_1)\otimes\cdots\otimes L(1/2,h_n)$ in V. In particular, all the multiplicities are finite. It was also shown in [DMZ] that $m_{h_1,\dots,h_n}$ is at most 1 if all h_i are different from $1/16$.

Definition 2.4. Let $U\cong L(1/2,h_1)\otimes\cdots\otimes L(1/2,h_n)$ be an irreducible module for T_n. We define the $1/16$-word (or τ-word) $\tau(U)$ of U as the binary word $\beta=(\beta_1,\dots,\beta_n)\in\mathbb{Z}_2^n$ such that

$$
\beta_i=\begin{cases}0 & \text{if } h_i=0 \text{ or } 1/2,\\ 1 & \text{if } h_i=1/16.\end{cases} \tag{2.2}
$$

For any $\beta\in\mathbb{Z}_2^n$, denote by V^β the sum of all irreducible submodules U of V such that $\tau(U)=\beta$.

Definition 2.5. Define $D:=\{\beta\in\mathbb{Z}_2^n \mid V^\beta\neq 0\}$. Then D becomes a binary code of length n and V can be written as a sum

$$
V=\bigoplus_{\beta\in D}V^\beta.
$$

For any $c=(c_1,\dots,c_n)\in\mathbb{Z}_2^n$, denote $M^c=m_{h_1,\dots,h_n}L(1/2,h_1)\otimes\cdots\otimes L(1/2,h_n)$ where $h_i=1/2$ if $c_i=1$ and $h_i=0$ elsewhere. Note that $m_{h_1,\dots,h_n}\le 1$ since $h_i\neq 1/16$.

Definition 2.6. Define $C:=\{c\in\mathbb{Z}_2^n \mid M^c\neq 0\}$. Then C also forms a binary code and $V^0=\bigoplus_{c\in C}M^c$.

Summarizing, there exists a pair of even linear codes (C, D) such that

$$V = \bigoplus_{\beta\in D} V^{\beta} \quad \text{and} \quad V^0 = \bigoplus_{c\in C} M^c.$$

The codes (C, D) are called the *structure codes* of a framed VOA V associated to the frame T_n. We also call the code D the $\frac{1}{16}$-code and the code C the $\frac{1}{2}$-code of V with respect to T_n.

The following theorem is also well-known (cf. [DGH, Theorem 2.9] and [M3, Theorem 6.1]):

Theorem 2.7. *1. D is triply even, i.e.,* $\mathrm{wt}(\alpha) \equiv 0 \mod 8$ *for all $\alpha \in D$.*
2. C is even.
3. A framed VOA V is holomorphic if and only if $C = D^{\perp}$.

In [LY], the structure of a general framed VOA has been studied in detail. It was shown that the structure codes (C, D) satisfy some duality conditions. In particular, the following result is established.

Theorem 2.8 (cf. Theorem 10 of [LY]). *Let D be a linear binary code of length $16k, k \in \mathbb{Z}^+$. Then D can be realized as the $\frac{1}{16}$-code of a holomorphic framed VOA of central charge $8k$ if and only if (1) D is triply even and (2) the all-one vector $\mathbf{1} \in D$.*

By the theorem above, the classification of the $\frac{1}{16}$-codes for holomorphic framed VOAs is equivalent to the classification of triply even codes of length $16k$.

It turns out that most triply even codes can be constructed by certain doubling processes or contained in some doublings [BM, DGH] (see also Section 3). However, in [BM], a very special triply even code D^{ex} of length 48 is constructed. It has dimension 9 and minimal weight 16 but it is not contained in any doublings. In this article, we shall constructed explicitly a holomorphic framed VOA which realizes D^{ex} as the $\frac{1}{16}$-code. We shall also construct several other VOAs using the subcodes of D^{ex}.

Our main method is by successive $\mathbb{Z}_2$-orbifoldings on certain lattice type framed VOAs. In general, it is difficult to determine if the Fock space obtained from $\mathbb{Z}_2$-orbifold construction has a VOA structure; however, for framed VOA, if an involution fixes the Virasoro frame pointwise, then the corresponding $\mathbb{Z}_2$-orbifold construction always gives a VOA (cf. [LY]).

2.2. **Miyamoto involutions and $\mathbb{Z}_2$-orbifold construction.** Next, we shall review the definition of Miyamoto involutions [M1] and the notion of $\mathbb{Z}_2$-orbifold construction.

Definition 2.9. Let V be a framed VOA with the structure codes (C, D), where $C, D \subset \mathbb{Z}_2^n$. For a binary word $\beta \in \mathbb{Z}_2^n$, we define

$$\tau_\beta(u) := (-1)^{\langle\alpha,\beta\rangle} u \quad \text{for } u \in V^{\alpha}. \tag{2.3}$$

Then by the fusion rules, τ_β defines an automorphism on V [M1].

