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Multiple Bernoulli polynomials and multiple zeta-functions of root systems[1]

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§1. Introduction

To give the explicit value of the following series was posed in 1644 and is called the Basel problem:

$$\boxed{\sum_{n=1}^{\infty}\frac{1}{n^2}=?}$$

In 1735, Euler gave the solution to the Basel problem, and its generalizations

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6},\qquad \sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{\pi^4}{90},\qquad \cdots$$

It is well known that these series are the origin of the Riemann zeta-function and the notion "zeta-functions" plays an important tool in modern mathematics.

Recently Witten [16] and Zagier [17] gave generalizations of the Basel problem:

> For $k\in\mathbb{Z}_{\geq 1}$,
> $$\sum_{\varphi}\frac{1}{(\dim\varphi)^{2k}}=?$$
> where the summation runs over all finite dimensional irreducible representations φ of a given Lie algebra $\mathfrak{g}$.

It is noted that these series were introduced to study the partition functions of two dimensional quantum gauge theories with compact semisimple Lie groups.

Witten and Zagier showed that their values are in $\mathbb{Q}\pi^{|\Delta_+|2k}$. Euler already established the solutions in the $\mathfrak{sl}_2$ case, since in this case, the problem reduces to the Basel problem. Subbarao-Sitaramachandrarao considered the $\mathfrak{sl}_3$ case in [14]. In [15], Szenes gave a certain algorithm for the computation in general cases, from the viewpoint of hyperplane arrangements. Gunnells-Sczech gave the explicit forms in the $\mathfrak{sl}_4$ case [1].

In this article, we will propose a new approach to this problem. We will introduce generalizations of Bernoulli polynomials and zeta-functions associated with root systems, which include the Riemann zeta-function, the Euler-Zagier zeta-functions and

[1]This is an updated version of our previous article [11].

the Witten zeta-functions. Furthermore we will develop a theory similar to that of the classical Riemann zeta-function.

§2. Review of Classical Theory

To state our results, first we recall the classical theory for the Riemann zeta-function and Bernoulli numbers.

The following is a well-known formula for the Riemann zeta-function and Bernoulli numbers.

For $k \in \mathbb{Z}_{\geq 1}$,

$$2\zeta(2k) = -B_{2k}\frac{(2\pi i)^{2k}}{(2k)!},$$

where for $t \in \mathbb{C}$ with $|t| < 2\pi$,

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}.$$

Using this formula, we obtain for $k \in \mathbb{Z}_{\geq 1}$,

$$\zeta(2k) + (-1)^{2k}\zeta(2k) = -B_{2k}\frac{(2\pi i)^{2k}}{(2k)!},$$

$$\zeta(2k+1) + (-1)^{2k+1}\zeta(2k+1) = -B_{2k+1}\frac{(2\pi i)^{2k+1}}{(2k+1)!} = 0.$$

Hence we have the important relations:

For $k \in \mathbb{Z}_{\geq 2}$,

$$\begin{array}{rcl} \zeta(k) + (-1)^k \zeta(k) & = & -B_k \dfrac{(2\pi i)^k}{k!}, \\ \text{value-relations} & = & \text{Bernoulli numbers.} \end{array}$$

These relations are generalized in the cases of Lerch zeta-functions and periodic Bernoulli functions. Let $\varphi(s,y)$ be the Lerch zeta-function defined by

$$\varphi(s,y) = \sum_{n=1}^{\infty} \frac{e^{2\pi i n y}}{n^s}.$$

Then a formula for Lerch zeta-functions implies

For $k \in \mathbb{Z}_{\geq 2}$ and $y \in \mathbb{R}$,

$$\begin{array}{rcl} \varphi(k,y) + (-1)^k \varphi(k,-y) & = & -B_k(\{y\}) \dfrac{(2\pi i)^k}{k!}, \\ \text{functional relations} & = & \text{periodic Bernoulli functions.} \end{array}$$

Here

$$\frac{te^{t\{y\}}}{e^t - 1} = \sum_{k=0}^{\infty} B_k(\{y\}) \frac{t^k}{k!},$$

and $\{y\} = y - [y]$ (i.e. fractional part).

Once we obtain periodic Bernoulli functions, we can calculate special values of L-functions.

For a primitive character χ of conductor f and $k \in \mathbb{Z}_{\geq 2}$ satisfying $(-1)^k\chi(-1) = 1$, we have

$$L(k,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^k} = \frac{(-1)^{k+1}}{2}\frac{(2\pi i)^k}{k!f^k} g(\chi) B_{k,\overline{\chi}},$$

where $g(\chi)$ is the Gauss sum and

$$B_{k,\chi} = f^{k-1}\sum_{a=1}^{f} \chi(a) B_k(a/f).$$

Our aim is to find a good class of multiple zeta-functions which generalize the theory above.

§3. Overview of Our Results

Based on the observation given in the previous section, we will construct multiple generalizations of Bernoulli polynomials and multiple zeta- and L-functions associated with arbitrary root systems. Before introducing the general theory, we give two simple theorems without using the terminology of root systems.

For $s_1, s_2, s_3 \in \mathbb{C}$ and $y_1, y_2 \in \mathbb{R}$, we consider the convergent series

$$\zeta_2(s_1,s_2,s_3,y_1,y_2;A_2) = \sum_{m,n=1}^{\infty} \frac{e^{2\pi i(my_1+ny_2)}}{m^{s_1} n^{s_2} (m+n)^{s_3}}.$$

Theorem A. *For $k_1, k_2, k_3 \in \mathbb{Z}_{\geq 2}$,*

$$\begin{aligned}
&\zeta_2(k_1,k_2,k_3,y_1,y_2;A_2) + (-1)^{k_1}\zeta_2(k_1,k_3,k_2,-y_1+y_2,y_2;A_2)\\
&\quad + (-1)^{k_2}\zeta_2(k_3,k_2,k_1,y_1,y_1-y_2;A_2) + (-1)^{k_2+k_3}\zeta_2(k_3,k_1,k_2,-y_1+y_2,-y_1;A_2)\\
&\quad + (-1)^{k_1+k_3}\zeta_2(k_2,k_3,k_1,-y_2,y_1-y_2;A_2) + (-1)^{k_1+k_2+k_3}\zeta_2(k_2,k_1,k_3,-y_2,-y_1;A_2)\\
&\qquad = (-1)^3 P(k_1,k_2,k_3,y_1,y_2;A_2)\frac{(2\pi i)^{k_1+k_2+k_3}}{k_1!k_2!k_3!},
\end{aligned}$$

where $P(k_1,k_2,k_3,y_1,y_2;A_2)$ is a multiple periodic Bernoulli function (defined later). In particular, we have

$$\zeta_2(2,2,2,0,0;A_2) = \frac{1}{6}(-1)^3 \frac{1}{3780}\frac{(2\pi i)^{2+2+2}}{2!2!2!} = \frac{\pi^6}{2835}.$$

cf.

$$\varphi(k,y) + (-1)^k\varphi(k,-y) = -B_k(\{y\})\frac{(2\pi i)^k}{k!}, \qquad \zeta(2) = \frac{1}{2}(-1)\frac{1}{6}\frac{(2\pi i)^2}{2!} = \frac{\pi^2}{6}.$$

For $s_1, s_2, s_3 \in \mathbb{C}$ and primitive Dirichlet characters χ_1, χ_2, χ_3, consider the convergent series

$$L_2(s_1,s_2,s_3,\chi_1,\chi_2,\chi_3;A_2) = \sum_{m,n=1}^{\infty} \frac{\chi_1(m)\chi_2(n)\chi_3(m+n)}{m^{s_1}n^{s_2}(m+n)^{s_3}}.$$

