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KERNEL FUNCTION AND QUANTUM ALGEBRAS

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ABSTRACT. We introduce an analogue $K_n(x,z;q,t)$ of the Cauchy-type kernel function for the Macdonald polynomials, being constructed in the tensor product of the ring $\Lambda_{\mathbb{F}}$ of symmetric functions and the commutative algebra $\mathcal{A}$ over the degenerate $\mathbb{CP}^1$. We show that a certain restriction of $K_n(x,z;q,t)$ with respect to the variable z is neatly described by the tableau sum formula of Macdonald polynomials. Next, we demonstrate that the level m representation of the Ding-Iohara quantum algebra $\mathcal{U}(q,t)$ naturally produces the currents of the deformed $\mathcal{W}_{q,p}(\mathfrak{sl}_n)$. Then we remark that the $K_n(x,z;q,t)$ emerges in the highest-to-highest correlation function of the deformed $\mathcal{W}_{q,p}(\mathfrak{sl}_n)$ algebra.

1. KERNEL FUNCTION

1.1. The algebra $\mathcal{A}$. We briefly recall the definition and the basic facts about the commutative algebra $\mathcal{A}$ introduced in [FHHSY]. Let q_1, q_2 be two independent indeterminates and set $q_3 := 1/q_1q_2$. We also use the symbols $\mathbb{F} := \mathbb{Q}(q_1, q_2)$, $\mathbb{N} := \{0, 1, 2, \dots\}$ and $\mathbb{N}_+ := \{1, 2, \dots\}$.

For $n, k \in \mathbb{N}_+$, we define two operators $\partial^{(0,k)}, \partial^{(\infty,k)}$ acting on the space of symmetric rational functions in n variables $x_1, \dots, x_n$ by

$$\partial^{(0,k)} \quad : \quad f \quad \mapsto \quad \frac{n!}{(n-k)!} \lim_{\xi \to 0} f(x_1, \dots, x_{n-k}, \xi x_{n-k+1}, \xi x_{n-k+2}, \dots, \xi x_n)$$

$$\partial^{(\infty,k)} \quad : \quad f \quad \mapsto \quad \frac{n!}{(n-k)!} \lim_{\xi \to \infty} f(x_1, \dots, x_{n-k}, \xi x_{n-k+1}, \xi x_{n-k+2}, \dots, \xi x_n)$$

whenever the limit exists. We also set $\partial^{(0,k)} c = 0, \partial^{(\infty,k)} c = 0$ for $c \in \mathbb{F}$. Finally we define $\partial^{(0,0)}$ and $\partial^{(\infty,0)}$ to be the identity operator.

Definition 1.1. For $n \in \mathbb{N}$, the vector space $\mathcal{A}_n = \mathcal{A}_n(q_1, q_2, q_3)$ is defined by the following conditions (i), (ii), (iii) and (iv).
(i) $\mathcal{A}_0 := \mathbb{F}$. For $n \in \mathbb{N}_+$, $f(x_1, \dots, x_n) \in \mathcal{A}_n$ is a rational function with coefficients in $\mathbb{F}$, and symmetric with respect to the x_i's.
(ii) For $n \in \mathbb{N}$, $0 \leq k \leq n$ and $f \in \mathcal{A}_n$, the limits $\partial^{(\infty,k)} f$ and $\partial^{(0,k)} f$ both exist and coincide: $\partial^{(\infty,k)} f = \partial^{(0,k)} f$ (degenerate $\mathbb{CP}^1$ condition).
(iii) The poles of $f \in \mathcal{A}_n$ are located only on the diagonal $\{(x_1, \dots, x_n) \mid \exists (i,j), i \neq j, x_i = x_j\}$, and the orders of the poles are at most two.
(iv) For $n \geq 3$, $f \in \mathcal{A}_n$ satisfies the wheel conditions

$$f(x_1, q_1x_1, q_1q_2x_1, x_4, \dots) = 0, \qquad f(x_1, q_2x_1, q_1q_2x_1, x_4, \dots) = 0.$$

Then we set the graded vector space $\mathcal{A} = \mathcal{A}(q_1, q_2, q_3) := \bigoplus_{n \geq 0} \mathcal{A}_n$.

Definition 1.2. For an m-variable symmetric rational function f and an n-variable symmetric rational function g, we define an $(m+n)$-variable symmetric rational function $f * g$ by

$$(f * g)(x_1, \dots, x_{m+n}) := \mathrm{Sym}\left[f(x_1, \dots, x_m) g(x_{m+1}, \dots, x_{m+n}) \prod_{\substack{1 \leq \alpha \leq m \\ m+1 \leq \beta \leq m+n}} \omega(x_\alpha, x_\beta) \right]. \tag{1.1}$$

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Here $\omega(x,y)$ is the rational function

$$\omega(x,y)=\omega(x,y;q_1,q_2,q_3):=\frac{(x-q_1y)(x-q_2y)(x-q_3y)}{(x-y)^3}, \tag{1.2}$$

and the symbol Sym means $\mathrm{Sym}(f(x_1,\dots,x_n)):=(1/n!)\ \sum_{\sigma\in\mathfrak{S}_n} f(x_{\sigma(1)},\dots,x_{\sigma(n)})$.

Fact 1.3 ([FHHSY, Theorem 1.5]). $\mathcal{A}$ is closed with respect to $*$, and the pair $(\mathcal{A},*)$ is a unital associative commutative algebra. The Poincaré series is $\sum_{n\geq 0}(\dim_{\mathbb{F}}\mathcal{A}_n)z^n=\prod_{m\geq 1}(1-z^m)^{-1}$.

1.2. **The ring $\Lambda_{\mathbb{F}}$ of symmetric functions.** As for the notations and definitions concerning the partitions, we basically follow the notation in [M]. A partition of $n\in\mathbb{N}$ is a sequence $\lambda=(\lambda_1,\lambda_2,\dots)$ of non-negative integers satisfying $\lambda_1\geq\lambda_2\geq\cdots$. We define $|\lambda|:=\lambda_1+\lambda_2+\cdots$, $\ell(\lambda):=\#\{i\mid\lambda_i\neq 0\}$, and write $\lambda\vdash n$ if $|\lambda|=n$. We denote the conjugate (transpose) of a partition λ by λ'. We work with the dominance partial ordering defined as : $\lambda\geq\mu \overset{\text{def}}{\Longleftrightarrow} |\lambda|=|\mu|,\ \lambda_1+\cdots+\lambda_i\geq \mu_1+\cdots+\mu_i$ for all $i\geq 1$.

We recall some basic facts about the ring of symmetric functions. As was in [FHHSY], we set $q_1=q^{-1},q_2=t$ (hence $q_3=qt^{-1}$) and $\mathbb{F}=\mathbb{Q}(q_1,q_2)=\mathbb{Q}(q,t)$. Let $\Lambda_{\mathbb{F}}$ be the ring of symmetric functions over the base field $\mathbb{F}$, constructed in the category of graded ring with the projection operators $\rho_{m,n}: f(x_1,\dots,x_m)\mapsto f(x_1,\dots,x_n,0\dots,0)$.

Let $p_n(x):=\sum_i x_i^n$ be the power sum function. For a partition $\lambda=(\lambda_1,\lambda_2,\dots)$, the monomial symmetric function is defined by $m_\lambda(x):=\sum_\alpha x^\alpha$, where α runs over all the distinct permutations of λ. The elementary symmetric function $e_n(x)$ is defined by the generating function $E(y):=\prod_i(1+x_iy)=\sum_{n\geq 0}e_n(x)y^n$. Set $G(y):=\prod_i\{(tx_iy;q)_\infty/(x_iy;q)_\infty\}=\sum_{n\geq 0}g_n(x;q,t)y^n$, where $(x;q)_\infty:=\prod_{i\geq 0}(1-q^ix)$. For a partition $\lambda=(\lambda_1,\lambda_2,\dots)$ set $p_\lambda:=p_{\lambda_1}p_{\lambda_2}\cdots$. Similarly we write $e_\lambda:=e_{\lambda_1}e_{\lambda_2}\cdots$ and $g_\lambda:=g_{\lambda_1}g_{\lambda_2}\cdots$. It is known that $\{p_\lambda\}$, $\{m_\lambda\}$, $\{e_\lambda\}$ and $\{g_\lambda\}$ form bases of $\Lambda_{\mathbb{F}}$.

Recall Macdonald's scalar product $\langle p_\lambda,p_\mu\rangle_{q,t}:=\delta_{\lambda,\mu}\prod_{i\geq 1}i^{m_i}m_i!\prod_{j\geq 1}(1-q^{\lambda_j})/(1-t^{\lambda_j})$, where m_i denotes the number of parts i in the partition λ. For any dual bases $\{u_\lambda\}$ and $\{v_\lambda\}$, we have

$$\Pi(x,y;q,t):=\prod_{i,j}\frac{(tx_iy_j;q)_\infty}{(x_iy_j;q)_\infty}=\sum_\lambda u_\lambda(x)v_\lambda(y). \tag{1.3}$$

It is known that $\{m_\lambda\}$ and $\{g_\lambda\}$ form dual bases, namely we have $\langle m_\lambda,g_\mu\rangle_{q,t}=\delta_{\lambda,\mu}$.

Macdonald polynomials $P_\lambda(x;q,t)$ are uniquely characterized by (a) the triangular expansion $P_\lambda=m_\lambda+\sum_{\mu<\lambda}a_{\lambda\mu}m_\mu$ $(a_{\lambda\mu}\in\mathbb{F})$, and (b) the orthogonality $\langle P_\lambda,P_\mu\rangle_{q,t}=0$ if $\lambda\neq\mu$.

Se set

$$b_\lambda(q,t):=\langle P_\lambda(z;q,t),P_\lambda(z;q,t)\rangle_{q,t}^{-1},\quad Q_\lambda(z;q,t):=b_\lambda(q,t)P_\lambda(z;q,t). \tag{1.4}$$

Then $\{Q_\lambda\}$ forms a dual basis to $\{P_\lambda\}$.

1.3. **The isomorphism $\iota:\Lambda_{\mathbb{F}}\to\mathcal{A}$.** Both $\Lambda_{\mathbb{F}}$ and $\mathcal{A}$ are commutative rings having the same Poincaré series $\sum_{n\geq 0}(\dim_{\mathbb{F}}\Lambda_{\mathbb{F}}^n)z^n=\sum_{n\geq 0}(\dim_{\mathbb{F}}\mathcal{A}_n)z^n=\prod_{m\geq 1}(1-z^m)^{-1}$, where $\Lambda_{\mathbb{F}}^n$ denotes the ring of symmetric functions of degree n. Moreover it was shown in [FHHSY] that there is a natural way to identify $\Lambda_{\mathbb{F}}$ and $\mathcal{A}$ from the point of view of the free field construction of the Macdonald operators. Based on the finding in [FHHSY] we give an isomorphism $\iota:\Lambda_{\mathbb{F}}\to\mathcal{A}$ as follows.

For $p\in\mathbb{F}$, let

$$\epsilon_n(z_1,z_2,\dots,z_n;p):=\prod_{1\leq i<j\leq n}\frac{(z_i-pz_j)(z_i-p^{-1}z_j)}{(z_i-z_j)^2}, \tag{1.5}$$

and set $\epsilon_\lambda(z;p):=(\epsilon_{\lambda_1}*\epsilon_{\lambda_2}*\cdots*\epsilon_{\lambda_l})(z;p)$ for a multi-index $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_l)$.

Fact 1.4 ([FHHSY, Propositions 2.20 & 2.23]). For $i=1,2,3$, $\{\epsilon_\lambda(z;q_i)\}_{\lambda\vdash n}$ forms a basis of $\mathcal{A}_n$.

Let us write the expansions of P_λ in the bases $\{e_\mu\}$ and$\{g_\mu\}$ by

$$P_\lambda(z;q,t) = \sum_{\mu \geq \lambda'} c^{e\to P}_{\lambda\mu}(q,t) e_\mu(z;q,t), \quad P_\lambda(x;q,t) = \sum_{\mu\geq\lambda} c^{g\to P}_{\lambda\mu}(q,t) g_\mu(x;q,t). \tag{1.6}$$

A detailed study of the algebra $\mathcal{A}$ with the help of the free field representation allowed us to establish the following equality.