Similarly, we can define an automorphism on V^0 by

$$\sigma_\beta(u) := (-1)^{\langle\alpha,\beta\rangle}u \quad \text{for } u \in M^\alpha,$$

where $V^0 = \oplus_{\alpha\in C} M^\alpha$.

Note that σ_β is just an automorphism of V^0. It does not necessarily lift to an automorphism of V. Nevertheless, the following holds.

Theorem 2.10 (cf. Theorem 12 of [LY]). *Let V be a framed VOA with the structure codes (C, D). Let $\xi \cdot \beta = (\xi_1 \cdot \beta_1, \dots, \xi_n \cdot \beta_n)$ be the coordinatewise product of ξ and β. For a binary word $\xi \in \mathbb{Z}_2^n$, there exists $g \in \mathrm{Aut}(V)$ such that $g|_{V^0} = \sigma_\xi$ if and only if $\xi \cdot \beta \in C$ for all $\beta \in D$. Moreover, g has order 2 if $\mathrm{wt}(\xi \cdot \beta) \equiv 0 \mod 4$ for all $\beta \in D$; otherwise, g has order 4.*

2.2.1. *$\mathbb{Z}_2$-orbifold construction.* Next, we shall recall $\mathbb{Z}_2$-orbifold construction for holomorphic framed VOAs.

Let $V = \oplus_{\beta\in D} V^\beta$ be a holomorphic framed VOA with the structure codes (C, D). For any $\delta \in \mathbb{Z}_2^n \setminus C$, denote

$$D^0 = \{\beta \in D \mid \langle\beta, \delta\rangle = 0\} \quad \text{and} \quad D^1 = \{\beta \in D \mid \langle\beta, \delta\rangle \neq 0\}.$$

Then the Miyamoto involution τ_δ has order 2 and the fixed point subspace $V^{\tau_\delta} = \oplus_{\beta\in D^0} V^\beta$.

Define

$$\tilde{V}(\tau_\delta) = \begin{cases} \Big(\bigoplus_{\beta\in D^0} V^\beta\Big) \oplus \Big(\bigoplus_{\beta\in D^1} M_{\delta+C} \boxtimes_{M_C} V^\beta\Big) & \text{if } \mathrm{wt}(\delta) \text{ is odd,} \\ \Big(\bigoplus_{\beta\in D^0} V^\beta\Big) \oplus \Big(\bigoplus_{\beta\in D^0} M_{\delta+C} \boxtimes_{M_C} V^\beta\Big) & \text{if } \mathrm{wt}(\delta) \text{ is even,} \end{cases}$$

where $\boxtimes_{M_C}$ denotes the fusion product over M_C.

Theorem 2.11 (cf. [LY]). *$\tilde{V}(\tau_\delta)$ is a holomorphic framed VOA. Moreover, the structure codes of $\tilde{V}(\tau_\delta)$ are given by (C, D) if $\mathrm{wt}(\delta)$ is odd and $(C\cup(\delta+C), D^0)$ if $\mathrm{wt}(\delta)$ is even.*

2.2.2. *σ-type involutions.* Next, let us consider another kind of $\mathbb{Z}_2$-orbifold construction.

Let $\xi \in \mathbb{Z}_2^n \setminus D$ such that $D' = \langle D, \xi\rangle$ is still triply even. Then for any $\beta, \gamma \in D$, it is clear that $\mathrm{wt}(\xi \cdot \beta) \equiv 0 \mod 4$ and $\mathrm{wt}(\xi \cdot \beta \cdot \gamma) \equiv 0 \mod 2$, i.e., $\xi \cdot \beta \in D^\perp = C$.

Therefore, by Theorem 2.10, there exists an automorphism $g \in \mathrm{Aut}(V)$ of order 2 such that $g = \sigma_\xi$. In this case, g fixes T and stabilizes each V^β for $\beta \in D$. Let

$$V^{\beta,+} = \{v \in V^\beta \mid gv = v\} \quad \text{and} \quad V^{\beta,-} = \{v \in V^\beta \mid gv = -v\}.$$

Set $C^0 = \{\alpha \in C \mid \langle\alpha, \xi\rangle = 0\}$ and $C^1 = \{\alpha \in C \mid \langle\alpha, \xi\rangle = 1\}$. Then we have $V^{0,+} = \{u \in M_C \mid \sigma_\xi u = u\} = M_{C^0}$ and $V^{0,-} = M_{C^1}$. Moreover, $V^{\beta,-} = V^{\beta,+} \boxtimes_{M_{C^0}} M_{C^1}$. Note also that $C = C^0 \cup C^1$ and $[C : C^0] = 2$.