Theorem B. *For $k \in \mathbb{Z}_{\geq 2}$ and a primitive Dirichlet character χ of conductor f such that $(-1)^k\chi(-1) = 1$,*

$$L_2(k,k,k,\chi,\chi,\chi;A_2) = \frac{(-1)^{3k+3}}{6}\left(\frac{(2\pi i)^k}{k!f^k}g(\chi)\right)^3 B_{k,k,k,\overline{\chi},\overline{\chi},\overline{\chi}}(A_2),$$

where $B_{k_1,k_2,k_3,\chi_1,\chi_2,\chi_3}(A_2)$ is a multiple generalized Bernoulli number (defined later). In particular, for $\rho_5 : \rho_5(1) = \rho_5(4) = 1, \rho_5(2) = \rho_5(3) = -1$, we have

$$L_2(2,2,2,\rho_5,\rho_5,\rho_5;A_2) = \frac{(-1)^{6+3}}{6}\left(\frac{(2\pi i)^2}{2!5^2}\sqrt{5}\right)^3\left(-\frac{28}{125}\right) = -\frac{112\sqrt{5}}{1171875}\pi^6.$$

cf.

$$L(k,\chi) = \frac{(-1)^{k+1}}{2}\frac{(2\pi i)^k}{k!f^k}g(\chi)B_{k,\overline{\chi}}, \qquad L(2,\rho_5) = \frac{(-1)^{2+1}}{2}\frac{(2\pi i)^2}{2!5^2}\sqrt{5}\,\frac{4}{5} = \frac{4\sqrt{5}}{125}\pi^2.$$

Theorems A and B are special cases of our main theorems. In the following sections, we will formulate these facts.

§4. Root Systems

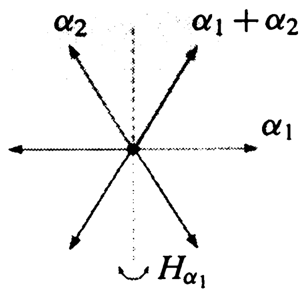
For reader's convenience, we give the definition and several examples of root systems.

§§4.1. Definitions

Let V be an r dimensional real vector space equipped with inner product $\langle\cdot,\cdot\rangle$.

A root system $\Delta \subset V$ is a set of vectors (roots):

(1) $|\Delta| < \infty$ and $0 \notin \Delta$,
(2) $\sigma_\alpha\Delta = \Delta$ for all $\alpha \in \Delta$,
(3) $\langle\alpha^\vee,\beta\rangle \in \mathbb{Z}$ for all $\alpha,\beta \in \Delta$,
(4) $\alpha, c\alpha \in \Delta \Longrightarrow c = \pm 1$,



where σ_α denotes the reflection with respect to the hyperplane H_α orthogonal to α and $\alpha^\vee = 2\alpha/\langle\alpha,\alpha\rangle$ (coroot).

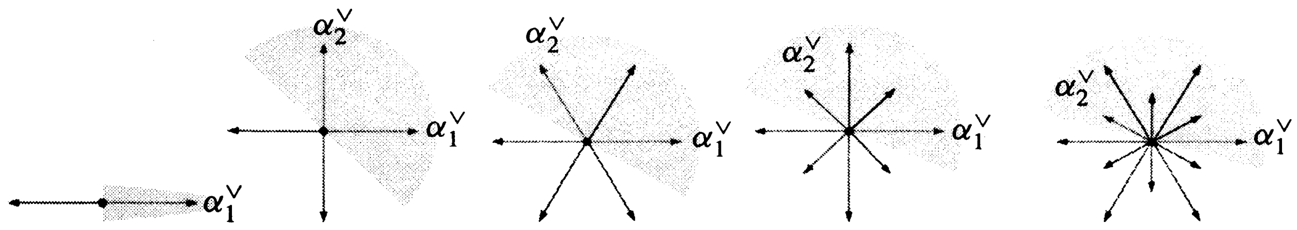
Let W be the Weyl group (the group generated by all σ_α). Let $\{\alpha_1,\ldots,\alpha_r\}$ be fundamental roots (a basis s.t. $\alpha = c_1\alpha_1 + \cdots + c_r\alpha_r \in \Delta$ with all $c_i \geq 0$ or $c_i \leq 0$). Let Δ_+ be positive roots (all roots $\alpha = c_1\alpha_1+\cdots+c_r\alpha_r \in \Delta$ with all $c_i \geq 0$) and P_{++}, strictly dominant weights ($= \bigoplus \mathbb{Z}_{\geq 1}\lambda_i$, $\{\lambda_1,\ldots,\lambda_r\}$ dual basis of $\{\alpha_1^\vee,\ldots,\alpha_r^\vee\}$). The key fact which plays an essential role is that the nice group W acts on Δ.

§§4.2. Examples

Since we mainly treat coroots, we give examples of root systems in terms of coroots. Note that if Δ is a root system, then $\Delta^\vee = \{\alpha^\vee \mid \alpha \in \Delta\}$ is also a root system.

There is only one root system of rank 1 and there are four root systems of rank 2:

A_1 | $A_1 \times A_1$ | A_2 | B_2 (or C_2) | G_2



$$\Delta_+^\vee = \{\alpha_1^\vee\} \quad \{\alpha_1^\vee, \alpha_2^\vee\} \quad \left\{\begin{matrix} \alpha_1^\vee, \alpha_2^\vee \\ \alpha_1^\vee + \alpha_2^\vee \end{matrix}\right\} \quad \left\{\begin{matrix} \alpha_1^\vee, \alpha_1^\vee + \alpha_2^\vee \\ \alpha_2^\vee, \alpha_1^\vee + 2\alpha_2^\vee \end{matrix}\right\} \quad \left\{\begin{matrix} \alpha_1^\vee, \alpha_1^\vee + \alpha_2^\vee \\ \alpha_2^\vee, \alpha_1^\vee + 2\alpha_2^\vee \\ \alpha_1^\vee + 3\alpha_2^\vee \\ 2\alpha_1^\vee + 3\alpha_2^\vee \end{matrix}\right\}$$

In this article, we use these root systems in examples for simplicity. It should be noted that root systems are classified as $A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2$ and our theory can be applied to all these root systems.

§5. Zeta-Functions of Root Systems

§§5.1. Witten Zeta-Functions

As prototypes of zeta-functions of root systems, we give the definition of Witten zeta-functions, which were originally introduced to calculate the volumes of certain moduli spaces.

Witten zeta-functions ([16, 17]): For a complex simple Lie algebra $\mathfrak{g}$ of type X_r,

$$\zeta_W(s; X_r) = \sum_{\varphi} (\dim \varphi)^{-s} = K(X_r)^s \sum_{\lambda \in P_{++}} \prod_{\alpha \in \Delta_+} \frac{1}{\langle \alpha^\vee, \lambda \rangle^s},$$

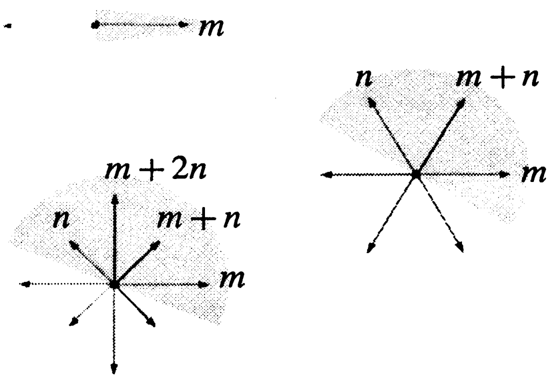
where the summation runs over all finite dimensional irreducible representations φ and $K(X_r) \in \mathbb{Z}_{\geq 1}$ is a constant.