Fact 1.5 ([FHHSY, §3 E]). Set the next two elements in $\mathcal{A}$.

$$f^{(q^{-1})}_\lambda(z;q,t) := \frac{t^{-|\lambda|}}{(1-t^{-1})^{|\lambda|}|\lambda|!} \sum_{\mu\geq\lambda'} c^{e\to P}_{\lambda\mu}(q,t)\epsilon_\mu(z;q) \frac{|\mu|!}{\prod_{i=1}^{\ell(\mu)} \mu_i!}, \tag{1.7}$$

$$f^{(t)}_\lambda(z;q,t) := \frac{(-1)^{|\lambda|}}{(1-q)^{|\lambda|}|\lambda|!} \sum_{\mu\geq\lambda} c^{g\to P}_{\lambda\mu}(q,t)\epsilon_\mu(z;t) \frac{|\mu|!}{\prod_{i=1}^{\ell(\mu)} \mu_i!}. \tag{1.8}$$

Then we have $f^{(q^{-1})}_\lambda(z;q,t) = f^{(t)}_\lambda(z;q,t)$[1].

Definition 1.6. Let $F_\lambda(z;q,t) := f^{(q^{-1})}_\lambda(z;q,t) = f^{(t)}_\lambda(z;q,t)$.

Definition 1.7. Define the isomorphism $\iota : \Lambda_{\mathbb{F}} \to \mathcal{A}$ by

$$\iota(e_\lambda) = \frac{t^{-|\lambda|}}{(1-t^{-1})^{|\lambda|}} \frac{1}{\prod_{i=1}^{\ell(\mu)} \lambda_i!} \epsilon_\lambda(z;q).$$

Proposition 1.8. (1) We have

$$\iota(g_\lambda) = \frac{(-1)^{-|\lambda|}}{(1-q)^{|\lambda|}} \frac{1}{\prod_{i=1}^{\ell(\mu)} \lambda_i!} \epsilon_\lambda(z;t).$$

(2) We have $\iota(P_\lambda) = F_\lambda(z;q,t)$.

Proof. (1) By the Wronski relation given in [FHHSY, Proposition 3.11].
(2) By (1.6) and the definitions of ι and F_λ. □

Remark 1.9. To explain the importance of the element $F_\lambda(z;q,t)$, we recall the Gordon filtration on $\mathcal{A}$. For $p \in \mathbb{F}$ and $\lambda = (\lambda_1, \ldots, \lambda_l) \vdash n$, we defined a linear map

$$\begin{array}{cccl} \varphi^{(p)}_\lambda : & \mathcal{A}_n & \longrightarrow & \mathbb{F}(y_1,\ldots,y_l) \\ & f(z_1,\ldots,z_n) & \mapsto & f(y_1, py_1, \ldots, p^{\lambda_1-1}y_1, \\ & & & y_2, py_2, \ldots, p^{\lambda_2-1}y_2, \\ & & & \cdots \\ & & & y_l, py_l \ldots, p^{\lambda_l-1}y_l), \end{array} \tag{1.9}$$

called the *specialization map*. The Gordon filtration is given by $\mathcal{A}^{(q_i)}_{n,\lambda} := \bigcap_{\mu\not\leq\lambda} \ker \varphi^{(q_i)}_\mu$ for $i = 1,2,3$. Then by [FHHSY, Theorem 1.19] , $\mathcal{A}^{(q^{-1})}_{n,\mu} \cap \mathcal{A}^{(t)}_{n,\mu'}$ is one dimensional and is spanned by $F_\lambda(z;q,t)$.

1.4. **The kernel function.** Now we are ready to study the kernel function from the point of view of the algebra $\mathcal{A}$.

Definition 1.10. Introduce $K_n(x,z;q,t) \in \Lambda^n_{\mathbb{F}} \otimes \mathcal{A}_n$ as

$$K_n(x,z;q,t) := \sum_{\lambda\vdash n} Q_\lambda(x) F_\lambda(z;q,t).$$

[1]Note that the first and second lines of Page 25 in [FHHSY] contains typos and should be read as (1.7) and (1.8).

Remark 1.11. The name "kernel function" comes from $\Pi(x,y)$ in (1.3). By Proposition 1.8 (2), we have, in a suitable completion of $\Lambda_{\mathbb{F}} \otimes \mathcal{A}$,

$$\sum_{n\geq 0} K_n(x,z;q,t) = \sum_{\lambda} Q_\lambda(x)\iota(P_\lambda(y)),$$

where λ runs over all the partitions of every non-negative integer. Thus K_n is a homogeneous component of the analogue of $\Pi(x,y)$.

Proposition 1.12. In $\Lambda_{\mathbb{F}} \otimes \mathcal{A}$ we have

$$K_n(x,z;q,t) = \frac{(-1)^n}{(1-q)^n n!} \sum_{\lambda\vdash n} m_\lambda(x)\epsilon_\lambda(z;t)\frac{|\lambda|!}{\prod_{i=1}^{\ell(\lambda)}\lambda_i!} \tag{1.10}$$

Proof. First we show

$$m_\lambda(x) = \sum_{\mu\leq\lambda} c_{\mu\lambda}^{g\to P}(q,t)Q_\mu(x;q,t). \tag{1.11}$$

Since $\{Q_\mu(x;q,t)\}$ is a basis of $\Lambda_{\mathbb{F}}$, we can expand $m_\lambda(x) = \sum_\nu c_{\nu\lambda}Q_\nu(x;q,t)$ with $c_{\nu\lambda} \in \mathbb{F}$. Then the pairing $\langle m_\lambda, P_\mu\rangle_{q,t}$ is calculated as $\langle m_\lambda, P_\mu\rangle_{q,t} = \langle \sum_\nu c_{\nu\lambda}Q_\nu(z;q,t), P_\mu\rangle_{q,t} = c_{\mu\lambda}$, where we used the fact that $\{P_\lambda\}$ and $\{Q_\lambda\}$ are dual. On the other hand, by (1.6), we have $\langle m_\lambda, P_\mu\rangle_{q,t} = \left\langle m_\lambda, \sum_{\nu\geq\mu} c_{\mu\nu}^{g\to P}(q,t)g_\mu\right\rangle_{q,t} = c_{\mu\lambda}^{g\to P}(q,t)$. Comparing both expressions, we obtain (1.11).

Then we have

$$\begin{aligned}\text{RHS of (1.10)} &= \frac{(-1)^n}{(1-q)^n n!}\sum_{\lambda\vdash n}\sum_{\mu\leq\lambda} c_{\mu\lambda}^{g\to P}(q,t)Q_\mu(x;q,t)\epsilon_\lambda(z;t)\frac{|\lambda|!}{\prod_{i=1}^{\ell(\lambda)}\lambda_i!}\\ &= \frac{(-1)^n}{(1-q)^n n!}\sum_{\mu\vdash n} Q_\mu(x;q,t)\sum_{\lambda\geq\mu} c_{\mu\lambda}^{g\to P}(q,t)\epsilon_\lambda(z;t)\frac{|\lambda|!}{\prod_{i=1}^{\ell(\lambda)}\lambda_i!} = \sum_{\mu\vdash n} Q_\mu(x;q,t)F_\mu(z;q,t).\end{aligned}$$

□

Consider the case of finitely many variables and set $x = (x_1, x_2, \ldots, x_m)$. Also let $z = (z_1, z_2, \ldots, z_n)$ be the set of variables for the elements in $\mathcal{A}_n$.

Proposition 1.13. We have

$$K_n(x,z;q,t) = \frac{(-1)^n}{(1-q)^n n!}\sum_{i_1=1}^{m}\sum_{i_2=1}^{m}\cdots\sum_{i_n=1}^{m} x_{i_1}x_{i_2}\cdots x_{i_n}\prod_{1\leq\alpha<\beta\leq n}\gamma_{i_\alpha,i_\beta}(z_\alpha,z_\beta;q,t), \tag{1.12}$$

where the function $\gamma_{i,j}(z,w;q,t)$ is given by

$$\gamma_{i,j}(z,w;q,t) := \begin{cases} \dfrac{(z-tw)(z-t^{-1}w)}{(z-w)^2} & i=j,\\ \dfrac{(z-q^{-1}w)(z-tw)(z-qt^{-1}w)}{(z-w)^3} & i<j,\\ \dfrac{(z-qw)(z-t^{-1}w)(z-q^{-1}tw)}{(z-w)^3} & i>j.\end{cases} \tag{1.13}$$

Proof. Note that we have

$$\gamma_{i,j}(z,w;q,t) = \begin{cases} \epsilon_2(z,w;t) & i=j,\\ \omega(z,w;q^{-1},t,qt^{-1}) & i<j,\\ \omega(z,w;q,t^{-1},q^{-1}t) = \omega(w,z;q^{-1},t,qt^{-1}) & i>j,\end{cases} \tag{1.14}$$

which is obtained from (1.2), (1.5) and (1.13). Thus we have

$$\sum_{i_1=1}^{m}\sum_{i_2=1}^{m}\cdots\sum_{i_n=1}^{m} x_{i_1}x_{i_2}\cdots x_{i_n}\prod_{1\le\alpha<\beta\le n}\gamma_{i_\alpha,i_\beta}(z_\alpha,z_\beta;q,t)$$

$$=\sum_{I_1,\dots,I_m} x_1^{a_1}x_2^{a_2}\cdots x_m^{a_m}\prod_{k=1}^{m}\epsilon_{a_k}(z_{I_k};t)\prod_{1\le i<j\le m}\prod_{\alpha\in I_i,\beta\in I_j}\omega(z_\alpha,z_\beta;q^{-1},t,qt^{-1}),$$

where I_k $(k=1,2,\dots,m)$ is a subset of $\{1,2,\dots,n\}$ such that $|I_k|=a_k$, $I_1\cup I_2\cup\cdots\cup I_m=\{1,\dots,n\}$. Using the multi-index notation $a=(a_1,\dots,a_m)\in\mathbb{N}^m$, we have

$$=\sum_{a\in\mathbb{N}^m,|a|=n} x^a\frac{n!}{\prod_{k=1}^m a_k!}\epsilon_a(z;t)$$

with $|a|:=a_1+\cdots+a_m$. Applying $\mathfrak{S}_m$ on the running index a and averaging them, we have

$$=\frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_m}\sum_{a\in\mathbb{N}^m,|a|=n} x^{\sigma(a)}\frac{n!}{\prod_{k=1}^m a_k!}\epsilon_{\sigma(a)}(z;t).$$

Dividing $\mathfrak{S}_m$ by the stabilizer $\mathrm{Stab}(a)$ of $a\in\mathbb{N}^m$ and using the commutativity of $\mathcal{A}$, we have

$$=\frac{1}{n!}\sum_{a\in\mathbb{N}^m,|a|=n}\#\mathrm{Stab}(a)\frac{n!}{\prod_{k=1}^m a_k!}\Big(\sum_{\bar\sigma\in\mathfrak{S}_m/\mathrm{Stab}(a)} x^{\bar\sigma(a)}\Big)\epsilon_a(z;t).$$

Then we obtain the result by taking a partition λ as the running index. □

1.5. **Macdonald's tableau sum formula.** We recall the tableau sum formula for the Macdonald polynomials.

Let $\mathrm{Tb}(\lambda;m)$ denotes the set of all the ways of drawing numbers $1,2,\dots,m$ into the Young diagram of shape λ *without any conditions.* Reading the numbers from left to right then top to bottom, namely in the English reading manner, we get a bijection between $\mathrm{Tb}(\lambda;m)$ and the set $\{1,2,\dots,m\}^n$.

Let $\mathrm{RTb}(\lambda;m)$ denotes the subset of $\mathrm{Tb}(\lambda;m)$ in which the numbers in each row are arranged in non-decreasing manner. The element of $\mathrm{RTb}(\lambda;m)$ is uniquely specified by the set of numbers $\theta_{i,j}$ which denote the number of j in the i-th row. We have $\lambda_i=\sum_{k=1}^n\theta_{i,k}$ for $1\le i\le n$. Next we introduce a sequence $\lambda^{(j)}=(\lambda_1^{(j)},\lambda_2^{(j)},\dots)$ by setting $\lambda_i^{(j)}:=\sum_{k=1}^j\theta_{i,k}$. It is clear that we have $\emptyset=\lambda^{(0)}\subset\lambda^{(1)}\subset\cdots\subset\lambda^{(m)}=\lambda$. Note that $\lambda^{(j)}$ may not be a partition.

Let $\mathrm{SSTb}(\lambda;m)$ be the set of semi-standard tableaux. A semi-standard tableau T is expressed as a sequence of partitions $\emptyset=\lambda^{(0)}\subset\lambda^{(1)}\subset\cdots\subset\lambda^{(m)}=\lambda$, where the skew diagrams $\lambda^{(k)}/\lambda^{(k-1)}$ $(k=1,2,\dots,m)$ are horizontal strips. We have $\theta_{i,j}=0$ for $i>j$, $\lambda_i=\sum_{k=i}^n\theta_{i,k}$ for $1\le i\le n$, and

$$0\le\theta_{i,j}\le\lambda_i-\lambda_{i+1}-\sum_{k=j+1}^{\ell(\lambda)}(\theta_{i,k}-\theta_{i+1,k})$$

for $1\le i<j\le\ell(\lambda)$.