Notation 2.12. For any $\alpha, \beta \in \mathbb{Z}_2^n$, we denote

$$L_{\frac{1}{2}}(\alpha) := L(\frac{1}{2}, \frac{1}{2}\alpha_1) \otimes \cdots \otimes L(\frac{1}{2}, \frac{1}{2}\alpha_n),$$
$$L_{\frac{1}{16}}(\beta) := L(\frac{1}{2}, \frac{1}{16}\beta_1) \otimes \cdots \otimes L(\frac{1}{2}, \frac{1}{16}\beta_n).$$

In this notation, every irreducible T-module with $\frac{1}{16}$-word β can be written as

$$L_{\frac{1}{16}}(\beta) \boxtimes_T L_{\frac{1}{2}}(\gamma \cdot (\mathbf{1} + \beta)) \qquad \text{for some } \gamma \in C.$$

Let U be an irreducible M_{C^0}-module which contains $L_{\frac{1}{16}}(\xi)$ as a T-submodule. Define

$$V^T(g) = \bigoplus_{\beta \in D} V^\beta \boxtimes_{M_{C^0}} U = \bigoplus_{\beta \in D} \left((V^{\beta,+} \boxtimes_{M_{C^0}} U) \oplus (V^{\beta,-} \boxtimes_{M_{C^0}} U) \right). \tag{2.4}$$

Then by [LY][Theorem 1], $V^T(g)$ is a g-twisted module for V. Moreover, the weights of $V^{\beta,-} \boxtimes_{M_{C^0}} U$ are integral if $\mathrm{wt}(\xi) \equiv 8 \mod 16$ and are in $\frac{1}{2} + \mathbb{Z}$ if $\mathrm{wt}(\xi) \equiv 0 \mod 16$.

Now let $U^\beta = V^{\beta,+}$ and

$$U^{\xi+\beta} = \begin{cases} V^{\beta,-} \boxtimes_{M_{C^0}} U & \text{if } \mathrm{wt}(\xi) \equiv 8 \mod 16 \\ V^{\beta,+} \boxtimes_{M_{C^0}} U & \text{if } \mathrm{wt}(\xi) \equiv 0 \mod 16. \end{cases}$$

for any $\beta \in D$

The following theorem also follows immediately from [LY][Theorem 8].

Theorem 2.13 ([LY]). *Let $D' = \langle D, \xi \rangle$. Then $\tilde{V}(g) = \bigoplus_{\beta \in D'} U^\beta$ is a holomorphic framed VOA. Moreover, the structure codes associated to the frame T for $\tilde{V}$ are given by (C^0, D').*

3. $\mathbb{Z}_4$-codes and framed VOAs

In this section, we shall recall a basic construction of framed VOAs from codes over $\mathbb{Z}_4$ (cf. [DGH]). In fact, almost all known examples of framed VOAs are constructed by this method.

Let $\mathcal{C}$ be a self-orthogonal $\mathbb{Z}_4$-code such that the Euclidean weights of all elements of $\mathcal{C}$ is divisible by 8. Define

$$A_4(\mathcal{C}) = \frac{1}{2} \{(x_1, \ldots, x_n) \in \mathbb{Z}^n | \, (x_1, \ldots, x_n) \in \mathcal{C} \mod 4\}.$$

Then $A_4(\mathcal{C})$ is an even lattice. It is also well-known that $A_4(\mathcal{C})$ is unimodular if and only if $\mathcal{C}$ is self-dual. Note that if $\mathcal{C} = 0$, then $A_4(\mathcal{C}) = (2\mathbb{Z})^n \cong (\sqrt{2}A_1)^n$.

In [DMZ] and [M2], it was shown that the lattice VOA $V_{\sqrt{2}A_1}$ is framed and

$$V_{\sqrt{2}A_1} \cong L(\frac{1}{2}, 0) \otimes L(\frac{1}{2}, 0) \oplus L(\frac{1}{2}, \frac{1}{2}) \otimes L(\frac{1}{2}, \frac{1}{2})$$

as an $L(\frac{1}{2}, 0) \otimes L(\frac{1}{2}, 0)$-module. Hence the lattice VOA

$$V_{A_4(\mathcal{C})} \supset V_{A_4(0)} \cong (V_{\sqrt{2}A_1})^{\otimes n}$$

is also framed for any self-orthogonal $\mathbb{Z}_4$-code $\mathcal{C}$.