From the second expression of the definition, we see that the explicit forms of Witten zeta-functions are obtained by formally replacing $\alpha_1^\vee$ and $\alpha_2^\vee$ by m and n respectively:

$$\zeta_W(s; A_1) = \sum_{m=1}^{\infty} \frac{1}{m^s} = \zeta(s),$$

$$\zeta_W(s; A_2) = 2^s \sum_{m,n=1}^{\infty} \frac{1}{m^s n^s (m+n)^s},$$

$$\zeta_W(s; B_2) = 6^s \sum_{m,n=1}^{\infty} \frac{1}{m^s n^s (m+n)^s (m+2n)^s}.$$



§§5.2. Zeta-Functions of Root Systems

Definition 1 ([6, 7, 8, 13]). Zeta-functions of root systems: For a root system Δ of type X_r, define

$$\zeta_r(\mathbf{s},\mathbf{y};X_r)=\sum_{\lambda\in P_{++}} e^{2\pi i\langle \mathbf{y},\lambda\rangle}\prod_{\alpha\in\Delta_+}\frac{1}{\langle\alpha^\vee,\lambda\rangle^{s_\alpha}},$$

where $\mathbf{s}=(s_\alpha)_{\alpha\in\Delta_+}\in\mathbb{C}^{|\Delta_+|}$ and $\mathbf{y}\in V$.

To define an action of the Weyl group, we extend $\mathbf{s}=(s_\alpha)_{\alpha\in\Delta_+}$ to $(s_\alpha)_{\alpha\in\Delta}$ by $s_\alpha=s_{-\alpha}$ and define $(w\mathbf{s})_\alpha=s_{w^{-1}\alpha}$. Then we have our first theorem.

Theorem 1 ([8]). *For* $\mathbf{s}=\mathbf{k}=(k_\alpha)_{\alpha\in\Delta_+}\in\mathbb{Z}_{\geq 2}^{|\Delta_+|}$, *we have*

$$\sum_{w\in W}\Bigg(\prod_{\alpha\in\Delta_+\cap w^{-1}\Delta_-}(-1)^{k_\alpha}\Bigg)\zeta_r(w^{-1}\mathbf{k},w^{-1}\mathbf{y};X_r)=(-1)^{|\Delta_+|}P(\mathbf{k},\mathbf{y};X_r)\Bigg(\prod_{\alpha\in\Delta_+}\frac{(2\pi i)^{k_\alpha}}{k_\alpha!}\Bigg),$$

where $P(\mathbf{k},\mathbf{y};X_r)$ *is a multiple periodic Bernoulli function (defined later).*

cf. $(X_r=A_1)$

$$\varphi(k,y)+(-1)^k\varphi(k,-y)=-B_k(\{y\})\frac{(2\pi i)^k}{k!}\qquad (W=\{\mathrm{id},\sigma_\alpha\}).$$

§6. Special Zeta-Values

Theorem 1 immediately implies the following theorem:

Theorem 2 ([8]). *For* $\mathbf{k}=(k_\alpha)_{\alpha\in\Delta_+}\in(2\mathbb{Z}_{\geq 1})^{|\Delta_+|}$ *satisfying* $w^{-1}\mathbf{k}=\mathbf{k}$ *for all* $w\in W$,

$$\zeta_r(\mathbf{k},\mathbf{0};X_r)=\frac{(-1)^{|\Delta_+|}}{|W|}P(\mathbf{k},\mathbf{0};X_r)\Bigg(\prod_{\alpha\in\Delta_+}\frac{(2\pi i)^{k_\alpha}}{k_\alpha!}\Bigg)\in\mathbb{Q}\pi^{\sum_{\alpha\in\Delta_+}k_\alpha}.$$

cf. $(X_r=A_1)$

$$\zeta(k)=\frac{-1}{2}B_k\frac{(2\pi i)^k}{k!}\in\mathbb{Q}\pi^k\qquad (k\in 2\mathbb{Z}_{\geq 1}).$$

In particular, $\mathbf{k}=(k)_{\alpha\in\Delta_+}$ with $k\in 2\mathbb{Z}_{\geq 1}$ (that is, all $k_\alpha=k$) satisfies the condition in Theorem 2. In this case, $\zeta_r(\mathbf{k},\mathbf{0};X_r)\in\mathbb{Q}\pi^{|\Delta_+|k}$ was shown by Witten and Zagier. Our statement is a true generalization of their results since we also have for example,

$$\begin{aligned}\zeta_2((2,4,4,2),\mathbf{0};B_2)&=\sum_{m,n=1}^{\infty}\frac{1}{m^2n^4(m+n)^4(m+2n)^2}\\&=\frac{(-1)^4}{2^2 2!}\frac{53}{1513512000}\left(\frac{(2\pi i)^2}{2!}\right)^2\left(\frac{(2\pi i)^4}{4!}\right)^2\\&=\frac{53\pi^{12}}{6810804000}.\end{aligned}$$

§7. Multiple Periodic Bernoulli Functions

In this section, we give the definitions of generating functions of multiple periodic Bernoulli functions. Let $\mathcal{V}$ be the set of all bases $\mathbf{V} \subset \Delta_+$, $\mathbf{V}^* = \{\mu_\beta^{\mathbf{V}}\}_{\beta\in\mathbf{V}}$, the dual basis of $\mathbf{V}^\vee = \{\beta^\vee\}_{\beta\in\mathbf{V}}$. Let $Q^\vee = \bigoplus_{i=1}^r \mathbb{Z}\alpha_i^\vee$ be the coroot lattice and $L(\mathbf{V}^\vee) = \bigoplus_{\beta\in\mathbf{V}} \mathbb{Z}\beta^\vee$, which is a sublattice of $Q^\vee$ with finite index ($|Q^\vee/L(\mathbf{V}^\vee)| < \infty$).

Fix a certain $\phi \in V$ and define a multiple generalization of fractional part as

$$\{\mathbf{y}\}_{\mathbf{V},\beta} = \begin{cases} \{\langle \mathbf{y}, \mu_\beta^{\mathbf{V}}\rangle\} & (\langle \phi, \mu_\beta^{\mathbf{V}}\rangle > 0), \\ 1 - \{-\langle \mathbf{y}, \mu_\beta^{\mathbf{V}}\rangle\} & (\langle \phi, \mu_\beta^{\mathbf{V}}\rangle < 0). \end{cases}$$

By using these definitions, we have

Definition 2 (generating function [8, 9, 10]). For $\mathbf{t} = (t_\alpha)_{\alpha\in\Delta_+} \in \mathbb{C}^{|\Delta_+|}$,

$$F(\mathbf{t},\mathbf{y};X_r) = \sum_{\mathbf{V}\in\mathcal{V}} \Big(\prod_{\gamma\in\Delta_+\setminus\mathbf{V}} \frac{t_\gamma}{t_\gamma - \sum_{\beta\in\mathbf{V}} t_\beta \langle \gamma^\vee, \mu_\beta^{\mathbf{V}}\rangle} \Big) \\ \times \frac{1}{|Q^\vee/L(\mathbf{V}^\vee)|} \sum_{q\in Q^\vee/L(\mathbf{V}^\vee)} \Big(\prod_{\beta\in\mathbf{V}} \frac{t_\beta \exp(t_\beta \{\mathbf{y}+q\}_{\mathbf{V},\beta})}{e^{t_\beta}-1} \Big).$$

It can be shown that the generating function $F(\mathbf{t},\mathbf{y};X_r)$ is holomorphic in the neighborhood of the origin in $\mathbf{t}$.