It is known that the $b_\lambda(q,t)$ in (1.4) has the factorized form.

$$Q_\lambda(x;q,t)=b_\lambda(q,t)P_\lambda(x;q,t),\quad b_\lambda(q,t)=\prod_{s\in\lambda}\frac{1-q^{a(s)}t^{\ell(s)+1}}{1-q^{a(s)+1}t^{\ell(s)}},\tag{1.15}$$

where for a box $s=(i,j)$ of λ, $a(s):=\lambda_i-j$ is the arm-length and $\ell(s):=\lambda'_j-i$ is the leg-length.

The $P_\lambda(x;q,t)$ has the tableau sum formula:

$$P_\lambda(x;q,t)=\sum_{T\in\mathrm{SSTb}(\lambda;m)}x^T\psi_T(q,t).$$

Here the coefficient $\psi(q,t) \in \mathbb{F}$ is determined by

$$\psi_T(q,t) := \prod_{k=1}^{m} \psi_{\lambda^{(k)}/\lambda^{(k-1)}}(q,t),$$
$$\psi_{\lambda/\mu}(q,t) := \prod_{1\le i\le j\le \ell(\mu)}^{n} \frac{f(q^{\mu_i-\mu_j}t^{j-i})f(q^{\lambda_i-\lambda_{j+1}}t^{j-i})}{f(q^{\lambda_i-\mu_j}t^{j-i})f(q^{\mu_i-\lambda_{j+1}}t^{j-i})}, \quad f(u) := \frac{(tu;q)_\infty}{(qu;q)_\infty}. \tag{1.16}$$

The next proposition is obtained by simple combinatorics, and we omit the proof for lack of space.

Proposition 1.14. Let $T \in \mathrm{RTb}(\lambda;m) \setminus \mathrm{SSTb}(\lambda;m)$ and regard T as a sequence $\lambda^{(j)}$ explained as above. Then $\psi_T(q,t)$ calculated from (1.16) vanishes.

1.6. **Tableau sum formula and $K_n(x,z;q,t)$.** Now we investigate the relationship between the function $K_n(x,z;q,t)$ and the tableau formula of Macdonald polynomial. We fix a natural number m and consider the case $x = (x_1,\dots,x_m)$.

In order to state the main result, we need to consider the composition of the specialization maps $\varphi_\lambda^{(p)}$ of (1.9). For a partition $\lambda = (\lambda_1,\dots,\lambda_l)$ of n and $\zeta \in \mathbb{F}$, we define the map $\widetilde{\varphi}_\lambda^{(\zeta)}$ by

$$\begin{aligned}\widetilde{\varphi}_\lambda^{(\zeta)} := \varphi_{(l)}^{(\zeta)} \circ \varphi_\lambda^{(q^{-1})} : \ \mathbb{F}(z_1,\dots,z_n) &\longrightarrow \mathbb{F}(y)\\ f(z_1,\dots,z_n) &\mapsto f(y, q^{-1}y, \dots, q^{-(\lambda_1-1)}y,\\ &\qquad \zeta y, q^{-1}\zeta y, \dots, q^{-(\lambda_2-1)}\zeta y,\\ &\qquad \dots,\\ &\qquad \zeta^{l-1}y, q^{-1}\zeta^{l-1}y, \dots, q^{-(\lambda_l-1)}\zeta^{l-1}y).\end{aligned} \tag{1.17}$$

Here the map $\varphi_{(l)}^{(\zeta)}$ denotes the substitution $\varphi_{(l)}^{(\zeta)} g(y_1,\dots,y_l) = g(y, \zeta y, \dots, \zeta^{l-1}y)$.

Theorem 1.15. For partitions μ, λ of n, $\widetilde{\varphi}_\lambda^{(\zeta)}(F_\mu/F_\lambda)$ is regular at $\zeta = t$ and its value is $\delta_{\lambda,\mu}$.

Our proof uses the tableau sum formula of $P_\lambda(x;q,t)$. Let us express the statement as

$$\lim_{\zeta\to t} \widetilde{\varphi}_\lambda^{(\zeta)} \frac{F_\mu(z;q,t)}{F_\lambda(z;q,t)} = \delta_{\lambda,\mu}.$$

Then by using Proposition 1.12, it can be rewritten into the next equivalent form.

$$\lim_{\zeta\to t} \widetilde{\varphi}_\lambda^{(\zeta)} \frac{K_n(x,z;q,t)}{F_\lambda(z;q,t)} = Q_\lambda(x;q,t). \tag{1.18}$$

Regard $T = (i_1, i_2, \dots, i_n) \in \{1,2,\dots,m\}^n$ as an element of $\mathrm{Tb}(\lambda;m)$. For simplicity we set

$$\gamma_T(z) := \prod_{1\le\alpha<\beta\le n} \gamma_{i_\alpha,i_\beta}(z_\alpha, z_\beta; q,t).$$

We also use the same symbol for the cases $T \in \mathrm{RTb}(\lambda;m)$ and $T \in \mathrm{SSTb}(\lambda;m)$. By Proposition 1.12, (1.18) is equivalent to

$$\frac{(-1)^n}{(1-q)^n n!} \sum_{T\in\mathrm{Tb}(\lambda;m)} x^T \lim_{\zeta\to t} \widetilde{\varphi}_\lambda^{(\zeta)} \frac{\gamma_T(z)}{F_\lambda(z;q,t)} = Q_\lambda(x;q,t).$$

It is easy to see from the definition of $\gamma_{i,j}$ that all the terms with $T \in \mathrm{Tb}(\lambda;m) \setminus \mathrm{RTb}(\lambda;m)$ vanish after the first specialization $\varphi_\lambda^{(q^{-1})}$. Thus we may replace $\sum_{T\in\mathrm{Tb}(\lambda;m)}$ by $\sum_{T\in\mathrm{RTb}(\lambda;m)}$.

Hence it is enough to show that for $T \in \mathrm{RTb}(\lambda;m)$ we have

$$\frac{(-1)^n}{(1-q)^n n!} \lim_{\zeta\to t} \widetilde{\varphi}_\lambda^{(\zeta)} \frac{\gamma_T(z)}{F_\lambda(z;q,t)} = b_\lambda(q,t)\psi_T(q,t).$$

We prove this in two steps.

Proposition 1.16. Let $D \in \mathrm{SSTb}(\lambda; m)$ given by $\theta_{i,i} = \lambda_i$ and $\theta_{i,j} = 0$ for $i \neq j$. Then we have

$$\frac{(-1)^n}{(1-q)^n n!} \lim_{\zeta \to t} \widetilde{\varphi}_\lambda^{(\zeta)} \frac{\gamma_D(z)}{F_\lambda(z; q, t)} = b_\lambda(q, t), \tag{1.19}$$

$$\lim_{\zeta \to t} \widetilde{\varphi}_\lambda^{(\zeta)} \frac{\gamma_T(z)}{\gamma_D(z)} = \psi_T(q, t). \tag{1.20}$$

Proof. The proof is postponed until §3.1. □

2. Ding-Iohara algebra and kernel function

In this section all objects are defined on $\widetilde{\mathbb{F}} := \mathbb{Q}(q^{1/2}, t^{1/2})$. We will also use $p := q/t$.

2.1. **Review of the Ding-Iohara algebra $\mathcal{U}(q,t)$.** Recall that the Ding-Iohara algebra [DI] was introduced as a generalization of the quantum affine algebra, which respects the structure of the Drinfeld coproduct. In [FHHSY, Appendix A], the authors introduced a version $\mathcal{U}(q,t)$ of the Ding-Iohara algebra having two parameters q and t.

Definition 2.1. Set

$$g(z) := \frac{G^+(z)}{G^-(z)}, \qquad G^\pm(z) := (1 - q^{\pm 1} z)(1 - t^{\mp 1} z)(1 - q^{\mp 1} t^{\pm 1} z).$$

Then we define $\mathcal{U}(q,t)$ to be a unital associative algebra generated by the Drinfeld currents

$$x^\pm(z) = \sum_{n \in \mathbb{Z}} x_n^\pm z^{-n}, \qquad \psi^\pm(z) = \sum_{\pm n \in \mathbb{N}} \psi_n^\pm z^{-n},$$

and the central element $\gamma^{\pm 1/2}$, satisfying the defining relations

$$\psi^\pm(z)\psi^\pm(w) = \psi^\pm(w)\psi^\pm(z), \qquad \psi^+(z)\psi^-(w) = \frac{g(\gamma^{+1} w/z)}{g(\gamma^{-1} w/z)} \psi^-(w)\psi^+(z),$$
$$\psi^+(z)x^\pm(w) = g(\gamma^{\mp 1/2} w/z)^{\mp 1} x^\pm(w)\psi^+(z), \qquad \psi^-(z)x^\pm(w) = g(\gamma^{\mp 1/2} z/w)^{\pm 1} x^\pm(w)\psi^-(z),$$
$$[x^+(z), x^-(w)] = \frac{(1-q)(1-1/t)}{1-q/t}\left(\delta(\gamma^{-1} z/w)\psi^+(\gamma^{1/2} w) - \delta(\gamma z/w)\psi^-(\gamma^{-1/2} w)\right),$$
$$G^\mp(z/w) x^\pm(z) x^\pm(w) = G^\pm(z/w) x^\pm(w) x^\pm(z).$$

Fact 2.2 ([FHHSY, Proposition A.2]). The algebra $\mathcal{U}(q,t)$ has a Hopf algebra structure with Coproduct Δ:

$$\Delta(\gamma^{\pm 1/2}) = \gamma^{\pm 1/2} \otimes \gamma^{\pm 1/2}, \qquad \Delta(x^+(z)) = x^+(z) \otimes 1 + \psi^-(\gamma_{(1)}^{1/2} z) \otimes x^+(\gamma_{(1)} z),$$
$$\Delta(\psi^\pm(z)) = \psi^\pm(\gamma_{(2)}^{\pm 1/2} z) \otimes \psi^\pm(\gamma_{(1)}^{\mp 1/2} z), \qquad \Delta(x^-(z)) = x^-(\gamma_{(2)} z) \otimes \psi^+(\gamma_{(2)}^{1/2} z) + 1 \otimes x^-(z),$$

where $\gamma_{(1)}^{\pm 1/2} = \gamma^{\pm 1/2} \otimes 1$ and $\gamma_{(2)}^{\pm 1/2} = 1 \otimes \gamma^{\pm 1/2}$.
Counit ε:

$$\varepsilon(\gamma^{\pm 1/2}) = 1, \qquad \varepsilon(\psi^\pm(z)) = 1, \qquad \varepsilon(x^\pm(z)) = 0.$$

Antipode a:

$$a(\gamma^{\pm 1/2}) = \gamma^{\mp 1/2}, \qquad a(x^+(z)) = -\psi^-(\gamma^{-1/2} z)^{-1} x^+(\gamma^{-1} z),$$
$$a(\psi^\pm(z)) = \psi^\pm(z)^{-1}, \qquad a(x^-(z)) = -x^-(\gamma^{-1} z)\psi^+(\gamma^{-1/2} z)^{-1}.$$

2.2. **Level one representation of $\mathcal{U}(q,t)$.** We say a representation of $\mathcal{U}(q,t)$ is of level k, if the central element γ is realized by the constant $(t/q)^{k/2} = p^{-k/2}$.