Now let us study the structure codes for the lattice VOA $V_{A_4(\mathcal{C})}$.
Let $\mathcal{C}$ be a self-dual $\mathbb{Z}_4$ code. Denote

$$\mathcal{C}_0 = \{(\alpha_1, \dots, \alpha_n) \in \{0,1\}^n \mid (2\alpha_1, \dots, 2\alpha_n) \in \mathcal{C}\},$$
$$\mathcal{C}_1 = \{\alpha \in \{0,1\}^n \mid \alpha \equiv \beta \mod 2 \text{ for some } \beta \in \mathcal{C}\}.$$

These codes $\mathcal{C}_0$ and $\mathcal{C}_1$ are called *torsion* and *residue* codes, respectively. Then both $\mathcal{C}_0$ and $\mathcal{C}_1$ are even binary codes. Moreover, $\mathcal{C}_1$ is doubly even and $\mathcal{C}_0 = \mathcal{C}_1^{\perp}$.

Now let us define three linear maps $d : \mathbb{Z}_2^n \to \mathbb{Z}_2^{2n}$, $\ell : \mathbb{Z}_2^n \to \mathbb{Z}_2^{2n}$ and $r : \mathbb{Z}_2^n \to \mathbb{Z}_2^{2n}$ such that

$$\begin{aligned} d(a_1, a_2, \dots, a_n) &= (a_1, a_1, a_2, a_2, \dots, a_n, a_n), \\ \ell(a_1, a_2, \dots, a_n) &= (a_1, 0, a_2, 0, \dots, a_n, 0), \\ r(a_1, a_2, \dots, a_n) &= (0, a_1, 0, a_2, \dots, 0, a_n), \end{aligned} \tag{3.1}$$

for any $(a_1, a_2, \dots, a_n) \in \mathbb{Z}_2^n$.

Proposition 3.1 (cf. [DGH]). *Let $\mathcal{C}$ be a self-dual $\mathbb{Z}_4$-code and $\mathcal{C}_0$ and $\mathcal{C}_1$ defined as above. Then the structure codes of the lattice VOA $V_{A_4(\mathcal{C})}$ are given by*

$$D = d(\mathcal{C}_1) \quad \textit{and} \quad C = D^{\perp} = \langle H, \ell(\mathcal{C}_0)\rangle,$$

where $H = d(\mathbb{Z}_2^n) = \{(00), (11)\}^n$ and $\langle H, \ell(\mathcal{C}_0)\rangle$ is the code generated by H and $\ell(\mathcal{C}_0)$.

Now set $\xi = (1010 \dots 10)$. Then for any $\beta = d(\alpha) \in d(\mathcal{C}_1) = D$, we have

$$\xi \cdot \beta = \ell(\alpha) \in \ell(C_1) \subset \ell(C_0) \subset C \quad \text{and} \quad \mathrm{wt}(\xi \cdot \beta) \equiv 0 \mod 4.$$

Therefore, by Theorem 2.10, there exists an automorphism $g \in \mathrm{Aut}(V_{A_4(\mathcal{C})})$ of order 2 such that $g = \sigma_\xi$. In fact, g is conjugate to the automorphism θ, which is the lift of the (-1)-map on the lattice $A_4(\mathcal{C})$, since σ_ξ acts as -1 on the weight one subspace of M_H, which generates the Heisenberg sub VOA in $V_{A_4(\mathcal{C})}$. (cf. [DGH, FLM])

By Theorem 2.13, we can construct the g-orbifold VOA $\tilde{V}_{A_4(\mathcal{C})}(g) = \oplus_{\beta \in D'} U^\beta$.

Theorem 3.2 ([LY]). *$\tilde{V}_{A_4(\mathcal{C})}$ is a holomorphic framed VOA. Moreover, the structure codes associated to the frame T for $\tilde{V}_{A_4(\mathcal{C})}$ are given by (C^0, D').*

Definition 3.3. Let C be a binary code of length n. We shall define

$$\mathcal{D}(C) = \langle d(C), (10)^n \rangle$$

to be the code generated by $d(C)$ and $(10)^n$. We call the code $\mathcal{D}(C)$ the *extended doubling (or simply the doubling)* of C.

Remark 3.4. By the discussion above, for a given Type II $\mathbb{Z}_4$-code $\mathcal{C}$, one can construct a framed VOA $\tilde{V}_{A_4(\mathcal{C})}$ whose $\frac{1}{16}$-code is given by $\mathcal{D}(C)$.

4. An exceptional [48, 9] triply even code

In this section, we shall recall the properties of the exceptional [48, 9] code constructed by [BM], which is not equivalent to any doublings.

4.1. **Triangular graph.** Let $X = \{1, 2, \ldots, 10\}$ be a set of 10 elements and let

$$\Omega := \binom{X}{2} = \{\ \{i, j\} \mid \{i, j\} \subset X\}$$

be the set of all 2-element subset of X. Then $|\Omega| = \binom{10}{2} = 45$.

The triangular graph on X is a graph whose vertex set is Ω and two vertices $S, S' \in \Omega$ are joined by an edge if and only if $|S \cap S'| = 1$.