Definition 3 (multiple periodic Bernoulli functions [8, 9, 10]).

$$F(\mathbf{t},\mathbf{y};X_r) = \sum_{\mathbf{k}\in\mathbb{Z}_{\geq 0}^{|\Delta_+|}} P(\mathbf{k},\mathbf{y};X_r) \prod_{\alpha\in\Delta_+} \frac{t_\alpha^{k_\alpha}}{k_\alpha!}.$$

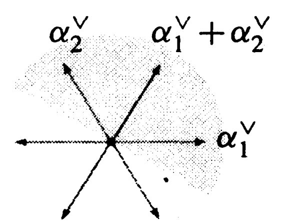
cf. ($X_r = A_1$)

$$F(t,y) = \frac{te^{t\{y\}}}{e^t - 1} = \sum_{k=0}^{\infty} B_k(\{y\}) \frac{t^k}{k!}.$$

§8. Example: A_2 Case

We calculate a multiple periodic Bernoulli function and its generating function in the case of the root system of type A_2.

We have the basic data as follows:



$\Delta_+^\vee = \{\alpha_1^\vee, \alpha_2^\vee, \alpha_1^\vee + \alpha_2^\vee\}$, $\mathcal{V} = \{\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3\}$,
$\mathbf{t} = (t_{\alpha_1}, t_{\alpha_2}, t_{\alpha_1+\alpha_2}) = (t_1, t_2, t_3)$,
$\mathbf{y} = y_1\alpha_1^\vee + y_2\alpha_2^\vee$.

Fix a sufficiently small $\varepsilon > 0$ and $\phi = \alpha_1^\vee + \varepsilon\alpha_2^\vee$. Then by using these data, we have the generating function and a multiple periodic Bernoulli function as

$$F(\mathbf{t},\mathbf{y};A_2) = \frac{t_3}{t_3-t_1-t_2}\frac{t_1e^{t_1\{y_1\}}}{e^{t_1}-1}\frac{t_2e^{t_2\{y_2\}}}{e^{t_2}-1} + \frac{t_2}{t_2+t_1-t_3}\frac{t_1e^{t_1\{y_1-y_2\}}}{e^{t_1}-1}\frac{t_3e^{t_3\{y_2\}}}{e^{t_3}-1} + \frac{t_1}{t_1+t_2-t_3}\frac{t_2e^{t_2(1-\{y_1-y_2\})}}{e^{t_2}-1}\frac{t_3e^{t_3\{y_1\}}}{e^{t_3}-1}$$

basis $\mathbf{V}\subset\Delta_+$, dual basis $\mathbf{V}^*$

$(\mathbf{V}_1^\vee=\{\alpha_1^\vee,\alpha_2^\vee\},\mathbf{V}_1^*=\{\lambda_1,\lambda_2\})$

$(\mathbf{V}_2^\vee=\{\alpha_1^\vee,\alpha_1^\vee+\alpha_2^\vee\},\mathbf{V}_2^*=\{\lambda_1-\lambda_2,\lambda_2\})$

$(\mathbf{V}_3^\vee=\{\alpha_2^\vee,\alpha_1^\vee+\alpha_2^\vee\},\mathbf{V}_3^*=\{\lambda_2-\lambda_1,\lambda_1\})$

For $\mathbf{k}=\mathbf{2}=(2,2,2)$,

$$\begin{aligned}P(\mathbf{2},(y_1,y_2);A_2) &= \frac{1}{3780}+\frac{1}{90}(\{y_1\}-\{y_1-y_2\}-\{y_2\})\\
&+\frac{1}{90}(-\{y_1\}^2-2\{y_1-y_2\}\{y_1\}+\{y_1-y_2\}^2-\{y_2\}^2+2\{y_1-y_2\}\{y_2\})\\
&+\frac{1}{18}(-\{y_1\}^3+3\{y_1-y_2\}\{y_1\}^2+3\{y_2\}^3+3\{y_1-y_2\}\{y_2\}^2)\\
&+\frac{1}{18}(\{y_1\}^4-2\{y_1-y_2\}\{y_1\}^3-3\{y_1-y_2\}^2\{y_1\}^2\\
&\qquad -5\{y_2\}^4-10\{y_1-y_2\}\{y_2\}^3-3\{y_1-y_2\}^2\{y_2\}^2)\\
&+\frac{1}{30}(\{y_1\}^5-5\{y_1-y_2\}\{y_1\}^4+10\{y_1-y_2\}^2\{y_1\}^3\\
&\qquad +5\{y_2\}^5+15\{y_1-y_2\}\{y_2\}^4+10\{y_1-y_2\}^2\{y_2\}^3)\\
&+\frac{1}{30}(-\{y_1\}^6+4\{y_1-y_2\}\{y_1\}^5-5\{y_1-y_2\}^2\{y_1\}^4\\
&\qquad -\{y_2\}^6-4\{y_1-y_2\}\{y_2\}^5-5\{y_1-y_2\}^2\{y_2\}^4).\end{aligned}$$

We have a functional relation corresponding to this multiple periodic Bernoulli function:

$$\begin{aligned}&\zeta_2(\mathbf{2},(y_1,y_2);A_2)+\zeta_2(\mathbf{2},(-y_1+y_2,y_2);A_2)+\zeta_2(\mathbf{2},(y_1,y_1-y_2);A_2)\\
&\quad+\zeta_2(\mathbf{2},(-y_2,y_1-y_2);A_2)+\zeta_2(\mathbf{2},(-y_1+y_2,-y_1);A_2)+\zeta_2(\mathbf{2},(-y_2,-y_1);A_2)\\
&\qquad=(-1)^3P(\mathbf{2},(y_1,y_2);A_2)\frac{(2\pi i)^6}{(2!)^3}.\end{aligned}$$

In particular if $(y_1,y_2)=(0,0)$, then

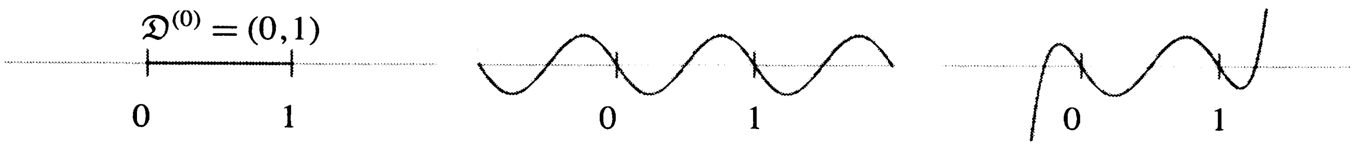
$$\zeta_2(\mathbf{2},(0,0);A_2)=\frac{1}{6}(-1)^3\frac{1}{3780}\frac{(2\pi i)^6}{(2!)^3}=\frac{\pi^6}{2835}.$$

cf. $(X_r=A_1)$

$$\zeta(2)=\frac{1}{2}(-1)\frac{1}{6}\frac{(2\pi i)^2}{2!}=\frac{\pi^2}{6},\qquad B_2(\{y\})=\frac{1}{6}-\{y\}+\{y\}^2.$$

§9. Multiple Bernoulli Polynomials

In the classical theory, Bernoulli polynomials can be derived by the analytic continuation of periodic Bernoulli functions. We explain this fact. Let $\mathfrak{H} = \{y \in \mathbb{R} \mid \{y\} \in \mathbb{Z}\} = \mathbb{Z}$ (discontinuous points of $\{y\}$). Let $\mathbb{R} \setminus \mathfrak{H} = \coprod_{\nu \in \mathbb{Z}} \mathfrak{D}^{(\nu)}$, where $\mathfrak{D}^{(\nu)} = (\nu, \nu+1)$. From each $\mathfrak{D}^{(\nu)}$ to $\mathbb{C}$, the function $B(\{y\})$ is analytically continued to a polynomial function $B_k^{(\nu)}(y) = B_k(y-\nu) \in \mathbb{Q}[y]$.