Fact 2.3 ([FHHSY, Proposition A.6]). Consider the Heisenberg Lie algebra $\mathfrak{h}$ over $\mathbb{F}$ with the generators a_n ($n \in \mathbb{Z}$) and the relations

$$[a_m, a_n] = m\frac{1-q^{|m|}}{1-t^{|m|}}\delta_{m+n,0}\, a_0. \tag{2.1}$$

Let $\mathfrak{h}^{\geq 0}$ (resp. $\mathfrak{h}^{<0}$) be the subalgebra generated by a_n for $n \geq 0$ (resp. $n < 0$). Consider the one dimensional representation $\widetilde{\mathcal{F}}$ of $\mathfrak{h}^{\geq 0}$, where a_n ($n > 0$) acts trivially and a_0 acts by some fixed element of $\widetilde{\mathcal{F}}$. Then one has the induced Fock representation $\mathcal{F} := \mathrm{Ind}_{\mathfrak{h}^{\geq 0}}^{\mathfrak{h}} \widetilde{\mathbb{F}}$ of $\mathfrak{h}$. Let us also introduce the following four vertex operators [FHHSY, (1.7),(3.23),(3.27),(3.28)].

$$\begin{aligned}
\eta(z) &:= \exp\Big(\sum_{n>0}\frac{1-t^{-n}}{n}a_{-n}z^n\Big)\exp\Big(-\sum_{n>0}\frac{1-t^{n}}{n}a_{n}z^{-n}\Big),\\
\xi(z) &:= \exp\Big(-\sum_{n>0}\frac{1-t^{-n}}{n}p^{-n/2}a_{-n}z^n\Big)\exp\Big(\sum_{n>0}\frac{1-t^{n}}{n}p^{-n/2}a_{n}z^{-n}\Big),\\
\varphi^+(z) &:= \exp\Big(-\sum_{n>0}\frac{1-t^{n}}{n}(1-p^{-n})p^{n/4}a_{n}z^{-n}\Big),\\
\varphi^-(z) &:= \exp\Big(\sum_{n>0}\frac{1-t^{-n}}{n}(1-p^{-n})p^{n/4}a_{-n}z^{n}\Big).
\end{aligned}$$

Then for a fixed $c \in \widetilde{\mathbb{F}}^{\times}$, we have a level one representation $\rho_c(\cdot)$ of $\mathcal{U}(q,t)$ on $\mathcal{F}$ by setting

$$\rho_c(\gamma^{\pm 1/2}) = p^{\mp 1/4},\quad \rho_c(\psi^{\pm}(z)) = \varphi^{\pm}(z),\quad \rho_c(x^+(z)) = c\,\eta(z),\quad \rho_c(x^-(z)) = c^{-1}\xi(z).$$

Remark 2.4. We can rephrase this fact as follows. Let us define b_n's by the expansion of $\psi^{\pm}$:

$$\psi^+(z) = \psi_0^+ \exp\Big(+\sum_{n>0} b_n\gamma^{n/2}z^{-n}\Big),\quad \psi^-(z) = \psi_0^- \exp\Big(-\sum_{n>0} b_{-n}\gamma^{n/2}z^{n}\Big). \tag{2.2}$$

Then we have

$$[b_m, b_n] = \frac{1}{m}(1-q^{-m})(1-t^m)(1-p^m)(\gamma^m - \gamma^{-m})\gamma^{-|m|}\delta_{m+n,0}, \tag{2.3}$$

and the coproduct for b_n reads

$$\Delta(b_n) = b_n \otimes \gamma^{-|n|} + 1 \otimes b_n. \tag{2.4}$$

Then the representation ρ_c is given by $\gamma^{\pm 1/2} \mapsto p^{\mp 1/4}$ and

$$b_n \mapsto -\frac{1-t^n}{|n|}(p^{|n|/2} - p^{-|n|/2})a_n,\quad \psi_0^{\pm} \mapsto 1,\quad x^+(z) \mapsto c\,\eta(z),\quad x^-(z) \mapsto c^{-1}\xi(z).$$

Definition 2.5. Consider the m-fold tensor representation $\rho_{y_1} \otimes \cdots \otimes \rho_{y_m}$ on $\mathcal{F}^{\otimes m}$ for $m \in \mathbb{Z}_{\geq 2}$. Define $\Delta^{(m)}$ inductively by

$$\Delta^{(2)} := \Delta,\quad \Delta^{(m)} := (\mathrm{id} \otimes \cdots \otimes \mathrm{id} \otimes \Delta) \circ \Delta^{(m-1)}.$$

Since we have $\rho_{y_1} \otimes \cdots \otimes \rho_{y_m}\Delta^{(m)}(\gamma) = \gamma_{(1)} \cdots \gamma_{(m)} = p^{-m/2}$, the level is m. We also define

$$\rho_y^{(m)} := \rho_{y_1} \otimes \cdots \otimes \rho_{y_m} \circ \Delta^{(m)} : \mathcal{U}(q,t) \to \mathcal{F}^{\otimes m}. \tag{2.5}$$

Lemma 2.6. We have

$$\rho_y^{(m)}(x^+(z)) = \sum_{i=1}^{m} y_i \widetilde{\Lambda}_i(z), \quad \rho_y^{(m)}(x^-(z)) = \sum_{i=1}^{m} y_i^{-1} \widetilde{\Lambda}_i^*(z),$$

where the $\widetilde{\Lambda}_i(z)$, $\widetilde{\Lambda}_i^*(z)$ are defined to be

$$\widetilde{\Lambda}_i(z) := \varphi^-(p^{-1/4}z) \otimes \varphi^-(p^{-3/4}z) \otimes \cdots \otimes \varphi^-(p^{-(2i-3)/4}z) \otimes \eta(p^{-(i-1)/2}z) \otimes 1 \otimes \cdots \otimes 1, \tag{2.6}$$

$$\widetilde{\Lambda}_i^*(z) := 1 \otimes \cdots \otimes 1 \otimes \xi(p^{-(m-i)/2}z) \otimes \varphi^+(p^{-(2m-2i-1)/4}z) \otimes \cdots \otimes \varphi^+(p^{-1/4}z), \tag{2.7}$$

where $\eta(p^{-(i-1)/2}z)$ and $\xi(p^{-(m-i)/2}z)$ sit in the i-th tensor component.

Proof. By the definition (2.5), Fact 2.3 and Remark 2.4. □

2.3. New currents $t(z)$ and $t^*(z)$.

Definition 2.7. We define

$$t(z) := \alpha(z)x^+(z)\beta(z), \quad t^*(z) := \alpha(p^{-1}z)^{-1}x^-(p^{-1}\gamma^{-1}z)\beta(\gamma^{-2}p^{-1}z)^{-1}. \tag{2.8}$$

Here we used auxiliary vertex operators

$$\alpha(z) := \exp\Big(-\sum_{n=1}^{\infty} \frac{1}{\gamma^n - \gamma^{-n}} b_{-n} z^n\Big), \quad \beta(z) := \exp\Big(\sum_{n=1}^{\infty} \frac{1}{\gamma^n - \gamma^{-n}} b_n z^{-n}\Big). \tag{2.9}$$

Here the part $1/(\gamma^n - \gamma^{-n})$ is considered to be the formal power sum $\sum_{i=0}^{\infty} \gamma^{-(2i+1)n}$.

Remark 2.8. The definition of $t^*(z)$ can be read as

$$t^*(\gamma p z) = \alpha(\gamma z)^{-1} x^-(z) \beta(\gamma^{-1}z)^{-1}.$$

This form is convenient in the actual calculations.

Proposition 2.9. (1) The elements $t(z)$ and $t^*(z)$ commutes with $\alpha(w)$, $\beta(w)$ and $\psi^{\pm}(w)$:

$$[t(z), \alpha(w)] = [t(z), \beta(w)] = [t^*(z), \alpha(w)] = [t^*(z), \beta(w)] = 0,$$
$$[t(z), \psi^{\pm}(w)] = [t^*(z), \psi^{\pm}(w)] = 0.$$

(2) Set

$$A(z) := \exp\Big(\sum_{n=1}^{\infty} \frac{1}{n} \frac{(1-q^n)(1-t^{-n})(1-p^{-n}\gamma^{-2n})}{1-\gamma^{-2n}} z^n\Big), \tag{2.10}$$

where the part $1/(1-\gamma^{-2n})$ is considered to be the formal power sum $\sum_{i=0}^{\infty} \gamma^{-2in}$. Then we have

$$A(\tfrac{w}{z})t(z)t(w) - A(\tfrac{z}{w})t(w)t(z) = \frac{(1-q)(1-t^{-1})}{1-p}[\delta(p^{-1}\tfrac{w}{z})t^{(2)}(z) - \delta(p\tfrac{w}{z})t^{(2)}(w)], \tag{2.11}$$

$$A(\tfrac{w}{z})t^*(z)t^*(w) - A(\tfrac{z}{w})t^*(w)t^*(z) = \frac{(1-q^{-1})(1-t)}{1-p^{-1}}[\delta(p\tfrac{w}{z})t^{*(2)}(z) - \delta(p^{-1}\tfrac{w}{z})t^{*(2)}(w)], \tag{2.12}$$

where $\delta(z) := \sum_{n\in\mathbb{N}} z^n + z^{-1}\sum_{n\in\mathbb{N}} z^{-n}$ is the formal delta function, and

$$t^{(2)}(z) := \alpha(pz)\alpha(z)x^+(pz)x^+(z)\beta(pz)\beta(z),$$
$$t^{*(2)}(z) := \alpha(\gamma pz)^{-1}\alpha(\gamma z)^{-1}x^-(pz)x^-(z)\beta(\gamma^{-1}pz)^{-1}\beta(\gamma^{-1}z)^{-1}.$$

(3) As in (2), set

$$B(z) := \exp\Big(\sum_{n=1}^{\infty} \frac{1}{n} \frac{(1-q^n)(1-t^{-n})(p^{-2n}\gamma^{-2n} - p^{-n}\gamma^{-2n})}{1-\gamma^{-2n}} z^n\Big). \tag{2.13}$$

Then

$$B(\tfrac{w}{z})t(z)t^*(w) - B(\gamma^2p^2\tfrac{z}{w})t^*(w)t(z) = \frac{(1-q)(1-t^{-1})}{1-p}\Big(\delta(p^{-1}\tfrac{w}{z})\psi_0^+ - \delta(\gamma^{-2}p^{-1}\tfrac{w}{z})\psi_0^-)\Big). \quad (2.14)$$

Proof. See §3.2. □

In the next subsection we show that the currents $t(z)$, $t^*(z)$ are connected to the realization of deformed $\mathcal{W}$ algebra in the Fock representation of $\mathcal{U}(q,t)$.

2.4. **Deformed algebra $\mathcal{W}_{q,p}(\mathfrak{sl}_m)$.** We basically follow the description of $\mathcal{W}_{q,p}(\mathfrak{sl}_m)$ in [FF, §4]. As for the connection between the singular vectors of the $\mathcal{W}_{q,p}(\mathfrak{sl}_m)$ and the Macdonald polynomials, see [SKAO, AKOS].

Definition 2.10. Set

$$f_{k,l}(z) := \exp\Big(\sum_{n=1}^{\infty}\frac{(1-q^n)(1-t^{-n})(p^{(k-1)n}-p^{(l-1)n})}{1-p^{ln}}z^n\Big).$$

Remark 2.11. Our functions $A(z)$ and $B(z)$ give special cases of this function under $\rho_y^{(m)}$, that is,

$$\rho_y^{(m)}(A(z)) = f_{1,m}(z), \quad \rho_y^{(m)}(B(z)) = f_{m-1,m}(z).$$

Definition 2.12. Set

$$T(z) = T_1(z) := \rho_y^{(m)}(t(z)), \quad T^*(z) = T_1^*(z) := \rho_y^{(m)}(t^*(z)). \quad (2.15)$$

Let us also define

$$\Lambda_i(z) := \rho_y^{(m)}(\alpha(z))\widetilde{\Lambda}_i(z)\rho_y^{(m)}(\beta(z)), \quad \Lambda_i^*(z) := \rho_y^{(m)}(\alpha(p^{-1}z)^{-1})\widetilde{\Lambda}_i^*(p^{(m-2)/2}z)\rho_y^{(m)}(\beta(\gamma^{-2}p^{-1}z)^{-1}). \quad (2.16)$$

Then by Definition 2.7 and Lemma 2.6 we have

$$T_1(z) = \sum_{i=1}^m y_i\Lambda_i(z), \quad T_1^*(z) = \sum_{i=1}^m y_i^{-1}\Lambda_i^*(z). \quad (2.17)$$

For $i = 2,\dots,m$, we further define

$$T_i(z) := \sum_{1\le j_1<\cdots<j_i\le m} y_{j_1}y_{j_2}\cdots y_{j_i} : \Lambda_{j_1}(z)\Lambda_{j_2}(zp)\cdots\Lambda_{j_i}(zp^{i-1}) :, \quad (2.18)$$

$$T_i^*(z) := \sum_{1\le j_1<\cdots<j_i\le m} y_{j_1}^{-1}y_{j_2}^{-1}\cdots y_{j_i}^{-1} : \Lambda_{j_1}^*(z)\Lambda_{j_2}^*(zp^{-1})\cdots\Lambda_{j_i}^*(zp^{-i+1}) : . \quad (2.19)$$

Proposition 2.13. (1) The operator product of $\Lambda_i(z)$ and $\Lambda_j(w)$ is given by

$$f_{1,m}(\tfrac{w}{z})\Lambda_i(z)\Lambda_j(w) =: \Lambda_i(z)\Lambda_j(w) : \times \begin{cases} 1 & i=j, \\ \gamma_+(z,w;q,p) & i<j, \\ \gamma_-(z,w;q,p) & i>j. \end{cases} \quad (2.20)$$

Here we used the symbol

$$\gamma_+(z,w;q,t) := \frac{(z-q^{-1}w)(z-qt^{-1}w)}{(z-w)(z-t^{-1}w)}, \quad \gamma_-(z,w;q,t) := \frac{(z-qw)(z-q^{-1}tw)}{(z-w)(z-tw)}. \quad (2.21)$$

(2) We have

$$:\Lambda_1(z)\Lambda_2(pz)\cdots\Lambda_m(p^{m-1}z) := 1.$$

Therefore $T_m(z) = y_1y_2\cdots y_m$.