We shall denote by T_{10} the binary code generated by the row vectors of the incidence matrix of the triangular graph on X.

Remark 4.1. Note that the entries of an incidence matrix are either 0 or 1 and we shall view 0 and 1 as integers modulo 2.

For $\{i, j\} \in \Omega$, let $\gamma_{\{i,j\}}$ be the binary word supported at $\{\{k, \ell\} \mid |\{i, j\} \cap \{k, \ell\}| = 1\}$, i.e., the set of all vertices joining to $\{i, j\}$. Note that $supp(\gamma_{\{i,j\}}) = \{\{i, k\}, \{i, k\} \mid k \in X \setminus \{i, j\}\}$ and $wt(\gamma_{\{i,j\}}) = 16$.

Lemma 4.2. *For any $i, j, k, \ell \in X$, we have*

(1) $\gamma_{\{i,j\}} + \gamma_{\{i,k\}} = \gamma_{\{j,k\}}$, *and*
(2) $wt(\gamma_{\{i,j\}} + \gamma_{\{k,\ell\}}) = 24$ *if* $\{i, j\} \cap \{k, \ell\} = \emptyset$.

Lemma 4.3. *Let $X = \{i_1, j_1\} \cup \{i_2, j_2\} \cup \cdots \cup \{i_5, j_5\}$ be a partition of X. Then we have $\gamma_{\{i_1,j_1\}} + \gamma_{\{i_2,j_2\}} + \cdots + \gamma_{\{i_5,j_5\}} = 0$.*

Lemma 4.4. *The set $\{\gamma_{\{1,j\}} \mid j = 2, 3, 4, \ldots, 9\}$ is a basis of T_{10}. In particular,* $\dim T_{10} = 8$.

Now let $\iota : \mathbb{Z}_2^{45} \to \mathbb{Z}_2^{48}$ be defined by $\iota(\alpha) = (\alpha, 0, 0, 0)$. Then we can embed T_{10} into $\mathbb{Z}_2^{48}$ using ι.

Definition 4.5. We shall denote by D^{ex} the binary code generated by $\iota(T_{10})$ and the all-one vector $\mathbf{1}$ in $\mathbb{Z}_2^{48}$. Clearly, $\dim D^{ex} = 9$.

Theorem 4.6 (cf. [BM])**.** *The binary code D^{ex} is a maximal triply even code of length 48, that means, it is not properly contained in any triply even code of length 48. Moreover, the weight enumerator of D^{ex} is given by $1 + 45x^{16} + 420x^{24} + 45x^{32} + x^{48}$.*

Remark 4.7. Note that $\{\gamma_{\{i,j\}} \mid \{i, j\} \in \Omega\}$ is exactly the set of all weight 16 vectors in D^{ex}.

Let D^{ex} be the triply even code defined in Definition 4.5 and let $C^{ex} = (D^{ex})^{\perp}$ be the dual code. Then $\dim C^{ex} = 39$ and the weight enumerator of C^{ex} is given by $1 + 6x^2 + 342x^4 + 4110x^6 + 23391x^8 + 60396x^{10} + 85652x^{12} + 60396x^{14} + 23391x^{16} + 4110x^{18} + 342x^{20} + 6x^{22} + x^{24}$.

Notation 4.8. For $\alpha = (\alpha_1, \cdots, \alpha_n), \beta = (\beta_1, \cdots, \beta_n) \in \mathbb{Z}_2^n$, we shall denote by $\alpha \cdot \beta$ the coordinatewise product of α and β, i.e., $\alpha \cdot \beta = (\alpha_1\beta_2, \ldots, \alpha_n\beta_n)$. We shall also use p_β to denote the natural projection of $\mathbb{Z}_2^{48}$ to the support of β, where $\beta \in \mathbb{Z}_2^{48}$.

Notation 4.9. For any positive integer n, let $\mathcal{E}_n$ be the subcode of $\mathbb{Z}_2^n$ consisting of all even codewords. We also denote the extended Hamming [8,4,4] code by H_8 and denote by d_{16}^+ the doubly even self-dual code of length 16 generated by

$$\begin{pmatrix} 1111\,0000\,0000\,0000 \\ 0011\,1100\,0000\,0000 \\ 0000\,1111\,0000\,0000 \\ 0000\,0011\,1100\,0000 \\ 0000\,0000\,1111\,0000 \\ 0000\,0000\,0011\,1100 \\ 0000\,0000\,0000\,1111 \\ 1010\,1010\,1010\,1010 \end{pmatrix}.$$

The following two lemmas can be proved easily by direct calculation.