$\mathbb{R} \setminus \mathfrak{H} = \coprod_{\nu \in \mathfrak{J}} \mathfrak{D}^{(\nu)}$ $\quad$ $B_k(\{y\})$ $\quad$ $B_k^{(0)}(y) = B_k(y)$

A similar procedure works well in general cases and we can define multiple generalizations of Bernoulli polynomials.

Let

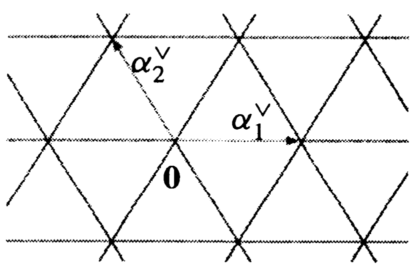
$$\mathfrak{H} = \bigcup_{\mathbf{V} \in \mathcal{V}} \bigcup_{q \in Q^\vee} \bigcup_{\beta \in \mathbf{V}} \{\mathbf{y} \in V \mid \{\mathbf{y}+q\}_{\mathbf{V},\beta} \in \mathbb{Z}\}$$

(discontinuous points of $\{\mathbf{y}+q\}_{\mathbf{V},\beta}$ appearing in the generating function).

Let

$$V \setminus \mathfrak{H} = \coprod_{\nu \in \mathfrak{J}} \mathfrak{D}^{(\nu)},$$

where $\mathfrak{D}^{(\nu)}$ is an open connected component, $\mathfrak{J}$ is a set of indices.

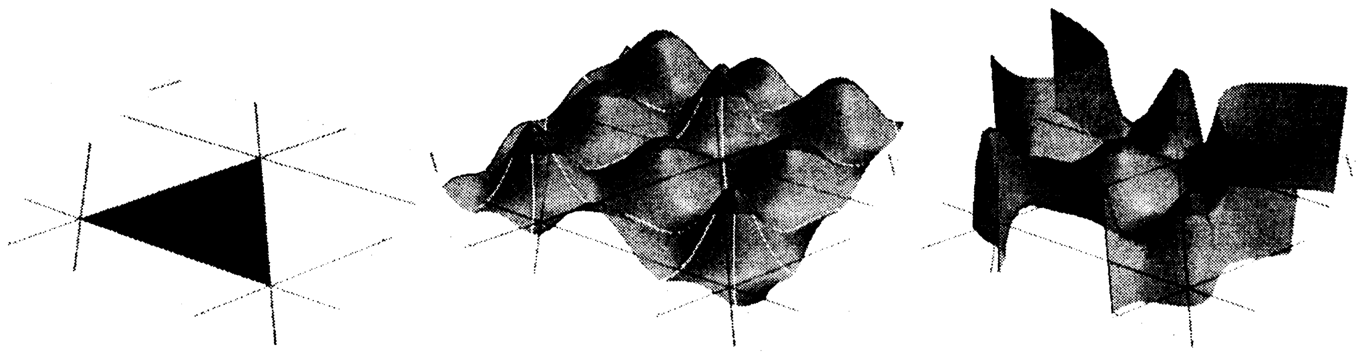


A_2 case

Theorem 3 ([8, 9, 10]). *From each region $\mathfrak{D}^{(\nu)}$ to the whole space $\mathbb{C} \otimes V$, $P(\mathbf{k},\mathbf{y};X_r)$ is analytically continued in $\mathbf{y}$ to a polynomial function $B_{\mathbf{k}}^{(\nu)}(\mathbf{y};X_r) \in \mathbb{Q}[\mathbf{y}]$ of total degree at most $|\mathbf{k}| = \sum_{\alpha \in \Delta_+} k_\alpha$, where $\mathbf{y} = \sum_{n=1}^r y_n \alpha_n^\vee$.*

§§9.1. Example: A_2 Case

The Bernoulli polynomial $B_{\mathbf{2}}^{(\mathbf{0})}(\mathbf{y};A_2)$ is obtained by the analytic continuation of the periodic Bernoulli function $P(\mathbf{2},\mathbf{y};A_2)$ from the region $\mathfrak{D}^{(\mathbf{0})}$.

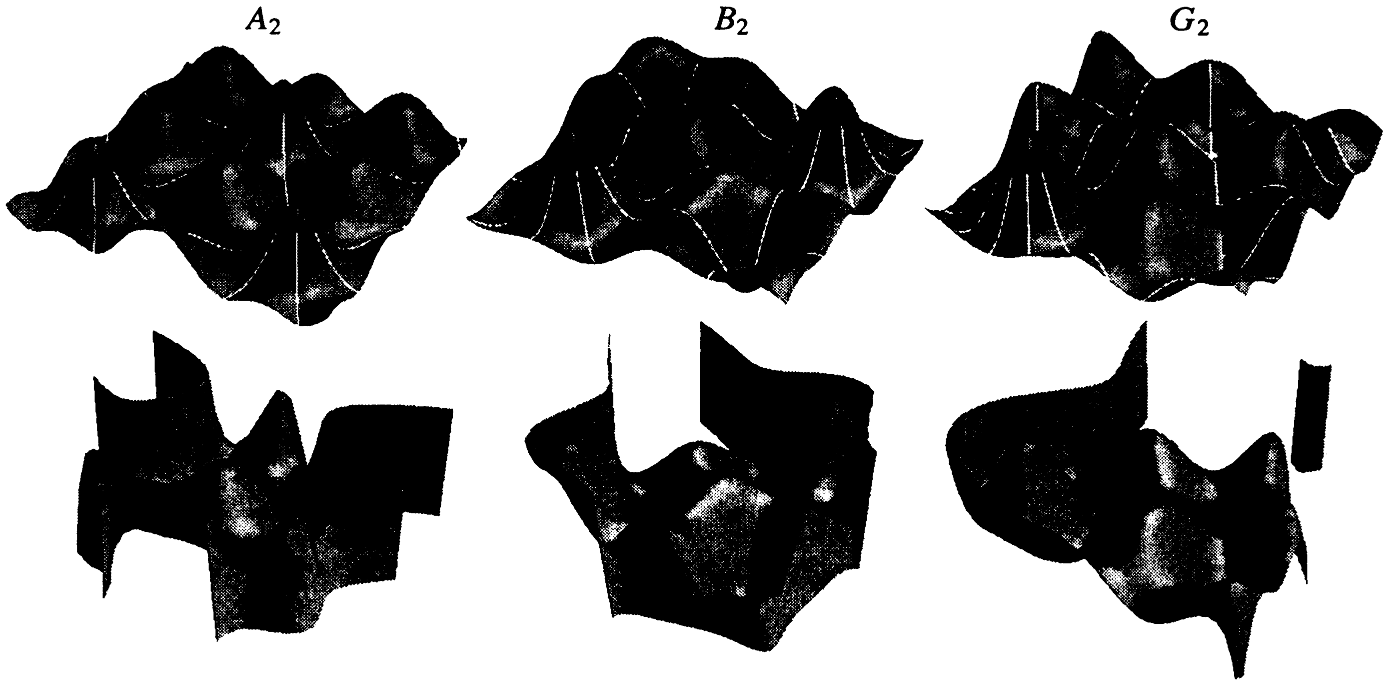


$V \setminus \mathfrak{H} = \coprod_{\nu \in \mathfrak{J}} \mathfrak{D}^{(\nu)}$ $\quad$ $P(\mathbf{2},\mathbf{y};A_2)$ (Periodic Bernoulli function) $\quad$ $B_{\mathbf{2}}^{(\mathbf{0})}(\mathbf{y};A_2)$ (Bernoulli polynomial)