(3) The $\Lambda_i(z)$ and $\Lambda_j^*(z)$ are connected by the following equation.

$$\Lambda_k^*(z) =: \prod_{i=1}^{k-1} \Lambda_i(p^{k-1}z) \prod_{l=k+1}^{m} \Lambda_l(p^{l-2}z) : . \tag{2.22}$$

Thus we also have

$$T_1^*(z) = y_1^{-1}y_2^{-1}\cdots y_m^{-1}T_{m-1}(z). \tag{2.23}$$

(4) The operator product of $\Lambda_i^*(z)$ and $\Lambda_j^*(w)$ is given by

$$f_{1,m}(\tfrac{w}{z})\Lambda_i^*(z)\Lambda_j^*(w) =: \Lambda_i^*(z)\Lambda_j^*(w) : \times \begin{cases} 1 & i=j, \\ \gamma_-(z,w;q,p) & i<j, \\ \gamma_+(z,w;q,p) & i>j. \end{cases} \tag{2.24}$$

(5) We have

$$: \Lambda_1^*(z)\Lambda_2^*(p^{-1}z)\cdots\Lambda_m^*(p^{-m+1}z) := 1.$$

Therefore $T_m^*(z) = y_1^{-1}y_2^{-1}\cdots y_m^{-1}$.

Proof. See §3.3. □

Proposition 2.14. We have

$$f_{1,m}(\tfrac{w}{z})T_1(z)T_i(w) - f_{1,m}(p^{1-i}\tfrac{z}{w})T_i(w)T_1(z) = \frac{(1-q)(1-t^{-1})}{1-p}[\delta(p^{-1}\tfrac{w}{z})T_{i+1}(z) - \delta(p^i\tfrac{w}{z})T_{i+1}(w)], \tag{2.25}$$

$$\begin{aligned} &f_{1,m}(\tfrac{w}{z})T_{m-1}(z)T_{m-1}(w) - f_{1,m}(\tfrac{z}{w})T_{m-1}(w)T_{m-1}(z) \\ &\quad = \frac{(1-q^{-1})(1-t)}{1-p^{-1}}[\delta(p\tfrac{w}{z})T_2^*(z) - \delta(p^{-1}\tfrac{w}{z})T_2^*(w)]. \end{aligned} \tag{2.26}$$

Proof. (2.25) follows from (2.17), (2.18) and (2.20). See [FF, Theorem 2] for detail[2].
(2.26) is also shown by the same method using (2.23), (2.19) and and (2.24). □

2.5. **Deformed $\mathcal{W}$ algebra and kernel function.** Our final consequence of this paper relates the vacuum expectation values of the deformed algebra $\mathcal{W}_{q,p}$ with the finite kernel function.

Theorem 2.15. Let $|0\rangle$ be the vacuum of $\mathcal{F}$, that is, $a_0|0\rangle = |0\rangle$ and $a_n|0\rangle = 0$ for $n>0$. Let $\langle 0|$ to be the dual vacuum. We denote the tensor $|0\rangle^{\otimes m} \in \mathcal{F}^{\otimes m}$ by the same symbol $|0\rangle$. We use the similar abbreviation for the tensored dual vacuum. Then, denoting $y=(y_1,\ldots,y_m)$, we have

$$\frac{(-1)^n}{(1-q)^n n!}\prod_{i<j} f_{1,m}(z_i/z_j)\,\langle 0|\,T_1(z_1)T_1(z_2)\cdots T_1(z_n)\,|0\rangle = K_n(y,z;q,p).$$

Proof. This follows from (2.17), the operator product (2.20) and the definition (1.12). □

[2]It seems that [FF] contains some typo. In (6.2) of that paper, the term $f_{m,N}(\frac{z}{w})$ should be $f_{m,N}(p^{1-m}\frac{z}{w})$.

3. Proofs of the propositions

3.1. Proof of Proposition 1.16. Using the $\gamma_\pm$ defined in (2.21), we have

$$\frac{\omega(z,w)}{\epsilon_2^{(t)}(z,w)}=\gamma_+(z,w;q,t),\quad \frac{\omega(w,z)}{\epsilon_2^{(t)}(w,z)}=\gamma_-(z,w;q,t),\quad \frac{\omega(w,z)}{\omega(z,w)}=\frac{\gamma_-(z,w;q,t)}{\gamma_+(z,w;q,t)}. \tag{3.1}$$

For later purpose, we prepare the following formulae. Let θ and ρ be natural numbers. Then

$$\prod_{1\le i<j\le\theta}\gamma_+(q^{-i}z,q^{-j}w;q,t)=\left(\frac{1-tz/w}{1-qz/w}\right)^\theta\frac{(qz/w)_\theta}{(tz/w)_\theta} \tag{3.2}$$

$$\prod_{1\le i<j\le\theta}\gamma_-(q^{-i}z,q^{-j}w;q,t)=\left(\frac{1-z/w}{1-qt^{-1}z/w}\right)^\theta\frac{(qt^{-1}z/w)_\theta}{(z/w)_\theta} \tag{3.3}$$

$$\prod_{l=1}^{\theta}\prod_{k=1}^{\rho}\gamma_+(q^{-l}z,q^{-k}w;q,t)=\frac{(q^{-\rho}w/z)_\theta}{(w/z)_\theta}\frac{(qt^{-1}w/z)_\theta}{(q^{-\rho+1}t^{-1}w/z)_\theta} \tag{3.4}$$

$$=\frac{(q^{\rho-\theta+1}z/w)_\theta}{(q^{-\theta+1}z/w)_\theta}\frac{(q^{-\theta}tz/w)_\theta}{(q^{\rho-\theta}tz/w)_\theta}, \tag{3.5}$$

$$\prod_{l=1}^{\theta}\prod_{k=1}^{\rho}\gamma_-(q^{-l}z,q^{-k}w;q,t)=\frac{(qw/z)_\theta}{(q^{-\rho+1}w/z)_\theta}\frac{(q^{-\rho}tw/z)_\theta}{(tw/z)_\theta} \tag{3.6}$$

$$=\frac{(q^{-\theta}z/w)_\theta}{(q^{\rho-\theta}z/w)_\theta}\frac{(q^{\rho-\theta+1}t^{-1}z/w)_\theta}{(q^{-\theta+1}t^{-1}z/w)_\theta}. \tag{3.7}$$

Here we used $(u)_n:=(u;q)_n=\prod_{i=1}^n(1-uq^{i-1})$. These equations are checked by simple calculations.

3.1.1. *Proof of* (1.19). By (1.7) we have

$$\frac{(1-q)^n n!}{(-1)^n}F_\lambda(z;q,t)=\left(\frac{1-q}{1-t}\right)^{|\lambda|}\sum_{\mu\ge\lambda'}c_{\lambda\mu}^{e\to P}(q,t)\epsilon_\mu(z;q)\frac{|\mu|!}{\prod_{i=1}^{\ell(\mu)}\mu_i!}.$$

Recalling the argument of [FHHSY, Proposition 2.19], we find that under the specialization $\varphi_\lambda^{(q^{-1})}$ only the term $\epsilon_{\lambda'}$ in $F_\lambda(z;q,t)$ survives and the other terms ϵ_μ vanish. The specialization result is [3]

$$\begin{aligned}\varphi_\lambda^{(q^{-1})}\epsilon_{\lambda'}(y)&=\frac{\prod_{h=1}^{\ell(\lambda)}\lambda_h'!}{n!}\prod_{i=1}^{\ell(\lambda)}\epsilon_{\lambda_i'}(y_1,\dots,y_{\lambda_i'};q)\prod_{1\le j<k\le\ell(\lambda')}\prod_{\alpha=1}^{\lambda_j'}\prod_{\beta=1}^{\lambda_k'}\omega(q^{-j+1}y_\alpha,q^{-k+1}y_\beta)\\&=\frac{\prod_{h=1}^{\ell(\lambda)}\lambda_h'!}{n!}\prod_{i=1}^{\ell(\lambda)}\epsilon_{\lambda_i'}(y_1,\dots,y_{\lambda_i'};q)\times\prod_{\alpha=1}^{\ell(\lambda)}\prod_{1\le i<j\le\lambda_\alpha}\omega(q^{-i+1}y_\alpha,q^{-j+1}y_\alpha)\\&\times\prod_{1\le\alpha<\beta\le\ell(\lambda)}\Bigg[\prod_{1\le i<j\le\lambda_\beta}\omega(q^{-i+1}y_\alpha,q^{-j+1}y_\beta)\omega(q^{-i+1}y_\beta,q^{-j+1}y_\alpha)\\&\qquad\times\prod_{i=1}^{\lambda_\beta}\prod_{j=\lambda_\beta+1}^{\lambda_\alpha}\omega(q^{-i+1}y_\beta,q^{-j+1}y_\alpha)\Bigg].\end{aligned}$$

We also note that $c_{\lambda'\lambda}^{e\to P}(q,t)=1$.

[3]This expression is given at the last equation in the proof of [FHHSY, Proposition 2.19], although it contains a typo. The range "$1\le j<k\le l$" of the third product should be "$1\le j<k\le\ell(\lambda')$"

Recalling (1.14), we can also calculate the first specialization $\varphi_\lambda^{(q^{-1})}$ of the numerator in (1.19) as

$$\varphi_\lambda^{(q^{-1})}\gamma_D(z) = \Big[\prod_{k=1}^{\ell(\lambda)}\prod_{1\le i<j\le\lambda_k}\epsilon_2(q^{-i},q^{-j};t)\Big]\Big[\prod_{\alpha=1}^{\ell(\lambda)}\prod_{\beta=\alpha}^{\ell(\lambda)}\prod_{i=1}^{\lambda_\alpha}\prod_{j=1}^{\lambda_\beta}\omega(q^{-i}y_\alpha,q^{-j}y_\beta)\Big]$$

$$= \Big[\prod_{k=1}^{\ell(\lambda)}\prod_{1\le i<j\le\lambda_k}\epsilon_2(q^{-i},q^{-j};t)\Big]\prod_{1\le\alpha<\beta\le\ell(\lambda)}\Big[\Big(\prod_{1\le i<j\le\lambda_\beta}\omega(q^{-i}y_\alpha,q^{-j}y_\beta)\Big)\Big(\prod_{1\le j<i\le\lambda_\beta}\omega(q^{-i}y_\alpha,q^{-j}y_\beta)\Big)$$

$$\Big(\prod_{j=1}^{\lambda_\beta}\prod_{i=\lambda_\beta+1}^{\lambda_\alpha}\omega(q^{-i}y_\alpha,q^{-j}y_\beta)\Big)\Big(\prod_{i=1}^{\lambda_\alpha}\omega(q^{-i}y_\alpha,q^{-i}y_\beta)\Big)\Big].$$

Thus we have

$$\frac{(-1)^n}{(1-q)^n n!}\widetilde{\varphi}_\lambda^{(\zeta)}\frac{\gamma_D(z)}{F_\lambda(z;q,t)} = \left(\frac{1-t}{1-q}\right)^{|\lambda|}\prod_{\alpha=1}^{\ell(\lambda)}\prod_{1\le i<j\le\lambda_\alpha}\frac{\epsilon_2^{(t)}(q^{-i+1}y_\alpha,q^{-j+1}y_\beta)}{\omega(q^{-i+1}y_\alpha,q^{-j+1}y_\beta)}\times$$

$$\prod_{1\le\alpha<\beta\le\ell(\lambda)}\Big[\Big(\prod_{1\le i<j\le\lambda_\beta}\frac{\omega(q^{-i+1}y_\alpha,q^{-j+1}y_\beta)}{\omega(q^{-i+1}y_\beta,q^{-j+1}y_\alpha)}\Big)\Big(\prod_{i=1}^{\lambda_\beta}\prod_{j=\lambda_\beta+1}^{\lambda_\alpha}\frac{\omega(q^{-j+1}y_\alpha,q^{-i+1}y_\beta)}{\omega(q^{-i+1}y_\beta,q^{-j+1}y_\alpha)}\Big)\Big(\frac{\omega(y_\alpha,y_\beta)}{\epsilon_2^{(q)}(y_\beta,y_\alpha)}\Big)^{\lambda_\beta}\Big].$$