Lemma 4.10. *Let $\beta \in D^{ex}$ with $\mathrm{wt}(\beta) = 16$ and let*

$$C_\beta = \{\alpha \in C^{ex} \mid supp(\alpha) \subset supp(\beta)\}.$$

Then we have

(1) $p_\beta(C_\beta) \cong d_{16}^+$. *In particular, it is a doubly even self-dual code.*
(2) $p_{\mathbf{1}+\beta}(C^{ex}) \cong \mathcal{E}_{32}$.

Lemma 4.11. *Let $\beta \in D^{ex}$ and $\mathrm{wt}(\beta) = 16$. Let U be any irreducible $M_{C^{ex}}$-module U with integral weights and $\tau(U) = \beta$. Then we have*

$$U = \bigoplus_{\gamma \in C^{ex}/C_\beta} L_{\frac{1}{16}}(\beta) \underset{T}{\boxtimes} L_{\frac{1}{2}}(\gamma \cdot (\mathbf{1} + \beta))$$

as a T-submodule. In particular, $\dim U_1 = 1$ *and* $U_1 \neq 0$.

5. Constructions of VOA

In this section, we shall give an explicit construction of a VOA V^{ex}, whose $\frac{1}{16}$-code is isomorphic to D^{ex}. The method is by successive orbifoldings from certain lattice VOAs. First, we shall find a subcode of D^{ex} which is isomorphic to a double of some doubly even code.

5.1. **Subcodes of D^{ex}.**

In this subsection, we shall study some subcodes of D^{ex}.

Notation 5.1. For any binary code C and a positive integer n, we denote

$$C(n) = \{\alpha \in C \mid \mathrm{wt}(\alpha) = n\}.$$

Notation 5.2. Let $\lambda = (\lambda_1, \dots, \lambda_m)$ be a partition of 10. Let $X_1, \dots, X_m$ be subsets of X such that $X = \cup_{i=1}^m X_i$ and $|X_i| = \lambda_i$ for $1 \le i \le m$.

Set $W_\lambda = \{\gamma_{\{i,j\}} \mid \{i,j\} \subset X_k, 1 \le k \le m\}$. Then W_λ and the all-one vector $\mathbf{1} \in \mathbb{Z}_2^{48}$ generates a subcode of D^{ex}. We shall denote this code by $D_{[\lambda_1,\dots,\lambda_m]}$ or simply by $[\lambda_1, \dots, \lambda_m]$. Note that $D^{ex} = D_{[10]} = [10]$.

We also define $C_{[\lambda_1,\dots,\lambda_m]} := (D_{[\lambda_1,\dots,\lambda_m]})^\perp$ and denote $K_{[\lambda_1,\dots,\lambda_m]} = supp(D_{[\lambda_1,\dots,\lambda_m]}(16))$ and $K'_{[\lambda_1,\dots,\lambda_m]} = \Omega \setminus K_{[\lambda_1,\dots,\lambda_m]}$.

Lemma 5.3. *Let D be a binary code of length $2n$. Then $D \cong d(C)$ for some binary code C if and only if there is an involution $g \in Sym_{2n}$ which acts fixed point free on $\{1, 2, \dots, 2n\}$ but fixes D pointwise.*

Lemma 5.4. *The subcode $[4,2,2,2]$ is isomorphic to $d(C)$ where C is generated by*

$$\begin{pmatrix} 1&1&1&0&1&0&1&1&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&1&0&0&1&1&0&0&0&0&1&1&0&1&0&0&1&0&0\\ 0&1&0&1&0&1&0&0&1&0&0&0&0&0&1&0&0&0&0&1&1&1&0&0\\ 0&0&1&0&0&0&0&1&1&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0\\ 1&1&1&1&0&1&0&0&1&0&0&0&1&0&0&1&0&0&0&0&0&0&0&0\\ 1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \end{pmatrix}$$

and the weight enumerator of C is $1 + 9x^8 + 44x^{12} + 9x^{16} + x^{24}$.

Now let $\mathcal{C}$ be a type II self-dual $\mathbb{Z}_4$-code such that the residue code $\mathcal{C}_1 = C$. We shall note that such a code $\mathcal{C}$ does exist.

Lemma 5.5. *For any $\alpha \in C(8)$, we have*

$$|\{\beta \in \mathcal{C} \mid \beta \equiv \alpha \mod 2, Ewt(\beta) = 8\}| = |\{\gamma \in C^\perp | supp(\gamma) \subset supp(\alpha)\}| = 8,$$

where $Ewt(\beta)$ denotes the Euclidean weight of β.

Proof. Let $\mathcal{K} = \{\beta \in \mathcal{C} \mid \beta \equiv \alpha \mod 2, Ewt(\beta) = 8\}$. Then for any $\beta, \beta' \in \mathcal{K}$, $\beta - \beta' \equiv 0 \mod 2$. Thus, $\beta - \beta' \in \mathcal{C}_0 = \mathcal{C}_1^\perp = C^\perp$.

