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CURVATURE OF SCHEMES OF FINITE VALENCY

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Contents

1. Definition of (association) schemes
2. The scheme ring
3. A theorem of Muzychuk and Ponomarenko
4. A theorem of Harvey Blau
5. The curvature of a scheme
6. A scheme of negative curvature
7. Schemes of positive curvature

1. Definition of association schemes. Let X be a set. We write 1_X to denote the set of all pairs (x,x) with $x \in X$. For each subset r of the cartesian product $X \times X$, we define r^* to be the set of all pairs (y,z) with $(z,y) \in r$. Whenever x stands for an element in X and r for a subset of $X \times X$, we define xr to be the set of all elements y in X such that $(x,y) \in r$.

Let S be a partition of $X \times X$ with $1_X \in S$, and assume that $s^* \in S$ for each element s in S. The set S is called an *association scheme* or simply a *scheme on* X if, for any three elements p, q, and r in S, there exists a cardinal number a_{pqr} such that, for any two elements y in X and z in yr, $|yp \cap zq^*| = a_{pqr}$. This last condition is called the *regularity condition.*

Assume that X is finite, and let S be a scheme on X. For each element s in S, we set $n_s := a_{ss^*1}$ and call this integer the *valency* of s. The integer $|X|$ is called the *valency* of S.

Since the empty set is not element of S, we have $1 \leq n_s$ for each element s in S. As a consequence, $|S| \leq |X|$. The scheme S is called *thin* if all elements of S have valency 1. Note that S is thin if and only if $|S| = |X|$.

2. The scheme ring. Let X be a finite set, let R be a field, and define RX to be the set of all maps from X to R. Then RX is a right R-module with basis X.

Let S be a scheme on X, and let s be an element in S. We define σ_s to be the uniquely determined R-endomorphism of RX which satisfies

$$\sigma_s(x) := \sum_{y \in xs} y$$

for each element x in X. The span of the set $\{\sigma_s \mid s \in S\}$ in $\mathrm{End}_R(RX)$ will be denoted by RS.

The regularity condition is equivalent to the fact that RS is a ring with respect to composition. This ring is called the *scheme ring* of S over R. The right RS-module RX is called the *standard module* of RS.

Now we assume that the characteristic of R does not divide any of the positive integers $|s|$ with $s \in S$. By [10; Theorem 9.1.5(iii)], this implies that RS is semisimple. We shall also assume that R is algebraically closed. If we speak about characters of S, we mean characters of RS.

By χ_X we denote the character of S afforded by (the standard module) RX. We call this character the *standard character* of S. By ρ we denote the *regular character* of S, that is the character of S afforded by the right RS-module RS. The linear map from RS to R that maps each element σ_s to n_s is a ring homomorphism. This homomorphism will be denoted by 1_S and is usually called the *principal character* of S.

For each irreducible character χ of S, we define m_χ to be the multiplicity of χ in χ_X. Recall that, for each irreducible character χ of S, $\chi(1)$ is the multiplicity of χ in ρ; cf. [10; Corollary 8.6.5].

By definition, we have $1_S(1) = 1$, and from [10; Lemma 9.1.8(ii)] we know that $m_{1_S} = 1$. It is also easy to see that, for each element x in X, there exists an RS-module monomorphism from RS to RX mapping each element σ_s with $s \in S$ to the sum of the elements in xs. Thus, $\chi(1) \leq m_\chi$ for each irreducible character χ of S.

For each irreducible character χ of S, the rational number

$$\frac{m_\chi}{\chi(1)}$$

will be called the *covalency* of χ.

According to what we saw before, the covalency of 1_S is 1, and, generally, irreducible characters of S have covalency at least 1. Note also that S is thin if and only if all irreducible characters of S have covalency 1.

In the following, the set of all irreducible characters of S will be denoted by $\mathrm{Irr}(S)$.

3. A theorem of Muzychuk and Ponomarenko. Let X be a finite set, let S be a scheme on X, and let R be an algebraically closed field the characteristic of which does not divide $|s|$ for any of the elements s in S.

The scheme S is called *pseudocyclic* if any two elements in $\mathrm{Irr}(S) \setminus \{1_S\}$ have the same covalency; cf. [9].