Then recalling (3.1) and using (3.2) and (3.3), one has

$$\prod_{1\le i<j\le\lambda_\alpha}\frac{\epsilon_2^{(t)}(q^{-i+1}y_\alpha,q^{-j+1}y_\beta)}{\omega(q^{-i+1}y_\alpha,q^{-j+1}y_\beta)} = \left(\frac{1-q}{1-t}\right)^{\lambda_\alpha}\frac{(t)_{\lambda_\alpha}}{(q)_{\lambda_\alpha}},$$

$$\prod_{1\le i<j\le\lambda_\beta}\frac{\omega(q^{-i+1}y_\alpha,q^{-j+1}y_\beta)}{\omega(q^{-i+1}y_\beta,q^{-j+1}y_\alpha)}\Big[\frac{\omega(y_\alpha,y_\beta)}{\epsilon_2^{(q)}(y_\beta,y_\alpha)}\Big]^{\lambda_\beta} = \frac{(ty_\beta/y_\alpha)_{\lambda_\alpha}}{(qy_\beta/y_\alpha)_{\lambda_\alpha}}\frac{(qt^{-1}y_\beta/y_\alpha)_{\lambda_\alpha}}{(y_\beta/y_\alpha)_{\lambda_\alpha}},$$

$$\prod_{i=1}^{\lambda_\beta}\prod_{j=\lambda_\beta+1}^{\lambda_\alpha}\frac{\omega(q^{-j+1}y_\alpha,q^{-i+1}y_\beta)}{\omega(q^{-i+1}y_\beta,q^{-j+1}y_\alpha)}$$

$$= \frac{(qy_\beta/y_\alpha)_{\lambda_\beta}}{(ty_\beta/y_\alpha)_{\lambda_\beta}}\frac{(y_\beta/y_\alpha)_{\lambda_\beta}}{(qt^{-1}y_\beta/y_\alpha)_{\lambda_\beta}}\frac{(q^{\lambda_\alpha-\lambda_\beta}ty_\beta/y_\alpha)_{\lambda_\beta}}{(q^{\lambda_\alpha-\lambda_\beta}y_\beta/y_\alpha)_{\lambda_\beta}}\frac{(q^{\lambda_\alpha-\lambda_\beta+1}t^{-1}y_\beta/y_\alpha)_{\lambda_\beta}}{(q^{\lambda_\alpha-\lambda_\beta+1}y_\beta/y_\alpha)_{\lambda_\beta}}.$$

Combining these factors, we obtain

$$\frac{(-1)^n}{(1-q)^n n!}\lim_{\zeta\to t}\widetilde{\varphi}_\lambda^{(\zeta)}\frac{\gamma_D(z;t)}{F_\lambda(z;q,t)} = \prod_{\alpha=1}^{\ell(\lambda)}\frac{(t)_{\lambda_\alpha}}{(q)_{\lambda_\alpha}}\prod_{1\le\alpha<\beta\le\ell(\lambda)}\frac{(q^{\lambda_\alpha-\lambda_\beta}ty_\beta/y_\alpha)_{\lambda_\beta}}{(q^{\lambda_\alpha-\lambda_\beta}y_\beta/y_\alpha)_{\lambda_\beta}}\frac{(q^{\lambda_\alpha-\lambda_\beta+1}t^{-1}y_\beta/y_\alpha)_{\lambda_\beta}}{(q^{\lambda_\alpha-\lambda_\beta+1}y_\beta/y_\alpha)_{\lambda_\beta}}$$

$$= \prod_{1\le\alpha<\beta\le\ell(\lambda)}\frac{(q^{\lambda_\alpha-\lambda_\beta}t^{\beta-\alpha+1})_{\lambda_\beta-\lambda_{\beta+1}}}{(q^{\lambda_\alpha-\lambda_\beta+1}t^{\beta-\alpha})_{\lambda_\beta-\lambda_{\beta+1}}}.$$

But one can easily find that the last expression equals to $b_\lambda(q,t)$ using the form (1.15).

3.1.2. *Proof of* (1.20). For a tableau $T\in\mathrm{RTb}(\lambda;m)$, define $\theta_{\alpha,k}$ and $\lambda_\alpha^{(k)}$ as explained in §1.5. Then by the direct calculation we have

$$\widetilde{\varphi}_\lambda^{(\zeta)}\frac{\gamma_T(z)}{\gamma_D(z)} = \prod_{1\le\alpha<\beta\le\ell(\lambda)}\prod_{i=1}^{\lambda_\alpha}\prod_{j=1}^{\lambda_\beta}\gamma_+(q^{-i}\zeta^\alpha,q^{-j}\zeta^\beta)^{-1} \qquad (3.8)$$

$$\times \prod_{k=1}^{m}\prod_{\alpha=1}^{\ell(\lambda)}\prod_{\beta=\alpha+1}^{\ell(\lambda)}\prod_{i=1}^{\theta_{\alpha,k}}\prod_{j=1}^{\lambda_\beta^{(k-1)}} \gamma_-(q^{-i-\lambda_\alpha^{(k-1)}}\zeta^\alpha, q^{-j}\zeta^\beta) \tag{3.9}$$

$$\times \prod_{k=1}^{m}\prod_{\alpha=1}^{\ell(\lambda)}\prod_{\beta=\alpha}^{\ell(\lambda)}\prod_{i=1}^{\theta_{\alpha,k}}\prod_{j=1+\lambda_\beta^{(k)}}^{\lambda_\beta} \gamma_+(q^{-i-\lambda_\alpha^{(k-1)}}\zeta^\alpha, q^{-j}\zeta^\beta) \tag{3.10}$$

By the formula (3.4) we find that

$$\lim_{\zeta\to t}(3.8) = \prod_{1\le\alpha<\beta\le\ell(\lambda)} \frac{(t^{\beta-\alpha})_{\lambda_\alpha}}{(q^{-\lambda_\beta}t^{\beta-\alpha})_{\lambda_\alpha}} \frac{(q^{-\lambda_\beta+1}t^{\beta-\alpha-1})_{\lambda_\alpha}}{(qt^{\beta-\alpha-1})_{\lambda_\alpha}}. \tag{3.11}$$

Note that the regularity of (3.8) at $\zeta = t$ is included in this equation. Similarly by the formula (3.6), (3.9) is regular at $\zeta = t$ and its value is

$$\lim_{\zeta\to t}(3.9) = \prod_{k=1}^{m}\prod_{\alpha=1}^{\ell(\lambda)}\prod_{\beta=\alpha+1}^{\ell(\lambda)} \frac{(q^{\lambda_\alpha^{(k-1)}+1}t^{\beta-\alpha})_{\theta_{\alpha,k}}}{(q^{\lambda_\alpha^{(k-1)}-\lambda_\beta^{(k-1)}}t^{\beta-\alpha+1})_{\theta_{\alpha,k}}} \frac{(q^{\lambda_\alpha^{(k-1)}-\lambda_\beta^{(k-1)}}t^{\beta-\alpha+1})_{\theta_{\alpha,k}}}{(q^{\lambda_\alpha^{(k-1)}-\lambda_\beta^{(k-1)}+1}t^{\beta-\alpha})_{\theta_{\alpha,k}}}. \tag{3.12}$$

The rest term (3.10) is calculated by the formula (3.4) and (3.5):

$$\begin{aligned}\lim_{\zeta\to t}(3.10) = \prod_{k=1}^{m}\prod_{\alpha=1}^{\ell(\lambda)} \Big[&\frac{(t)_{\theta_{\alpha,k}}}{(q)_{\theta_{\alpha,k}}} \frac{(q^{\lambda_\alpha-\lambda_\alpha^{(k)}+1})_{\theta_{\alpha,k}}}{(q^{\lambda_\alpha-\lambda_\alpha^{(k)}})_{\theta_{\alpha,k}}} \\ &\prod_{\beta=\alpha+1}^{\ell(\lambda)} \frac{(q^{\lambda_\alpha^{(k-1)}-\lambda_\beta^{(k-1)}+1}t^{\beta-\alpha-1})_{\theta_{\alpha,k}}}{(q^{\lambda_\alpha^{(k-1)}-\lambda_\beta^{(k)}}t^{\beta-\alpha})_{\theta_{\alpha,k}}} \frac{(q^{\lambda_\alpha^{(k-1)}-\lambda_\beta}t^{\beta-\alpha})_{\theta_{\alpha,k}}}{(q^{\lambda_\alpha^{(k-1)}-\lambda_\beta+1}t^{\beta-\alpha-1})_{\theta_{\alpha,k}}} \Big].\end{aligned} \tag{3.13}$$

Note that some parts of (3.12) and (3.13) are combined into the next form.

$$\begin{aligned}
&\Big[\prod_{k=1}^{m}\prod_{\alpha=1}^{\ell(\lambda)}\prod_{\beta=\alpha+1}^{\ell(\lambda)} \frac{(q^{\lambda_\alpha^{(k-1)}+1}t^{\beta-\alpha})_{\theta_{\alpha,k}}}{(q^{\lambda_\alpha^{(k-1)}-\lambda_\beta^{(k-1)}}t^{\beta-\alpha+1})_{\theta_{\alpha,k}}}\Big] \\
&\times \Big[\prod_{k=1}^{m}\prod_{\alpha=1}^{\ell(\lambda)} \frac{(q^{\lambda_\alpha-\lambda_\alpha^{(k)}+1})_{\theta_{\alpha,k}}}{(q^{\lambda_\alpha-\lambda_\alpha^{(k)}})_{\theta_{\alpha,k}}} \prod_{\beta=\alpha+1}^{\ell(\lambda)} \frac{(q^{\lambda_\alpha^{(k-1)}-\lambda_\beta}t^{\beta-\alpha})_{\theta_{\alpha,k}}}{(q^{\lambda_\alpha^{(k-1)}-\lambda_\beta+1}t^{\beta-\alpha-1})_{\theta_{\alpha,k}}}\Big] \\
&= \Big[\prod_{\alpha=1}^{\ell(\lambda)} \frac{(q)_{\lambda_\alpha}}{(t)_{\lambda_\alpha}}\Big]\Big[\prod_{1\le\alpha<\beta\le\ell(\lambda)} \frac{(q^{-\lambda_\beta}t^{\beta-\alpha})_{\lambda_\alpha}}{(q^{-\lambda_\beta+1}t^{\beta-\alpha-1})_{\lambda_\alpha}} \frac{(qt^{\beta-\alpha})_{\lambda_\alpha}}{(t^{\beta-\alpha+1})_{\lambda_\alpha}}\Big] \\
&= \Big[\prod_{1\le\alpha<\beta\le\ell(\lambda)} \frac{(q^{-\lambda_\beta}t^{\beta-\alpha})_{\lambda_\alpha}}{(q^{-\lambda_\beta+1}t^{\beta-\alpha-1})_{\lambda_\alpha}}\Big]\Big[\prod_{1\le\alpha<\beta\le\ell(\lambda)+1} \frac{(qt^{\beta-\alpha+1})_{\lambda_\alpha}}{(t^{\beta-\alpha})_{\lambda_\alpha}}\Big] = (3.13)^{-1} \times \prod_{\alpha=1}^{\ell(\lambda)} \frac{(qt^{\ell(\lambda)-\alpha})_{\lambda_\alpha}}{(t^{\ell(\lambda)-\alpha+1})_{\lambda_\alpha}}.
\end{aligned}$$