Since β, β' both have Euclidean weight 8 and $\beta \equiv \beta' \equiv \alpha \mod 2$, both β and β' are supported at $supp(\alpha)$. Hence

$$\mathcal{K} = \beta_0 + \{2\gamma \mid \gamma \in (C^\perp)_\alpha\}$$

and we have $|\mathcal{K}| = |C_\alpha^\perp| = 2^3$. ∎

Proposition 5.6. *Let $\mathcal{C}$ be a type II self-dual $\mathbb{Z}_4$-code such that the residue code $\mathcal{C}_1 = C$. Let*

$$N = A_4(\mathcal{C}) = \frac{1}{2}\{(x_1, \dots, x_{24}) \in \mathbb{Z}^{24} \mid (x_1, \dots, x_{24}) \in \mathcal{C} \mod 4\}.$$

Then the kissing number of N is 96 and thus N is isometric to the Niemeier lattice of type A_3^8.

Proof. The kissing number of N is equal to the number of codewords with Euclidean weight 8. Thus, by Lemma 5.5, it is equal to

$$|C(8)| \cdot 8 + |C^\perp(2)| \cdot 4 = 9 \times 8 + 6 \times 4 = 96$$

as desired. ∎

Theorem 5.7. *There is a Virasoro frame T of $V_{N(A_3^8)}$ such that the $1/16$-code of $V_{N(A_3^8)}$ is isomorphic to $[4,2,2,2]$.*

Let $\xi_0 = \gamma_{\{1,2\}} + \gamma_{\{3,4\}}$, $\xi_1 = \gamma_{\{1,2\}}$ and $\xi_2 = \gamma_{\{7,8\}}$. Set

$$\begin{aligned}
\mathcal{D}^0 &= \langle [4,2,2,2], \xi_0 \rangle, \cong \mathcal{D}(C),\\
\mathcal{D}^1 &= \langle \mathcal{D}^0, \xi_1 \rangle \cong [6,4],\\
\mathcal{D}^2 &= \langle \mathcal{D}^1, \xi_2 \rangle \cong [10] \cong D^{ex}.
\end{aligned}$$

By Theorem 2.11, there exists holomorphic framed VOAs

$$\begin{aligned}
V(\mathcal{D}^0) &= \bigoplus_{\beta\in\mathcal{D}^0} W^\beta \cong \tilde{V}_{N(A_3^8)},\\
V(D_{[6,4]}) = V(\mathcal{D}^1) &= \bigoplus_{\beta\in\mathcal{D}^1} U^\beta,\\
V(D^{ex}) = V(\mathcal{D}^2) &= \bigoplus_{\beta\in\mathcal{D}^2} V^\beta
\end{aligned}$$

such that

$$\begin{aligned}
U^\beta &= V^\beta \oplus V^\beta \boxtimes M_{\delta_2+C_2} &&\text{for } \beta\in\mathcal{D}^1 \text{ and}\\
W^\beta &= U^\beta \oplus U^\beta \boxtimes M_{\delta_1+C^1} &&\text{for } \beta\in\mathcal{D}^0,
\end{aligned}$$

where $C_0 = C_1 \cup (\delta_1 + C_1)$ and $C_1 = C_2 \cup (\delta_2 + C_2)$.

5.2. **Lie algebra structure for $V(D^{ex})_1$.** Next we shall determine the Lie algebra structure for the weight one subspaces. First we shall recall a theorem by Dong and Mason.

Theorem 5.8 (Dong-Mason). *Let V be a C_2-cofinite holomorphic VOA of CFT type. Suppose the central charge c of V is 24. Then the Lie algebra V_1 has rank less than or equal to 24 and is either abelian (including 0) or semi-simple.*

Now denote

$$V^{ex} = V(D^{ex}) = \bigoplus_{\beta\in D^{ex}} V^\beta.$$

For any $\beta \in D^{ex}$ with $\mathrm{wt}(\beta) = 16$, let v_β be a highest weight vector of V^β such that $(v_\beta, v_\beta) = 1$. Since $\dim(V_1^\beta) = 1$, v_β is unique up to a multiplication of ± 1.

For $\alpha \in C^{ex}(2)$, let q_α be a highest weight vector of $L_{\frac12}(\alpha)$ with $(q_\alpha, q_\alpha) = 1$. Again, q_α is unique up to a multiplication of ± 1.

Proposition 5.9. *The set $\{v_\beta \mid \beta \in D^{ex}(16)\} \cup \{q_\alpha \mid \alpha \in C^{ex}(2)\}$ forms a basis for V_1. In particular,* $\dim V_1 = 48$.

