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A LOWER LARGE DEVIATION BOUND FOR NON-UNIFORMLY HYPERBOLIC DYNAMICAL SYSTEMS

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ABSTRACT. Let f be a diffeomorphism of a manifold preserving a hyperbolic Borel probability measure μ having transversal intersections for almost every pairs of stable and unstable manifolds. Then we obtain a lower bound for the large deviation rate.

1. INTRODUCTION

The theory of large deviations for dynamical systems is an object of intense study. See [1, 3, 6, 7, 10, 13, 15]. Here we obtain a lower large deviation bounds for dynamical systems preserving a hyperbolic measure satisfying the weak transversality condition.

We consider a $C^{1+\alpha}(\alpha > 0)$ diffeomorphism $f\colon M \to M$ of a compact smooth Riemannian manifold M preserving a *hyperbolic* Borel probability measure μ, i.e., all the Lyapunov exponents are non-zero at μ-almost every point. We impose an additional hypothesis that for $\mu \otimes \mu$-almost every pair $(x, y) \in M \times M$ there exist integers $p, q \in \mathbb{Z}$ and a point $z \in \mathcal{W}^u(f^p(x)) \cap \mathcal{W}^s(f^q(y))$ such that

$$T_z\mathcal{W}^u(f^p(x)) \oplus T_z\mathcal{W}^s(f^q(y)) = T_zM.$$

Here $\mathcal{W}^s(z)$ and $\mathcal{W}^u(z)$ are stable and unstable manifolds at z, respectively (see the definition in §3). Such a measure is said to satisfy the *weak transversality condition* (WTC for short).

We denote by $\mathcal{M}$ the collection of all probability measures on M and by $\mathcal{M}_f$ the collection of all f-invariant probability measures on M. It is well known that $\mathcal{M}$ is compact convex metrizable with respect to the weak* topology and $\mathcal{M}_f$ is a non-empty compact convex subset of $\mathcal{M}$. Denote by m the Riemannian volume on M.

It follows from Birkhoff's ergodic theorem that we have the following limit for μ-almost every $x \in M$:

$$\chi^+(x) = \lim_{n\to\infty} \frac{1}{n} \log |\det(D_x f^n | T_x\mathcal{W}^u(x))|. \tag{1.1}$$

And this coincides with the sum of all the positive Lyapunov exponents at x, counted with multiplicity (see §3).

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Theorem 1.1. *Let $f\colon M \to M$ be a $C^{1+\alpha}$ diffeomorphism of a compact smooth Riemannian manifold M preserving a hyperbolic Borel probability measure μ. Suppose that μ satisfies the WTC. Then for any open neighborhood $\mathcal{G} \subset \mathcal{M}$ of μ,*

$$\liminf_{n\to\infty} \frac{1}{n} \log m(\{x \in M\colon \delta_n(x) \in \mathcal{G}\}) \geq h_\mu(f) - \int \chi^+(x)\, d\mu(x),$$

where $\delta(y)$ is the Dirac measure at y and $\delta_n(x) = \sum_{i=0}^{n-1} \delta(f^i(x))/n$.

This theorem is known in the ergodic case ([13], [15]) or for some topological dynamics with (a weaker form of) the specification property ([10]).

Let us remark that the weak transversality condition can be checked in the following cases:

- *ergodic* hyperbolic probability measures (as an immediate consequence of Propositions 2.4 and 2.5 in [4]);
- invariant probability measures on basic sets of Axiom A diffeomorphisms ([5, Proposition 18.3.10]);
- hyperbolic probability measures which are invariant under partially hyperbolic diffeomorphisms admitting *minimal* strong stable foliations, and whose stable manifolds coincide with the strong stable leaves almost everywhere.

A foliation is said to be *minimal* provided every leaf of this foliation is dense in M. Recently, Pujals and Sambarino ([11]) gave a sufficient condition (called *Property SH*) for the strong stable foliation to remain minimal under C^1 perturbations, and presented several examples of partially hyperbolic (but non-hyperbolic) diffeomorphisms satisfying this property. Furthermore, it is easy to check that the Property SH also guarantees the existence of the hyperbolic invariant measures in the third case above.

The specification property enables us to obtain a lower bound for the large deviation rate around non-ergodic measures ([10], [15]), but does not hold for generic non-hyperbolic systems ([12]). So, the WTC is likely to be available for studying a large class of non-hyperbolic systems.