Therefore we have

$$\begin{aligned}\lim_{\zeta\to t}\widetilde{\varphi}_\lambda^{(\zeta)} \frac{\gamma_T(z)}{\gamma_D(z)} = &\Big[\prod_{\alpha=1}^{\ell(\lambda)} \frac{(qt^{\ell(\lambda)-\alpha})_{\lambda_\alpha}}{(t^{\ell(\lambda)-\alpha+1})_{\lambda_\alpha}} \prod_{k=1}^{m} \frac{(t)_{\theta_{\alpha,k}}}{(q)_{\theta_{\alpha,k}}}\Big] \times \\ &\prod_{k=1}^{m}\prod_{1\le\alpha<\beta\le\ell(\lambda)} \frac{(q^{\lambda_\alpha^{(k-1)}-\lambda_\beta^{(k-1)}}t^{\beta-\alpha+1})_{\theta_{\alpha,k}}}{(q^{\lambda_\alpha^{(k-1)}-\lambda_\beta^{(k-1)}+1}t^{\beta-\alpha})_{\theta_{\alpha,k}}} \frac{(q^{\lambda_\alpha^{(k-1)}-\lambda_\beta^{(k-1)}+1}t^{\beta-\alpha+1})_{\theta_{\alpha,k}}}{(q^{\lambda_\alpha^{(k-1)}-\lambda_\beta^{(k)}}t^{\beta-\alpha})_{\theta_{\alpha,k}}}\end{aligned}$$

$$=\prod_{k=1}^{m}\prod_{1\le\alpha\le\beta\le\ell(\lambda)}\frac{(q^{\lambda_\alpha^{(k-1)}-\lambda_\beta^{(k-1)}}t^{\beta-\alpha+1})_{\theta_{\alpha,k}}}{(q^{\lambda_\alpha^{(k-1)}-\lambda_\beta^{(k-1)}+1}t^{\beta-\alpha})_{\theta_{\alpha,k}}}\times\prod_{k=1}^{m}\prod_{1\le\alpha\le\beta\le\ell(\lambda)}\frac{(q^{\lambda_\alpha^{(k-1)}-\lambda_{\beta+1}^{(k)}+1}t^{\beta-\alpha})_{\theta_{\alpha,k}}}{(q^{\lambda_\alpha^{(k-1)}-\lambda_{\beta+1}^{(k)}+1}t^{\beta-\alpha+1})_{\theta_{\alpha,k}}}. \quad (3.14)$$

Note that the function $f(u) := (tu)_\infty/(qu)_\infty$ satisfies $f(u)/f(q^{-\theta}u) = (q^{-\theta+1}u)_\infty/(q^{-\theta}tu)_\infty$. Then (3.14) can be rewritten into

$$(3.14)=\prod_{k=1}^{m}\prod_{1\le\alpha\le\beta\le\ell(\lambda)}\frac{f(q^{\lambda_\alpha^{(k-1)}-\lambda_\beta^{(k-1)}}t^{\beta-\alpha})}{f(q^{\lambda_\alpha^{(k)}-\lambda_\beta^{(k-1)}}t^{\beta-\alpha})}\frac{f(q^{\lambda_\alpha^{(k)}-\lambda_{\beta+1}^{(k)}}t^{\beta-\alpha})}{f(q^{\lambda_\alpha^{(k-1)}-\lambda_{\beta+1}^{(k)}}t^{\beta-\alpha})}. \quad (3.15)$$

Finally, if $T\in \mathrm{SSTb}(\lambda;m)$, we have $k\ge\ell(\lambda^{(k)})$, Therefore if $\beta\ge k$ then $\lambda_{\beta+1}^{(k)}=\lambda_\beta^{(k-1)}=0$. Thus one can see that

$$(3.15)=\prod_{k=1}^{m}\prod_{1\le\alpha\le\beta\le\ell(\lambda^{(k-1)})}\frac{f(q^{\lambda_\alpha^{(k-1)}-\lambda_\beta^{(k-1)}}t^{\beta-\alpha})}{f(q^{\lambda_\alpha^{(k)}-\lambda_\beta^{(k-1)}}t^{\beta-\alpha})}\frac{f(q^{\lambda_\alpha^{(k)}-\lambda_{\beta+1}^{(k)}}t^{\beta-\alpha})}{f(q^{\lambda_\alpha^{(k-1)}-\lambda_{\beta+1}^{(k)}}t^{\beta-\alpha})}=\prod_{k=1}^{m}\psi_{\lambda^{(k)}/\lambda^{(k-1)}}(q,t),$$

which is $\psi_T(q,t)$. On the other hand if $T\in\mathrm{RTb}(\lambda;m)\setminus\mathrm{SSTb}(\lambda;m)$, one can see that $(3.15)=0$. Using Proposition 1.14, we have the desired equality.

3.2. **Proof of Proposition 2.9.** First we rewrite the relation of $\psi^\pm(z)$ and $x^\pm(w)$ given in Definition 2.1 into the next adjoint form.

$$\exp\Big(\sum_{n>0}\mathrm{ad}_{b_n}\,\gamma^{n/2}z^{-n}\Big)x^\pm(w)=\exp\Big(\mp\sum_{n>0}\frac{1}{n}(1-q^n)(1-t^{-n})(1-p^{-n})\gamma^{\mp n/2}\big(\frac{w}{z}\big)^n\Big)x^\pm(w),$$

$$\exp\Big(-\sum_{n>0}\mathrm{ad}_{b_{-n}}\,\gamma^{n/2}z^{n}\Big)x^\pm(w)=\exp\Big(\pm\sum_{n>0}\frac{1}{n}(1-q^n)(1-t^{-n})(1-p^{-n})\gamma^{\mp n/2}\big(\frac{w}{z}\big)^n\Big)x^\pm(w).$$

Here we used the exponential form (2.2) of $\psi^\pm$. Then we see that

$$\begin{aligned}\alpha(z)x^\pm(w)\alpha(z)^{-1}&=\exp\Big(-\sum_{n>0}\mathrm{ad}_{b_{-n}}\frac{z^n}{\gamma^n-\gamma^{-n}}\Big)x^\pm(w)\\&=\exp\Big(\pm\sum_{n>0}\frac{1}{n}\frac{(1-q^n)(1-t^{-n})(1-p^{-n})}{\gamma^n-\gamma^{-n}}\gamma^{-n/2\mp n/2}\big(\frac{z}{w}\big)^n\Big)x^\pm(w),\end{aligned}\quad (3.16)$$

$$\begin{aligned}\beta(z)x^\pm(w)\beta(z)^{-1}&=\exp\Big(\sum_{n>0}\mathrm{ad}_{b_n}\frac{z^{-n}}{\gamma^n-\gamma^{-n}}\Big)x^\pm(w)\\&=\exp\Big(\mp\sum_{n>0}\frac{1}{n}\frac{(1-q^n)(1-t^{-n})(1-p^{-n})}{\gamma^n-\gamma^{-n}}\gamma^{-n/2\mp n/2}\big(\frac{w}{z}\big)^n\Big)x^\pm(w).\end{aligned}\quad (3.17)$$

We also prepare the operator product of $\alpha(w)$ and $\beta(z)$, which is easily obtained from the definitions (2.9) and the commutation relations (2.3) of b_n's:

$$\beta(z)\alpha(w)=\alpha(w)\beta(z)\exp\Big(\sum_{n>0}\frac{1}{n}\frac{(1-q^n)(1-t^{-n})(1-p^{-n})}{\gamma^n-\gamma^{-n}}\gamma^{-n}\big(\frac{w}{z}\big)^n\Big). \quad (3.18)$$

3.2.1. *Proof of (1).* Using (3.18) and (3.16), we see that

$$\begin{aligned}&\alpha(z)t(w)\alpha(z)^{-1}=\alpha(z)\alpha(w)x^+(w)\beta(w)\alpha(z)^{-1}\\&=\alpha(w)\alpha(z)x^+(w)\alpha(z)^{-1}\beta(w)\times\exp\Big(-\sum_{n>0}\frac{1}{n}\frac{(1-q^n)(1-t^{-n})(1-p^{-n})}{\gamma^n-\gamma^{-n}}\gamma^{-n}\big(\frac{z}{w}\big)^n\Big)\\&=\alpha(w)x^+(w)\beta(w)\times\exp\Big(-\sum_{n>0}\frac{1}{n}\frac{(1-q^n)(1-t^{-n})(1-p^{-n})}{\gamma^n-\gamma^{-n}}\gamma^{-n}\big(\frac{z}{w}\big)^n\end{aligned}$$

$$+\sum_{n>0}\frac{1}{n}\frac{(1-q^n)(1-t^{-n})(1-p^{-n})}{\gamma^n-\gamma^{-n}}\gamma^{-n}\left(\frac{z}{w}\right)^n\Big) = t(w).$$

Thus we have $[t(z),\alpha(w)]=0$. The other relations $[t(z),\beta(w)]=0$, $[t^*(z),\alpha(w)]=[t^*(z),\beta(w)]=0$, $[t(z),\psi^{\pm}(w)]=[t^*(z),\psi^{\pm}(w)]=0$ also follow from equations (3.16)-(3.18) and we omit the detail.

3.2.2. *Proof of (2).* Using the commutativity $[t(z),\alpha(w)]=0$ given in (1), we have

$$A(w/z)t(z)t(w) = A(w/z)\alpha(z)x^+(z)\beta(z)\alpha(w)x^+(w)\beta(w) = \alpha(z)\alpha(w)x^+(z)x^+(w)\beta(z)\beta(w)$$
$$\times \exp\Big(\sum_{n>0}\frac{1}{n}\frac{(1-q^n)(1-t^{-n})(1-p^{-n}\gamma^{-2n})}{1-\gamma^{-2n}}\left(\frac{w}{z}\right)^n - \sum_{n>0}\frac{1}{n}\frac{(1-q^n)(1-t^{-n})(1-p^{-n})}{\gamma^n-\gamma^{-n}}\gamma^{-n}\left(\frac{w}{z}\right)^n\Big).$$

Here the first summation in the exponential comes from the $A(w/z)$, and the second from transposition of $\beta(w)x^+(z)$ using (3.17). Thus we have

$$A(w/z)t(z)t(w) = \alpha(z)\alpha(w)x^+(z)x^+(w)\beta(z)\beta(w)\times\exp\Big(\sum_{n>0}\frac{1}{n}(1-q^n)(1-t^{-n})\left(\frac{w}{z}\right)^n\Big)$$
$$= \frac{(1-q\frac{w}{z})(1-t^{-1}\frac{w}{z})}{(1-\frac{w}{z})(1-p\frac{w}{z})}\alpha(z)\alpha(w)x^+(z)x^+(w)\beta(z)\beta(w).$$

Then

$$A(w/z)t(z)t(w) - A(z/w)t(w)t(z) = \alpha(z)\alpha(w)$$
$$\times\Big[\frac{(1-q\frac{w}{z})(1-t^{-1}\frac{w}{z})}{(1-\frac{w}{z})(1-p\frac{w}{z})}x^+(z)x^+(w) - \frac{(1-q\frac{z}{w})(1-t^{-1}\frac{z}{w})}{(1-\frac{z}{w})(1-p\frac{z}{w})}x^+(w)x^+(z)\Big] \tag{3.19}$$
$$\times\beta(z)\beta(w).$$

Now recall the relation of $x^+(z)$ and $x^+(w)$ given in Definition 2.1:

$$-\left(\tfrac{z}{w}\right)^3G^+\left(\tfrac{z}{w}\right)x^+(z)x^+(w) = G^+\left(\tfrac{z}{w}\right)x^+(w)x^+(z). \tag{3.20}$$

Using this equation, the line (3.19) is rewritten into

$$(3.19) = \Big[\frac{1}{(1-\frac{w}{z})(1-p\frac{w}{z})(1-p^{-1}\frac{w}{z})} + \frac{(\frac{z}{w})^3}{(1-\frac{z}{w})(1-p\frac{z}{w})(1-p^{-1}\frac{z}{w})}\Big]G^+\left(\tfrac{w}{z}\right)x^+(w)x^+(z)$$
$$= \Big[\frac{\delta(\frac{w}{z})}{(1-p^{-1})(1-p)} + \frac{\delta(p\frac{w}{z})}{(1-p^{-1})(1-p^{-2})} + \frac{\delta(p^{-1}\frac{w}{z})}{(1-p)(1-p^2)}\Big]G^+\left(\tfrac{w}{z}\right)x^+(w)x^+(z).$$

Now from (3.20) and $G^+(1)\neq 0$, we see that $\delta(w/z)G^+(w/z)x^+(w)x^+(z)=0$. We also find from (3.20) and $G^+(p^{-1})=0$ that $\delta(p\frac{w}{z})G^+(\frac{w}{z})x^+(w)x^+(z) = \delta(p\frac{w}{z})G^+(p^{-1})x^+(pw)x^+(w)$. Similarly from (3.20) and $G^+(p)\neq 0$ we have $\delta(p^{-1}\frac{w}{z})G^+(\frac{w}{z})x^+(w)x^+(z) = \delta(p^{-1}\frac{w}{z})G^+(p)x^+(pz)x^+(z)$. Thus after a short calculation we have

$$(3.19) = \frac{(1-t^{-1})(1-q)}{1-p}\Big[\delta(p^{-1}\tfrac{w}{z})x^+(pz)x^+(z) - \delta(p\tfrac{w}{z})x^+(pw)x^+(w)\Big].$$

Then we have the desired consequence (2.11).