There are two large classes of pseudocyclic schemes. Firstly, if S thin, we have $\chi(1) = m_\chi$ for each element χ in $\mathrm{Irr}(S)$, so that all elements in $\mathrm{Irr}(S) \setminus \{1_S\}$ have covalency 1. Secondly, one can show that S is pseudocyclic if $S \cong G//H$ for some Frobenius group G with one-point stabilizer H.[1]

In [9; Theorem 2.2], Mikhail Muzychuk and Ilia Ponomarenko proved the following theorem for which we shall give a slightly different proof here.

Theorem 1 *Assume S to be pseudocyclic. Then*

$$\frac{|X|-1}{|S|-1} = \frac{m_\chi}{\chi(1)} = n_s = \rho(\sigma_s) + 1 = \sum_{r \in S} a_{rsr} + 1 = \sum_{r \in S} a_{rr^*s} + 1$$

for any two elements s in $S \setminus \{1\}$ and χ in $\mathrm{Irr}(S) \setminus \{1_S\}$.

[1] See [10; Section 4.1] for the definition of the quotient scheme $G//H$.

Proof. For the standard character χ_X of S we have

$$\sum_{\chi\in\mathrm{Irr}(S)\setminus\{1_S\}} m_\chi\chi(1) = \chi_X(1) - 1_S(1) = |X| - 1.$$

For the regular character ρ of RS we have

$$\sum_{\chi\in\mathrm{Irr}(S)\setminus\{1_S\}} \chi(1)\chi(1) = \rho(1) - 1_S(1) = |S| - 1.$$

Thus,

$$\frac{m_\chi}{\chi(1)} = \frac{|X|-1}{|S|-1}$$

for each non-principal irreducible character χ of S.

Among the elements in $S \setminus \{1\}$ we fix s such that n_s is as small as possible. Then

$$\sum_{\chi\in\mathrm{Irr}(S)\setminus\{1_S\}} m_\chi\chi(\sigma_s) = \chi_X(\sigma_s) - 1_S(\sigma_s) = -n_s$$

and

$$\sum_{\chi\in\mathrm{Irr}(S)\setminus\{1_S\}} \chi(1)\chi(\sigma_s) = \rho(\sigma_s) - 1_S(\sigma_s) = \rho(\sigma_s) - n_s.$$

Thus,

$$\frac{m_\chi}{\chi(1)} = \frac{n_s}{n_s - \rho(\sigma_s)}.$$

Together this yields

$$\frac{|X|-1}{|S|-1} = \frac{n_s}{n_s - \rho(\sigma_s)},$$

so that

$$(|X|-1)(n_s - \rho(\sigma_s)) = (|S|-1)n_s.$$

Thus, as $\rho(\sigma_s)$ is an integer, $|X|-1$ divides $(|S|-1)n_s$. Thus, the minimal choice of s forces

$$|X| - 1 \leq (|S|-1)n_s \leq |X| - 1.$$

It follows that each element in $S\setminus\{1\}$ has valency n_s $(= \rho(\sigma_s)+1)$, and this finishes the proof of the theorem.

Muzychuk and Ponomarenko also proved the following partial converse of Theorem 1.

Theorem 2 *Assume that $n_p = n_q$ for any two elements p and q in $S \setminus \{1\}$ and that*

$$n_s = \sum_{r\in S} a_{rr^*s} + 1$$

for each element s in $S \setminus \{1\}$. Then S is pseudocyclic.

4. A theorem of Harvey Blau. Let X be a finite set, let S be a scheme on X, and let R be an algebraically closed field the characteristic of which does not divide $|s|$ for each element s in S.

Set

$$\eta := \sum_{\chi\in\mathrm{Irr}(S)\setminus\{1_S\}} \chi.$$

In [1; Theorem 1], Harvey Blau proved the following theorem.[2]

Theorem 3 *Assume that $m_\phi = m_\psi$ for any two elements ϕ and ψ in $\mathrm{Irr}(S)\setminus\{1_S\}$. Then the following hold.*

(i) *The scheme S is commutative.*

(ii) *For any two elements s in $S\setminus\{1\}$ and χ in $\mathrm{Irr}(S)\setminus\{1_S\}$, we have $m_\chi = n_s$.*

(iii) *For each element s in $S\setminus\{1\}$, $\eta(\sigma_s) = -1$.*

PROOF. We are assuming that there exists a positive integer m such that $m_\chi = m$ for each $\chi \in \mathrm{Irr}(S)\setminus\{1_S\}$. Thus,

$$\chi\chi = 1_S + m\eta.$$

Thus,

$$|X| = \chi\chi(1) = 1 + m\eta(1)$$

and, for each element s in $S\setminus\{1\}$,

$$0 = \chi\chi(\sigma_s) = n_s + m\eta(\sigma_s).$$

Thus, $|X| - 1 = m\eta(1)$ and $n_s = m(-\eta(\sigma_s))$.