The explicit form of the Bernoulli polynomial $B_{\mathbf{2}}^{(\mathbf{0})}(\mathbf{y};A_2)$ is given as follows:

$$
\begin{aligned}
B_{\mathbf{2}}^{(\mathbf{0})}(\mathbf{y};A_2) = {} & \frac{1}{3780}+\frac{1}{45}(y_1y_2-y_1^2-y_2^2)+\frac{1}{18}(3y_1y_2^2-3y_1^2y_2+2y_1^3) \\
& +\frac{1}{9}(-2y_1y_2^3-3y_1^2y_2^2+4y_1^3y_2-2y_1^4+y_2^4) \\
& +\frac{1}{30}(-5y_1y_2^4+10y_1^2y_2^3+10y_1^3y_2^2-15y_1^4y_2+6y_1^5) \\
& +\frac{1}{30}(6y_1y_2^5-5y_1^2y_2^4-5y_1^4y_2^2+6y_1^5y_2-2y_1^6-2y_2^6) \in \mathbb{Q}[\mathbf{y}].
\end{aligned}
$$

§§9.2. Further Examples: A_2, B_2, G_2 Cases



The graphs in the upper (resp. lower) row are those of periodic Bernoulli functions (resp. Bernoulli polynomials).

We summarize what we have obtained: we have constructed periodic Bernoulli functions so that they describe functional-relations of multiple zeta-functions of root systems, which can be calculated by using the generating function; Bernoulli polynomials are obtained by the analytic continuation of periodic Bernoulli functions.

$$\sum_{w\in W}\Bigl(\prod_{\alpha\in\Delta_+\cap w^{-1}\Delta_-}(-1)^{k_\alpha}\Bigr)\zeta_r(w^{-1}\mathbf{k},w^{-1}\mathbf{y};X_r)=(-1)^{|\Delta_+|}P(\mathbf{k},\mathbf{y};X_r)\Bigl(\prod_{\alpha\in\Delta_+}\frac{(2\pi i)^{k_\alpha}}{k_\alpha!}\Bigr),$$

$$F(\mathbf{t},\mathbf{y};X_r)=\sum_{\mathbf{k}\in\mathbb{Z}_{\geq 0}^{|\Delta_+|}}P(\mathbf{k},\mathbf{y};X_r)\prod_{\alpha\in\Delta_+}\frac{t_\alpha^{k_\alpha}}{k_\alpha!},$$

$$P(\mathbf{k},\mathbf{y};X_r)\Longleftrightarrow B_{\mathbf{k}}^{(\nu)}(\mathbf{y};X_r)\in\mathbb{Q}[\mathbf{y}].$$

§10. L-Functions of Root Systems

We give an application of periodic Bernoulli functions or equivalently Bernoulli polynomials. For this purpose, we define an L-analogue of zeta-functions of root systems.

Definition 4 ([9, 10]). L-functions of root systems: For a root system Δ of type X_r, define

$$L_r(\mathbf{s},\chi;X_r)=\sum_{\lambda\in P_{++}}\prod_{\alpha\in\Delta_+}\frac{\chi_\alpha(\langle\alpha^\vee,\lambda\rangle)}{\langle\alpha^\vee,\lambda\rangle^{s_\alpha}},$$

where $\chi=(\chi_\alpha)_{\alpha\in\Delta_+}$ is a set of primitive Dirichlet characters of conductors $f_\alpha\in\mathbb{Z}_{\geq 1}$.

We extend $\chi=(\chi_\alpha)_{\alpha\in\Delta_+}$ to $(\chi_\alpha)_{\alpha\in\Delta}$ by $\chi_\alpha=\chi_{-\alpha}$ and define $(w\chi)_\alpha=\chi_{w^{-1}\alpha}$. Then we have value-relations of L-functions.

Theorem 4 ([9, 10]). *For* $\mathbf{s}=\mathbf{k}=(k_\alpha)_{\alpha\in\Delta_+}\in\mathbb{Z}_{\geq 2}^{|\Delta_+|}$,

$$\sum_{w\in W}\Big(\prod_{\alpha\in\Delta_+\cap w^{-1}\Delta_-}(-1)^{k_\alpha}\chi_\alpha(-1)\Big)L_r(w^{-1}\mathbf{k},w^{-1}\chi;X_r)$$
$$=(-1)^{|\Delta_+|}\Big(\prod_{\alpha\in\Delta_+}\chi_\alpha(-1)g(\chi_\alpha)\frac{(2\pi i)^{k_\alpha}}{k_\alpha!f^{k_\alpha}}\Big)B_{\mathbf{k},\overline{\chi}}(X_r),$$

where $B_{\mathbf{k},\chi}(X_r)$ is a multiple generalized Bernoulli number (defined later).

cf. $(X_r=A_1)$

$$L(k,\chi)+(-1)^k\chi(-1)L(k,\chi)=-\chi(-1)g(\chi)\frac{(2\pi i)^k}{k!f^k}B_{k,\overline{\chi}}.$$

§11. Special L-Values

Theorem 4 immediately implies a formula for special values of L-functions:

Theorem 5 ([9, 10]). *For* $\mathbf{k}\in(\mathbb{Z}_{\geq 2})^{|\Delta_+|}$ *and* χ *s.t.* $w^{-1}\mathbf{k}=\mathbf{k}$, $w^{-1}\chi=\chi$ *for all* $w\in W$ *and* $(-1)^{k_\alpha}\chi_\alpha(-1)=1$ *for all* $\alpha\in\Delta_+$,

$$L_r(\mathbf{k},\chi;X_r)=\frac{(-1)^{|\mathbf{k}|+|\Delta_+|}}{|W|}\Big(\prod_{\alpha\in\Delta_+}\frac{(2\pi i)^{k_\alpha}}{k_\alpha!f_\alpha^{k_\alpha}}g(\chi_\alpha)\Big)B_{\mathbf{k},\overline{\chi}}(X_r).$$

cf. $(X_r=A_1)$

$$L(k,\chi)=\frac{(-1)^{k+1}}{2}\frac{(2\pi i)^k}{k!f^k}g(\chi)B_{k,\overline{\chi}}.$$

As an example, let ρ_7 be the Dirichlet character of conductor 7 defined by $\rho_7(1)=\rho_7(6)=1$, $\rho_7(2)=\rho_7(5)=e^{2\pi i/3}$, $\rho_7(3)=\rho_7(4)=e^{4\pi i/3}$. Then the Gauss sum is

$g(\rho_7) = 2\big(\cos(2\pi/7) + e^{2\pi i/3}\cos(4\pi/7) + e^{4\pi i/3}\cos(6\pi/7)\big)$ and we have

$$L_2((2,4,4,2),(1,\rho_7,\rho_7,1);B_2) = \sum_{m,n=1}^{\infty} \frac{\rho_7(n)\rho_7(m+n)}{m^2n^4(m+n)^4(m+2n)^2}$$
$$= \frac{(-1)^{12+4}}{2^2 2!}\left(\frac{(2\pi i)^2}{2!}\right)^2\left(\frac{(2\pi i)^4}{4!7^4}g(\rho_7)\right)^2\left(\frac{69967019}{6988350600} + \frac{102810289\sqrt{-3}}{6988350600}\right)$$
$$= g(\rho_7)^2\pi^{12}\left(\frac{69967019}{181289027372537700} + \frac{102810289\sqrt{-3}}{181289027372537700}\right).$$

We give two more examples. Let ρ_5 be the quadratic character of conductor 5. Then we have

$$L_2((2,2,2,2),(\rho_5,\rho_5,\rho_5,\rho_5);B_2) = \frac{92}{29296875}\pi^8;$$
$$L_3((2,2,2,2,2,2),(\rho_5,\rho_5,\rho_5,\rho_5,\rho_5,\rho_5);A_3) = -\frac{1856}{213623046875}\pi^{12}.$$

The latter can be regarded as a character analogue of the formula in [1, Prop. 8.5].