2. Upper bounds for large deviations

We remark that the WTC is not sufficient to obtain nontrivial upper bounds for the large deviation estimate. Indeed, consider a diffeomorphism f on a two-sphere S^2 with a hyperbolic fixed point p of saddle type such that the stable manifold of p coincides with its unstable manifold and $|\det D_p f| < 1$ (see [4, 14]). Since the stable and unstable manifolds of p intersect transversally at p itself, the point mass $\delta(p)$ satisfies the WTC. And it is known that every point x sufficiently close to the stable manifold of p satisfies $\lim_{n\to\infty} \delta_n(x) = \delta(p)$ ([14]), which implies that

- $\displaystyle\lim_{n\to\infty} \frac{1}{n} \log m(\{x \in S^2\colon \delta_n(x) \in \mathcal{F}\}) = 0$

for each neighborhood $\mathcal{F} \subset \mathcal{M}$ of $\delta(p)$. We note here that 0 is a trivial upper bound (independent of the choice of measures) for the large deviation estimate. On the other hand, the lower estimate in Theorems 1.1 is strictly less than the true values for $\delta(p)$ as follows:

- $h_{\delta(p)}(f) - \int \chi^+ \, d\delta(p) = -\chi_1(p) \ (< 0)$.

It is known ([1]) that if f is a C^2 diffeomorphism exhibiting a partially hyperbolic non-uniformly expanding attracting set, we can obtain a nontrivial upper large deviation bounds for f.

3. Definitions

Let M be a compact smooth Riemannian manifold with a norm $\|\cdot\|$, $f\colon M \to M$ a $C^{1+\alpha}(\alpha > 0)$ diffeomorphism of M preserving a Borel probability measure μ and $Df : TM \to TM$ the derivative of f. As always, we let d be the distance on M induced by the Riemannian metric.

A point $x \in M$ is said to be *Lyapunov regular* if there exist real numbers $\chi_1(x) > \chi_2(x) > \cdots > \chi_{r(x)}(x)$ and a $D_x f$-invariant decomposition $T_x M = E_1(x) \oplus E_2(x) \oplus \cdots \oplus E_{r(x)}(x)$ such that for each $i = 1, 2, \ldots, r(x)$

$$\lim_{n \to \pm\infty} \frac{1}{n} \log \|D_x f^n(v)\| = \chi_i(x) \qquad (v \in E_i(x) \setminus \{0\})$$

exists, and

$$\lim_{n \to \pm\infty} \frac{1}{n} \log |\det(D_x f^n)| = \sum_{i=1}^{r(x)} \chi_i(x) \dim E_i(x).$$

We denote by $\Gamma = \Gamma^\mu$ the set of Lyapunov regular points. By the multiplicative ergodic theorem ([8]) Γ has full μ-measure. The numbers $\chi_i(x)$ are called the *Lyapunov exponents* of f at the point x. The functions $x \mapsto \chi_i(x)$, $r(x)$ and $\dim E_i(x)$ are Borel measurable and f-invariant. If the invariant measure is supposed to be ergodic, then we denote the constants by $\chi_1, \chi_2, \ldots, \chi_r$ and $\dim E_i$.

We call the measure μ *hyperbolic* if none of the Lyapunov exponents for μ vanish and there exist Lyapunov exponents with different signs for μ-almost everywhere. In what follows we always assume that μ is hyperbolic, and we will denote $u(x) = \max\{i\colon \chi_i(x) > 0\}$ and $s(x) = \min\{i\colon \chi_i(x) < 0\}$ for μ-almost every $x \in M$. Note that $s(x) = u(x) + 1$. The associated decomposition is represented as $T_x M = E^u(x) \oplus E^s(x)$, where $E^u(x) = \bigoplus_{i=1}^{u(x)} E_i(x)$, $E^s(x) = \bigoplus_{i=s(x)}^{r(x)} E_i(x)$ for μ-almost every $x \in M$. For $x \in \Gamma$, we define the *unstable* and *stable manifolds* at x as

$$\mathcal{W}^u(x) = \left\{ y \in M\colon \limsup_{n \to \infty} \frac{1}{n} \log d(f^{-n}(x), f^{-n}(y)) < 0 \right\},$$

$$\mathcal{W}^s(x) = \left\{ y \in M\colon \limsup_{n \to \infty} \frac{1}{n} \log d(f^{n}(x), f^{n}(y)) < 0 \right\}.$$

Then $\mathcal{W}^u(x)$ and $\mathcal{W}^s(x)$ are injectively immersed manifolds satisfying

$$T_x\mathcal{W}^u(x) = E^u(x) \quad \text{and} \quad T_x\mathcal{W}^s(x) = E^s(x),$$

respectively. See [2, 9]. Note that

$$\begin{aligned}\chi^+(x) &= \lim_{n\to\infty} \frac{1}{n} \log|\det(D_x f^n|T_x\mathcal{W}^u(x))| \\ &= \lim_{n\to\infty} \frac{1}{n} \log|\det(D_x f^n|E^u(x))| \\ &= \sum_{i=1}^{u(x)} \chi_i(x) \dim E_i(x).\end{aligned}$$

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