Among the elements in $S\setminus\{1\}$ we fix s such that n_s is as small as possible. Then

$$m(-\eta(\sigma_s))(|S|-1) = n_s(|S|-1) \leq |X| - 1 = m\eta(1) \leq m(|S|-1).$$

Thus, as $\eta(\sigma_s)$ is integral, $n_s = m$ and $|S| - 1 = \eta(1)$. The latter equation means that S is commutative, and this finishes the proof of the theorem.

It might be worth mentioning that Theorem 3(i) provides a shortcut in the original proof of the commutativity of schemes of prime valency that was given by Akihide Hanaki and Katsuhiro Uno in [7; Theorem 3.3].

Corollary *Assume that $m_\phi = m_\psi$ for any two elements ϕ and ψ in $\mathrm{Irr}(S)\setminus\{1_S\}$. Then*

$$\sum_{s\in S}\chi(\sigma_{s^*})\chi(\sigma_s) = n_S - m_\chi + 1$$

for each element χ in $\mathrm{Irr}(S)\setminus\{1_S\}$.

PROOF. Let χ be an element in $\mathrm{Irr}(S)$. Then $\chi(1) = 1$; cf. Theorem 3(i). Thus, by [10; Theorem 9.1.7(ii)],

$$\frac{1}{n_S}\sum_{s\in S}\frac{1}{n_{s^*}}\chi(\sigma_{s^*})\chi(\sigma_s) = \frac{1}{m_\chi}.$$

Now recall that, by Theorem 3(ii) $m_\chi = n_s$ for each element s in $S\setminus\{s\}$. Thus,

$$\frac{1}{n_S} + \frac{1}{n_S}\sum_{s\in S\setminus\{1\}}\frac{1}{m_\chi}\chi(\sigma_{s^*})\chi(\sigma_s) = \frac{1}{m_\chi}.$$

Multiplying this equation by $n_S m_\chi$ we now obtain

$$m_\chi + \sum_{s\in S\setminus\{1\}}\chi(\sigma_{s^*})\chi(\sigma_s) = n_S.$$

The claim of the corollary follows immediately from this equation.

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[2] In fact, Blau proved his theorem for a wider class of rings than the class of the scheme rings.

Muzychuk and Ponomarenko assume that the ratio

$$\frac{m_\chi}{\chi(1)}$$

is the same for each element χ in $\mathrm{Irr}(S) \setminus \{1_S\}$. Blau assumes that m_χ is the same for each element χ in $\mathrm{Irr}(S) \setminus \{1_S\}$. What if $\chi(1)$ is the same for each element χ in $\mathrm{Irr}(S) \setminus \{1_S\}$? This question seems to be interesting but difficult.

5. The curvature of a scheme. Let X be a finite set, and let S be a scheme on X. We set

$$\chi(S) := \frac{1}{|S|-1} \sum_{s \in S \setminus \{1\}} n_s$$

and call this positive rational number the *characteristic* of S. Thus, the characteristic of a scheme is defined to be the average of the valencies of its non-trivial elements.

Note that

$$\chi(S) = \frac{|X|-1}{|S|-1}.$$

Thus, $1 \leq \chi(S)$, and S is thin if and only if $\chi(S) = 1$.

Let R be an algebraically closed field the characteristic of which does not divide $|s|$ for any of the elements s in S. We define

$$\chi^*(S) := \frac{1}{|\mathrm{Irr}(S)|-1} \sum_{\chi \in \mathrm{Irr}(S) \setminus \{1_S\}} \frac{m_\chi}{\chi(1)}$$

and call this positive rational number the *cocharacteristic* of S. Thus, the cocharacteristic of a scheme is the average of the covalencies of its non-principal irreducible characters.

Recall from Section 2 that the covalency of an irreducible character of S is at least 1 and that S is thin if and only if all irreducible characters of S have covalency 1. Thus, $1 \leq \chi^*(S)$, and S is thin if and only if $\chi^*(S) = 1$.