§12. Multiple Generalized Bernoulli Numbers

The generating function of multiple generalized Bernoulli numbers is given in terms of that of multiple Bernoulli polynomials as in the classical theory.

Definition 5 (generating function [9, 10]). For $\mathbf{t} = (t_\alpha)_{\alpha\in\Delta_+}$,

$$G(\mathbf{t},\chi;X_r) = \sum_{\substack{a_\alpha=1\\ \alpha\in\Delta_+}}^{f_\alpha} \Big(\prod_{\alpha\in\Delta_+} \frac{\chi_\alpha(a_\alpha)}{f_\alpha}\Big) F(\mathbf{ft},\mathbf{y}(\mathbf{a};\mathbf{f});X_r),$$

where $F(\mathbf{t},\mathbf{y};X_r)$ is the generating function of multiple periodic Bernoulli functions and $\mathbf{ft} = (f_\alpha t_\alpha)_{\alpha\in\Delta_+}$, $\mathbf{y}(\mathbf{a};\mathbf{f}) = \sum_{\alpha\in\Delta_+} a_\alpha\alpha^\vee/f_\alpha$.

Definition 6 (multiple generalized Bernoulli numbers [9, 10]).

$$G(\mathbf{t},\chi;X_r) = \sum_{\mathbf{k}\in\mathbb{Z}_{\geq 0}^{|\Delta_+|}} B_{\mathbf{k},\chi}(X_r) \prod_{\alpha\in\Delta_+} \frac{t_\alpha^{k_\alpha}}{k_\alpha!},$$

$$B_{\mathbf{k},\chi}(X_r) = \Big(\prod_{\alpha\in\Delta_+} f_\alpha^{k_\alpha-1}\Big) \sum_{\substack{a_\alpha=1\\ \alpha\in\Delta_+}}^{f_\alpha} \Big(\prod_{\alpha\in\Delta_+} \chi_\alpha(a_\alpha)\Big) P(\mathbf{k},\mathbf{y}(\mathbf{a};\mathbf{f});X_r).$$

cf. $(X_r = A_1)$

$$G(t,\chi) = \sum_{a=1}^{f} \frac{\chi(a)}{f} F(ft,a/f) = \sum_{a=1}^{f} \frac{\chi(a)}{f} \frac{fte^{ft\{a/f\}}}{e^{ft}-1} = \sum_{k=0}^{\infty} B_{k,\chi}\frac{t^k}{k!}.$$

$$B_{k,\chi} = f^{k-1}\sum_{a=1}^{f} \chi(a)B_k(\{a/f\}).$$

§§12.1. Properties

Theorem 6 ([9, 10]). *Assume that $f_\alpha > 1$ if Δ is of type A_1. Then for $w \in W$,*

$$B_{w^{-1}\mathbf{k},w^{-1}\chi}(X_r) = \Big(\prod_{\alpha\in\Delta_+\cap w^{-1}\Delta_-} (-1)^{k_\alpha}\chi_\alpha(-1)\Big) B_{\mathbf{k},\chi}(X_r).$$

Hence $B_{\mathbf{k},\chi}(X_r) = 0$ if there exists an element $w \in W_{\mathbf{k}} \cap W_\chi$ such that

$$\prod_{\alpha\in\Delta_+\cap w^{-1}\Delta_-} (-1)^{k_\alpha}\chi_\alpha(-1) \neq 1,$$

where $W_{\mathbf{k}}$ and W_χ are the stabilizers of $\mathbf{k}$ and χ respectively.

cf. ($X_r = A_1$)

$$B_{k,\chi} = 0 \quad \text{if } (-1)^k\chi(-1) \neq 1.$$

Several other properties in the classical theory such as

$$F(t,y) = F(-t,-y) \text{ for } y \in \mathbb{R}\setminus\mathbb{Z}, \quad B_k(1-y) = (-1)^k B_k(y),$$
$$\frac{1}{t}\frac{\partial}{\partial y}F(t,y) = F(t,y)$$

can be reinterpreted in terms of root systems and Weyl groups.

§13. Zeta-Functions for Lie Groups

Recall that Witten zeta-functions were originally introduced for compact semisimple Lie groups. It is known that there is one-to-one correspondence between finite dimensional representations of complex semisimple Lie algebra $\mathfrak{g}$ and those of simply connected compact semisimple Lie group G. In the cases of general compact semisimple Lie groups, we need analytically integral forms L for a maximal torus of G, which satisfies $Q \subset L \subset P$.

Definition 7 (Zeta-functions of Lie groups). For a root system Δ of type X_r, define

$$\zeta_r(\mathbf{s},\mathbf{y};X_r;L) = \sum_{\lambda\in L\cap P_{++}} e^{2\pi i\langle y,\lambda\rangle} \prod_{\alpha\in\Delta_+} \frac{1}{\langle\alpha^\vee,\lambda\rangle^{s_\alpha}}$$

$$F(\mathbf{t},\mathbf{y};X_r;L) = \sum_{\mu\in P^\vee/Q^\vee} \hat{\chi}_L(\mu) F(\mathbf{t},\mathbf{y}+\mu;X_r)$$

$$= \sum_{\mathbf{k}\in\mathbb{Z}_{\geq 0}^{|\Delta_+|}} P(\mathbf{k},\mathbf{y};X_r;L) \prod_{\alpha\in\Delta_+} \frac{t_\alpha^{k_\alpha}}{k_\alpha!}$$

where $\hat{\chi}_L : P^\vee/Q^\vee \to \mathbb{C}^*$ is given by $\hat{\chi}_L(\mu) = \dfrac{1}{|P/Q|}\sum_{\lambda\in L/Q} e^{-2\pi i\langle\mu,\lambda\rangle}$.

Note that these definitions are based on the origin of L-functions, that is, Dirichlet's theorem on arithmetic progressions.

As an application, we obtain for example,

$$\zeta_2(\mathbf{2},\mathbf{0};A_2;Q)=\sum_{2m-n,2n-m>0}\frac{1}{(2m-n)^2(2n-m)^2(m+n)^2}$$
$$=\frac{(-1)^3}{3!}\frac{187}{918540}\Big(\frac{(2\pi i)^2}{2!}\Big)^3=\frac{187\pi^6}{688905}.$$

§14. Integral Representation

The analytic continuations of multiple zeta-functions were already obtained by Essouabri [3], Matsumoto [12], de Crisenoy [2], etc. However we give yet another method which is a generalization of the formula

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}=\frac{1}{\Gamma(s)(e^{2\pi is}-1)}\int_C\frac{z^{s-1}}{e^z-1}dz \qquad (C\text{: Hankel contour}).$$

For $\boldsymbol{\xi}\in\mathbb{C}^R$, $\boldsymbol{a},\boldsymbol{s}\in\mathbb{C}^N$ and $\boldsymbol{b}\in\mathbb{C}^{N\times R}$, consider the multiple series

$$\zeta(\boldsymbol{\xi},\boldsymbol{a},\boldsymbol{b},\boldsymbol{s})=$$
$$\sum_{m_1=0}^{\infty}\cdots\sum_{m_R=0}^{\infty}\frac{e^{\xi_1m_1}\cdots e^{\xi_Rm_R}}{(a_1+b_{11}m_1+\cdots+b_{1R}m_R)^{s_1}\cdots(a_N+b_{N1}m_1+\cdots+b_{NR}m_R)^{s_N}}.$$

Theorem 7 ([4, 5]).

$$\zeta(\boldsymbol{\xi},\boldsymbol{a},\boldsymbol{b},\boldsymbol{s})=\frac{1}{\Gamma(s_1)\cdots\Gamma(s_N)}\prod_{t\in S}\frac{1}{e^{2\pi it(s)}-1}\times$$
$$\int_\Sigma\frac{e^{(b_{11}+\cdots+b_{1R}-a_1)z_1}\cdots e^{(b_{N1}+\cdots+b_{NR}-a_N)z_N}z_1^{s_1-1}\cdots z_N^{s_N-1}}{(e^{z_1b_{11}+\cdots+z_Nb_{N1}}-e^{\xi_1})\cdots(e^{z_1b_{1R}+\cdots+z_Nb_{NR}}-e^{\xi_R})}dz_1\wedge\cdots\wedge dz_N,$$

where Σ is essentially a union of surfaces and S is a set of linear functionals on $\mathbb{C}^N$.