Proof. First we shall note that $(V^\beta)_1 = 0$ for $\mathrm{wt}(\beta) > 16$ and is spanned by v_β if $\mathrm{wt}(\beta) = 16$. Moreover, $V_1^0 = (M_{C^{ex}})_1$ is spanned by $\{q_\alpha | \alpha \in C^{ex}(2)\}$. Since $V_1 = \oplus_{\beta\in D}(V^\beta)_1$, we have the desired result. ∎

Note that $|supp(D^{ex}(16))| = 45$, $|supp(C^{ex}(2))| = 3$, and

$$supp(D^{ex}(16)) \cap supp(C^{ex}(2)) = \emptyset.$$

Therefore, for any $\beta \in D^{ex}(16)$ and $\alpha \in C^{ex}(2)$, we have

$$(q_\alpha)_0 v_\beta = 0,$$

since $M_\alpha \boxtimes_T L_{\frac{1}{16}}(\beta)$ has the minimal weight $\gneqq 1$.

Now let $\mathfrak{g}_1 = span\{v_\beta \mid \beta \in D^{ex}(16)\}$ and $\mathfrak{g}_2 = span\{q_\alpha \mid \alpha \in C^{ex}(2)\}$.

Lemma 5.10. *The Lie algebra $\mathfrak{g}_1$ commutes with $\mathfrak{g}_2$ and hence $V_1^{ex} \cong \mathfrak{g}_1 \oplus \mathfrak{g}_2$.*

By the fusion rules, it is easy to verify the following two lemmas.

Lemma 5.11. *For $\alpha, \alpha' \in C^{ex}(2)$, we have*

$$(q_\alpha)_0(q_{\alpha'}) = \begin{cases} \pm q_{\alpha+\alpha'} & \text{if } \mathrm{wt}(\alpha+\alpha') = 2, \\ 0 & \text{if } \mathrm{wt}(\alpha+\alpha') \neq 2. \end{cases}$$

Lemma 5.12. *Let $\beta, \beta' \in D^{ex}(16)$. Then we have*

$$(v_\beta)_0(v_{\beta'}) = \begin{cases} \pm v_{\beta+\beta'} & \text{if } |\beta\cap\beta'| = 8, \\ 0 & \text{if } |\beta\cap\beta'| = 4 \text{ or } 16. \end{cases}$$

In particular, $span\{v_\beta \mid \beta \in D^{ex}(16)\}$ forms a Lie subalgebra for V_1.

Lemma 5.13. *The Lie subalgebra $\mathfrak{g}_2$ generated by $\{q_\alpha \mid \alpha \in C^{ex}(2)\}$ is isomorphic to $sl_2(\mathbb{C})$.*

Proof. Since $\dim \mathfrak{g}_2 = 3$ and $\mathfrak{g}_2$ has no non-trivial ideals, it is clear that $\mathfrak{g}_2 \cong sl_2(\mathbb{C})$. ∎

Lemma 5.14. *Let $\beta_1 = \gamma_{\{1,6\}}, \beta_2 = \gamma_{\{2,7\}}, \beta_3 = \gamma_{\{3,8\}}, \beta_4 = \gamma_{\{4,9\}}, \beta_5 = \gamma_{\{5,10\}}$ and let $\mathfrak{h} = span\{v_{\beta_1}, \dots, v_{\beta_5}\}$. Then $\mathfrak{h}$ is a maximal abelian subalgebra of $\mathfrak{g}_1 = span\{v_\beta \mid \beta \in D^{ex}(16)\}$.*

Proof. By Lemma 5.12, it is clear that $\mathfrak{h}$ is abelian. Let $u = \sum_\beta a_\beta v_\beta \in \mathfrak{g}_1$ such that $[\mathfrak{h}, u] = 0$. Then

$$(v_{\beta_i})_0 u = \sum_{|\alpha\cap\beta_i|=1} a_\alpha v_\alpha = 0 \quad \text{for all } i = 1,2,3,4,5.$$

It implies that $a_\alpha = 0$ unless $\alpha = \beta_1, \cdots, \beta_5$ and hence $u \in \mathfrak{h}$. ∎

Theorem 5.15. *The Lie algebra $\mathfrak{g}_1$ spanned by $\{v_\beta \mid \beta \in D^{ex}(16)\}$ is isomorphic to $o_{10}(\mathbb{C})$, i.e., of the type D_5.*

Proof. Since $\mathfrak{g}_1$ is semi-simple, has rank 5 and $\dim \mathfrak{g}_1 = 45$, the only possibility is $o_{10}(\mathbb{C})$. ∎

Theorem 5.16. *The Lie algebra V_1^{ex} is isomorphic to $sl_2(\mathbb{C}) \oplus o_{10}(\mathbb{C})$, i.e., of the type A_1D_5.*

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