If S is commutative, we have $|\mathrm{Irr}(S)| = |S|$ and $\chi(1) = 1$ for each element χ in $\mathrm{Irr}(S)$. Thus, we have

$$\chi^*(S) = \frac{|X|-1}{|S|-1}$$

in this case.

We define

$$\gamma(S) := \ln\left(\frac{\chi^*(S)}{\chi(S)}\right)$$

and call this number the *curvature* of S.

From what we saw above one obtains that thin schemes and commutative schemes have curvature 0. From Theorem 1 one also obtains that pseudocyclic schemes have curvature 0. (Recall that thin schemes are pseudocyclic.)

Among the pseudocyclic schemes there are examples which are neither thin nor commutative. This follows from [9; Theorem 2.1]. This theorem that says that each Frobenius group with non-commutative kernel provides a non-thin and non-commutative pseudocyclic scheme.

It would be interesting to know if the class of all pseudocyclic schemes covers the class of all non-commutative schemes of curvature 0. In other words, one would like to know if non-commutative schemes of curvature 0 are necessarily pseudocyclic. If not, is there a different way to characterize the schemes of curvature 0? This question seems to be interesting but difficult.

Looking at the list [6] of schemes of small valencies one realizes that there is no big difference between the number of schemes of positive valency and the number of schemes of negative valency.

6. A scheme of negative curvature. Let S be a scheme isomorphic to the scheme number 176 in the list of schemes of valency 28 in [6], and let T denote the thin residue of S.[3] Then $|T| = 4$, and all elements of T have valency 1. Moreover, all elements in $S \setminus T$ have valency 2. It follows that $|S| = 16$. Thus, as $|X| = 28$,

$$\chi(S) = \frac{|X|-1}{|S|-1} = \frac{27}{15} = \frac{9}{5}.$$

From $|X| = 28$ and $|T| = 4$ we obtain that the quotient scheme $S//T$, viewed as a group, is cyclic of order 7. Thus, $S//T$ has six linear characters

$$\lambda_1, \ldots, \lambda_6$$

different from $1_{S//T}$. According to [3; Theorem 3.5], these irreducible characters can be viewed as linear characters of S having kernel T.[4]

Assume that S has a non-principal linear character different from $\lambda_1, \ldots, \lambda_6$. Then S has either two different irreducible characters of degree 2 or five different non-principal linear characters different from $\lambda_1, \ldots, \lambda_6$; cf. [10; Corollary 8.6.5]. In both cases, there exists, for each element χ in $\mathrm{Irr}(S) \setminus \{1_S, \lambda_1, \ldots, \lambda_6\}$, an element i in $\{1, \ldots, 6\}$ such that $\lambda_i\chi = \chi$; cf. [5; Theorem 3.3].[5] Thus, as none of the characters $\lambda_i, \ldots, \lambda_6$ vanishes on $\{\sigma_s \mid s \in S\}$, all non-principal irreducible characters of S different from $\lambda_1, \ldots, \lambda_6$ vanish on $\{\sigma_s \mid s \in S \setminus T\}$.

Since $|T| = 4$, each non-principal linear character of S different from $\lambda_1, \ldots, \lambda_6$ has a kernel of order 2. Thus, S cannot have five different linear characters vanishing on $S \setminus T$. It follows that S has two different irreducible characters of degree 2.

Let ϕ be an irreducible character of S which has degree 2 and vanishes on $\{\sigma_s \mid s \in S \setminus T\}$. Then applying the orthogonality relations [10; Theorem 9.1.7(ii)] to ϕ one obtains $m_\phi = 7$. Thus, as we are assuming that S has at least eight linear characters, S cannot have two different irreducible character of degree 2.

What we have seen so far is that $\lambda_1, \ldots, \lambda_6$ are the only non-principal linear characters of S. Thus, by [10; Corollary 8.6.5], S possesses an irreducible character ϕ of degree 3 such that

$$\mathrm{Irr}(S) = \{1_S, \lambda_1, \ldots, \lambda_6, \phi\}.$$

[3]See [10; Section 3.2] for the definition of the thin residue of a scheme.

[4]See [4; Section 3] for the definition of the kernel of a scheme character.

[5]See [5; Theorem 3.3] for the definition of products of linear scheme characters with irreducible scheme characters.