From the integrand, we can construct generating functions of Bernoulli numbers for nonpositive domain.

§15. Possibilities of Generalizations to Elliptic Analogues

Lastly we give two possibilities of generalizations to "elliptic" analogues by regarding $\zeta_r(\mathbf{s},\mathbf{y};X_r)$ as "rational" or "trigonometric" versions.

The first is Eisenstein analogue. The Eisenstein series is defined by

$$G_k(\tau;x,y)=\sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}}\frac{e^{2\pi i(mx+ny)}}{(m+n\tau)^k}.$$

Let $(x,y)\in\mathbb{R}^2\setminus\mathbb{Z}^2$ and

$$e^{2\pi ixt}\frac{\theta'(0;\tau)\theta(t+x\tau-y;\tau)}{\theta(t;\tau)\theta(x\tau-y;\tau)}=\sum_{k=0}^{\infty}\mathcal{H}_k(x,y;\tau)\frac{(2\pi i)^kt^{k-1}}{k!}.$$

Then we have the following elliptic analogue.

Theorem 8 (Katayama (1978)). *For $k \in \mathbb{N}$ with $k \geq 2$, we have*

$$G_k(\tau;x,y) = -\mathcal{H}_k(x,y;\tau)\frac{(2\pi i)^k}{k!}.$$

The second is q-analogue. Instead of Weyl's dimension formula, we employ the character formula. Let $q = e^{-2\pi i/\tau}$ with $\Im\tau > 0$ and

$$\zeta_q(s,z;x) = \sum_{m=1}^{\infty} \frac{e^{2\pi i m x} q^{mz}}{[m]_q^s}, \qquad [m]_q = \frac{1-q^m}{1-q}.$$

Let

$$\psi(t) = \frac{\tau}{2\pi i}\frac{e^{2\pi i t/\tau}-1}{e^{2\pi i t z/\tau}} = t + O(t^2) \qquad \text{(i.e. local coordinate around 0).}$$

Define

$$e^{2\pi i x t}\frac{\theta'(0;\tau)\theta(t+x\tau-y;\tau)}{\theta(t;\tau)\theta(x\tau-y;\tau)} = \sum_{k=0}^{\infty} \mathcal{Q}_k(x,y,z;\tau)\frac{(2\pi i/\tau)^k\psi'(t)\psi(t)^{k-1}}{(q;q)^k}.$$

Then

Theorem 9. *For $k \in \mathbb{N}$, $y+kz \in \mathbb{Z}$, $0 < z < 1$ and $x \in \mathbb{R}$, we have*

$$\zeta_q(k,k(1-z);x) + (-1)^k\zeta_q(k,kz;-x) = -\mathcal{Q}_k(x,y,z;\tau)\frac{1}{[k]_q!}.$$

In particular, for $\tau = i$,

$$\zeta_q(2,1;0) = (1-e^{-2\pi})^2\frac{\pi-3}{24\pi}, \qquad \zeta_q(4,2;0) = (1-e^{-2\pi})^4\frac{30\pi^3 - 11\pi^4 + 3\varpi^4}{1440\pi^4}.$$

Our future work is to construct generalizations to arbitrary root systems.

References

[1] P. E. Gunnells and R. Sczech, Evaluation of Dedekind sums, Eisenstein cocycles, and special values of L-functions, Duke Math. J. **118** (2003), 229-260.

[2] M. de Crisenoy, *Values at T-tuples of negative integers of twisted multivariable zeta series associated to polynomials of several variables*, Compos. Math. **142** (2006), 1373–1402.

[3] D. Essouabri, *Singularité des séries de Dirichlet associées à des polynômes de plusieurs variables et applications en théorie analytique des nombres*, Ann. Inst. Fourier (Grenoble) **47** (1997), no. 2, 429–483.

[4] Y. Komori, *An integral representation of Mordell-Tornheim double zeta function and its values at non-positive integers*, Ramanujan J. **17** (2008) no. 2, 163–183.

[5] Y. Komori, *An integral representation of multiple Hurwitz-Lerch zeta functions and generalized multiple Bernoulli numbers*, Quart. J. Math., to appear.

[6] Y. Komori, K. Matsumoto and H. Tsumura, Zeta-functions of root systems, in "Proceedings of the Conference on L-functions" (Fukuoka, 2006), L. Weng and M. Kaneko (eds), World Scientific, 2007, pp. 115–140.

[7] Y. Komori, K. Matsumoto and H. Tsumura, *On Witten multiple zeta-functions associated with semisimple Lie algebras II*, J. Math. Soc. Japan, to appear.

[8] Y. Komori, K. Matsumoto and H. Tsumura, *On Witten multiple zeta-functions associated with semisimple Lie algebras III*, preprint, submitted for publication.

[9] Y. Komori, K. Matsumoto and H. Tsumura, *On multiple Bernoulli polynomials and multiple L-functions of root systems*, Proc. London Math. Soc., to appear.

[10] Y. Komori, K. Matsumoto and H. Tsumura, *Zeta and L-functions and Bernoulli polynomials of root systems*, Proc. Japan Acad. Ser. A Math. Sci. **84** (2008) no. 5, 57–62.

[11] Y. Komori, K. Matsumoto and H. Tsumura, *On multiple Bernoulli polynomials and multiple L-functions of root systems*, in 'Analytic Number Theory and Related Areas', RIMS Kokyuroku **1665**, Kyoto Univ., 2009, pp. 114–126.

[12] K. Matsumoto, *Analytic properties of multiple zeta-functions in several variables*, in 'Number Theory: Tradition and Modrnization', Proc. 3rd China-Japan Seminar, W. Zhang and Y. Tanigawa (eds.), Springer, 2006, pp. 153–173.

[13] K. Matsumoto and H. Tsumura, *On Witten multiple zeta-functions associated with semisimple Lie algebras I*, Ann. Inst. Fourier **56** (2006), 1457–1504.

[14] M. V. Subbarao and R. Sitaramachandrarao, *On some infinite series of L. J. Mordell and their analogues*, Pacific J. Math. **119** (1985), 245-255.

[15] A. Szenes, *Iterated residues and multiple Bernoulli polynomials*, Internat. Math. Res. Notices **18** (1998), 937-958.

[16] E. Witten, *On quantum gauge theories in two dimensions*, Comm. Math. Phys. **141** (1991), 153–209.

[17] D. Zagier, *Values of zeta functions and their applications*, in 'First European Congress of Mathematics' Vol. II, A. Joseph et al. (eds.), Progr. Math. **120**, Birkhäuser, 1994, pp. 497–512.