Now recall from [3; Theorem 4.1] that $m_{\chi_i} = 1$ for each element i in $\{1, \ldots, 6\}$. Thus, $m_\phi = 7$, and we obtain

$$\chi^*(S) = \frac{1}{7}(6 + \frac{7}{3}) = \frac{25}{21}.$$

It follows that

$$\gamma(S) := \ln(\frac{\chi^*(S)}{\chi(S)}) = \ln(\frac{125}{189}) < 0,$$

and that means that S has negative curvature.[6]

7. Schemes of positive curvature. Let S be a scheme, and let h and k be involutions of S such that S is a Coxeter scheme over $\{h, k\}$.[7]

In the following, we assume that $n_h \neq 1$ and $n_k \neq 1$. Then, by a theorem of Walter Feit and Graham Higman,

$$|S| \in \{4, 6, 8, 12, 16\};$$

cf. [2; Theorem 1]. If $|S| = 4$, S is commutative by [8; (4.1)]. Thus, we have $\gamma(S) = 0$ in this case. In the remainder of this section, we compute the curvature of S in the cases where $|S| = 6$ and $|S| = 8$.

Assume that $|S| = 6$. In this case, one easily obtains $n_h = n_k$. We set $n := n_h$. Then, by [10; Theorem 12.5.1(i)], $n_S = (n^2 + n + 1)(n + 1)$. Thus,

$$\chi(S) = \frac{n(n^2 + 2n + 2)}{5}.$$

Let R be an algebraically closed field the characteristic of which does not divide $|s|$ for each element s in S. Then, by [10; Lemma 12.4.1(ii), (iii)], RS possesses a non-principal linear character st and an irreducible character ϕ of degree 2 such that

$$\mathrm{Irr}(S) = \{1_S, st, \phi\}.$$

From [10; Lemma 12.5.1(ii)] we also know that $m_{\mathrm{st}} = n^3$ and from [10; Lemma 12.5.1(iii)] that $m_\phi = n(n + 1)$. Thus,

$$\chi^*(S) = \frac{1}{2}(n^3 + \frac{n(n + 1)}{2}) = \frac{n(2n^2 + n + 1)}{4}.$$

It follows that

$$\gamma(S) = \ln(\frac{5(2n^2 + n + 1)}{4(n^2 + 2n + 2)}).$$

In particular, S has positive curvature.

If $n = 2$, we have

$$\gamma(S) = \ln(\frac{11}{8}).$$

In general, we have $\gamma(S) \to \ln(2.5)$ as $n \to \infty$.

Now let $|S| = 8$. Then, by [10; Lemma 12.5.2(i)],

$$n_S = (n_h + 1)(n_k + 1)(n_h n_k + 1).$$

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[6]The values and multiplicities of the characters of S in this section could have been taken from [6]. I owe the above reference to [5; Theorem 3.3] to Mikhail Muzychuk.

[7]See [10; Section 12.3] for the definition of Coxeter schemes.

Thus,

$$\chi(S) = \frac{n_h^2 n_k^2 + n_h^2 n_k + n_h n_k^2 + 2n_h n_k + n_h + n_k}{7}.$$

From [10; Lemma 12.4.1(ii), (iii)] Theorem 12.5.2(ii)] we know that S has three non-principal linear characters st, λ_h, λ_k and an irreducible character ϕ of degree 2 such that

$$\mathrm{Irr}(S) = \{1_S, st, \lambda_h, \lambda_k, \phi\},$$

$m_{\mathrm{st}} = n_h^2 n_k^2$, and

$$m_{\lambda_h} = \frac{n_k^2(n_h n_k + 1)}{n_h + n_k}, \quad m_{\lambda_k} = \frac{n_h^2(n_k n_h + 1)}{n_k + n_h}, \quad m_\phi = \frac{n_h n_k(n_h + 1)(n_k + 1)}{n_h + n_k}.$$

Thus,

$$\chi^*(S) = \frac{1}{4}\left(\frac{n_h^2 n_k^2}{1} + \frac{n_k^2(n_h n_k + 1)}{n_h + n_k} + \frac{n_h^2(n_h n_k + 1)}{n_h + n_k} + \frac{n_h n_k(n_h + 1)(n_k + 1)}{2(n_h + n_k)}\right).$$

If $n_h = 2$ and $n_k = 2$,

$$\gamma(S) = \ln\left(\frac{427}{352}\right).$$

In general, $\gamma(S) \to \ln(1.75)$ as $n_h \to \infty$ and $n_k \to \infty$.

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