The equation (2.12) can be similarly shown, so that we omit the detail.

3.2.3. *Proof of (3).* We apply the same method as in (2). Recalling Remark (2.8), we calculate $B(\gamma pw/z)t(z)t^*(\gamma pw) - B(\gamma^{-1}p^{-1}z/w)t^*(\gamma pw)t(z)$. From the definition (2.13) of $B(z)$, the commutativity $[t(z),\alpha(w)]=0$ given in (1) and the formula (3.18), we have

$$B(\gamma pw/z)t(z)t^*(\gamma pw) = B(\gamma p\tfrac{w}{z})\alpha(z)x^+(z)\beta(z)\alpha(\gamma w)^{-1}x^-(w)\beta(\gamma^{-1}w)^{-1}$$
$$= \alpha(z)\alpha(\gamma w)^{-1}x^+(z)x^-(w)\beta(z)\beta(\gamma^{-1}w)^{-1}$$

$$\times \exp\Big(\sum_{n>0}\frac{1}{n}\frac{(1-q^n)(1-t^{-n})(1-p^{-n})}{\gamma^n-\gamma^{-n}}\big(\frac{w}{z}\big)^n$$
$$+\sum_{n>0}\frac{1}{n}\frac{(1-q^n)(1-t^{-n})(\gamma^{-2n}p^{-2n}-\gamma^{-2n}p^{-n})}{1-\gamma^{-2n}}\gamma^n p^{-n}\big(\frac{w}{z}\big)^n\Big)$$
$$=\alpha(z)\alpha(\gamma w)^{-1}x^+(z)x^-(w)\beta(z)\beta(\gamma^{-1}w)^{-1}.$$

A similar calculation shows that $B(\gamma p\frac{z}{w})t^*(\gamma pw)t(z) = \alpha(z)\alpha(\gamma w)^{-1}x^-(w)x^+(z)\beta(z)\beta(\gamma^{-1}w)^{-1}$. Thus we have

$$B(\gamma p\tfrac{w}{z})t(z)t^*(\gamma pw) - B(\gamma p\tfrac{z}{w})t^*(\gamma pw)t(z)$$
$$=\alpha(z)\alpha(\gamma w)^{-1}[x^+(z)x^-(w)-x^-(w)x^+(z)]\beta(z)\beta(\gamma^{-1}w)^{-1}$$

Using the expression of $[x^+(z), x^-(w)]$ given in Definition 2.1, the expansion (2.2) and the defnition (2.9), one may immediately find that

$$B(\gamma p\tfrac{w}{z})t(z)t^*(\gamma pw) - B(\gamma p\tfrac{z}{w})t^*(\gamma pw)t(z) = \frac{(1-q)(1-t^{-1})}{1-p}\Big(\delta(\gamma^{-1}\tfrac{z}{w})\psi_0^+ - \delta(\gamma\tfrac{z}{w})\psi_0^-\Big).$$

Replacing w in the above equation with $\gamma^{-1}p^{-1}w$, we have the desired equation (2.14).

3.3. **Proof of Proposition 2.13.** Let us define $a_{n,(i)} := 1\otimes\cdots\otimes 1\otimes a_n\otimes 1\otimes\cdots 1$, where a_n sits in the i-th tensor component. Then from (2.4) and (2.5) one finds that $\rho_y^{(m)}(b_n) = -\sum_{i=1}^m a_{n,(i)}(1-t^n)(1-p^{-|n|})p^{(m-i+1)|n|/2}/|n|$. Thus we have

$$\rho_y^{(m)}(\alpha(z)) = \prod_{i=1}^m \alpha_{(i)}^m(z),\quad \alpha_{(i)}^m(z) := \exp\Big(\sum_{n>0}\frac{1}{n}\frac{p^{(m-i+1)n/2}(1-t^{-n})(1-p^{-n})}{p^{-mn/2}-p^{mn/2}}a_{-n,(i)}z^n\Big),\tag{3.21}$$
$$\rho_y^{(m)}(\beta(z)) = \prod_{i=1}^m \beta_{(i)}^m(z),\quad \beta_{(i)}^m(z) := \exp\Big(-\sum_{n>0}\frac{1}{n}\frac{p^{(m-i+1)n/2}(1-t^{n})(1-p^{-n})}{p^{-mn/2}-p^{mn/2}}a_{n,(i)}z^{-n}\Big).\tag{3.22}$$

3.3.1. *Proof of (1).* We calculate each tensor component of $\Lambda_i(z)\Lambda_j(w)$. First assume $i=j$.

If $k>i$, then the k-th tensor component comes from $\alpha_{(k)}^m(z)\beta_{(k)}^m(z)\alpha_{(k)}^m(w)\beta_{(k)}^m(z)$. Under the normal ordering, the following coefficient arises.

$$\exp\Big(-\sum_{n>0}\frac{1}{n}(1-q^n)(1-t^{-n})\big(\frac{1-p^{-n}}{1-p^{mn}}\big)^2 p^{(2m-k+1)n}\big(\frac{w}{z}\big)^n\Big).\tag{3.23}$$

For $k=i$, the i-th tensor component comes from $\alpha_{(k)}^m(z)\eta(p^{-(i-1)/2}z)\beta_{(k)}^m(z)\alpha_{(k)}^m(w)\eta(p^{-(i-1)/2}w)\beta_{(k)}^m(w)$. Under the normal ordering, the following coefficient arises.

$$\exp\Big(-\sum_{n>0}\frac{1}{n}(1-q^n)(1-t^{-n})\big(\frac{w}{z}\big)^n \Big\{\big(\frac{1-p^{-n}}{1-p^{mn}}\big)^2p^{(2m-i+1)n}+\frac{1-p^{-n}}{1-p^{mn}}p^{mn}+\frac{1-p^{-n}}{1-p^{mn}}p^{(m-i+1)n}+1\Big\}\Big).\tag{3.24}$$

If $k<i$, then the k-th tensor component is $\alpha_{(k)}^m(z)\varphi^-(p^{-(2k-1)/4}z)\beta_{(k)}^m(z)\alpha_{(k)}^m(w)\varphi^-(p^{-(2k-1)/4}w)$ $\beta_{(k)}^m(w)$. The normal ordering coefficient is

$$\exp\Big(-\sum_{n>0}\frac{1}{n}(1-q^n)(1-t^{-n})\big(\frac{w}{z}\big)^n\Big\{\big(\frac{1-p^{-n}}{1-p^{mn}}\big)^2p^{(2m-k+1)n}+\frac{(1-p^{-n})^2}{1-p^{mn}}p^{(m-k+1)n}\Big\}\Big).\tag{3.25}$$

By simple calculations, the product of (3.23), (3.24) and (3.25) is shown to be $f_{1,m}(w/z)^{-1}$. Thus the statement holds.

Next we consider the case $i < j$. If $k < i$, then the normal order coefficient is the same as (3.25). For $k = i$, the normal order coefficient is

$$\begin{aligned}&\exp\Big(-\sum_{n>0}\frac{1}{n}(1-q^n)(1-t^{-n})\big(\frac{w}{z}\big)^n\\ &\qquad\{\big(\frac{1-p^{-n}}{1-p^{mn}}\big)^2 p^{(2m-i+1)n}+\frac{1-p^{-n}}{1-p^{mn}}p^{mn}+\frac{(1-p^{-n})^2}{1-p^{mn}}p^{(m-i+1)n}+1-p^{-n}\}\Big).\end{aligned} \tag{3.26}$$

If $i < k < j$, then the normal order coefficient is

$$\exp\Big(-\sum_{n>0}\frac{1}{n}(1-q^n)(1-t^{-n})\big(\frac{w}{z}\big)^n\{\big(\frac{1-p^{-n}}{1-p^{mn}}\big)^2 p^{(2m-k+1)n}+\frac{(1-p^{-n})^2}{1-p^{mn}}p^{(m-k+1)n}\}\Big). \tag{3.27}$$

If $k = j$, then the normal order coefficient is

$$\exp\Big(-\sum_{n>0}\frac{1}{n}(1-q^n)(1-t^{-n})\big(\frac{w}{z}\big)^n\{\big(\frac{1-p^{-n}}{1-p^{mn}}\big)^2 p^{(2m-j+1)n}+\frac{1-p^{-n}}{1-p^{mn}}p^{(m-j+1)n}\}\Big). \tag{3.28}$$

If $k > j$, then the normal order coefficient is (3.23). The product of (3.25), (3.26), (3.27), (3.28), (3.23) is equal to $f_{1,m}(w/z)^{-1}\gamma_+(z,w;q,p)$. Thus we obtain the result.

The case $i > j$ is similar, so we omit the detail.

3.3.2. *Proof of (2).* The desired equation is equivalent to

$$\rho_y^{(m)}(\alpha(z)\cdots\alpha(p^{m-1}z)) : \prod_{k=1}^{m}\widetilde{\Lambda}_k(p^{k-1}z) : \rho_y^{(m)}(\beta(z)\cdots\beta(p^{m-1}z)) = 1.$$

We will show this equation by comparing each tensor component.

By (3.21), the k-th tensor component of $\rho_y^{(m)}(\alpha(z)\cdots\alpha(p^{m-2}z))$ is equal to

$$\exp\Big(-\sum_{n>0}\frac{1}{n}(1-t^{-n})p^{(2m-k-1)n/2}a_{-n}z^n\Big). \tag{3.29}$$

Similarly, the k-th tensor component of $\rho_y^{(m)}(\beta(z)\cdots\beta(p^{m-1}z))$ is equal to

$$\exp\Big(\sum_{n>0}\frac{1}{n}(1-t^{-n})p^{(-k+1)n/2}a_n z^{-n}\Big). \tag{3.30}$$

The k-th tensor component of $: \prod_{k=1}^{m}\widetilde{\Lambda}_k(p^{k-1}z) :$ is

$$\begin{aligned}&:\eta(p^{-(k-1)/2}p^{k-1}z)\varphi^-(p^{-(2k-1)/4}p^k z)\varphi^-(p^{-(2k-1)/4}p^{k+1}z)\cdots\varphi^-(p^{-(2k-1)/4}p^{m-1}z):\\ &=\exp\Big(\sum_{n>0}\frac{1-t^{-n}}{n}p^{n(2m-k-1)/2}a_{-n}z^n\Big)\exp\Big(-\sum_{n>0}\frac{1-t^n}{n}p^{-n(k-1)/2}a_n z^{-n}\Big)\end{aligned} \tag{3.31}$$

It is easy to see that (3.29),(3.30) and (3.31) cancel. Thus we have the consequnce.

3.3.3. *Proof of (3).* The desired equation is equivalent to

$$\begin{aligned}&\rho_y^{(m)}(\alpha(p^{-1}z)\cdots\alpha(p^{m-2}z)) : \prod_{k=1}^{i-1}\widetilde{\Lambda}_k(p^{k-1}z)\prod_{l=i+1}^{m}\widetilde{\Lambda}_l(p^{l-2}z) : \rho_y^{(m)}(\beta(z)\cdots\beta(p^{m-1}z))\\ &=\widetilde{\Lambda}_i^*(p^{(m-2)/2}z).\end{aligned} \tag{3.32}$$

We will show this equation by comparing each tensor component.

As in (3.29) , the k-th tensor component of $\rho_y^{(m)}(\alpha(p^{-1}z)\cdots\alpha(p^{m-2}z))$ is equal to

$$\exp\Big(-\sum_{n>0}\frac{1}{n}(1-t^n)p^{(2m-k-3)n/2}a_{-n}z^n\Big). \tag{3.33}$$

The k-th tensor component of $\rho_y^{(m)}(\beta(z)\cdots\beta(p^{m-1}z))$ is given by (3.30).

The k-th tensor component of $:\prod_{k=1}^{i-1}\widetilde{\Lambda}_k(p^{k-1}z)\prod_{l=i+1}^{m}\widetilde{\Lambda}_l(p^{l-2}z):$ depends on k. If $k=i$, then by Lemma 2.6 and some simple calculations, the component turns out to be

$$\begin{aligned}&\varphi^-(p^{-(2i-1)/4}p^{i-1}z)\varphi^-(p^{-(2i-1)/4}p^{i}z)\cdots\varphi^-(p^{-(2i-1)/4}p^{m-2}z)\\&=\exp\Big(-\sum_{n>0}\frac{1-t^{-n}}{n}p^{(i-3)/2}(1-p^{n(m-i)})a_{-n}z^n\Big).\end{aligned} \tag{3.34}$$

Similarly, if $k<i$, then by Lemma 2.6 the component is

$$\begin{aligned}&:\eta(p^{-(k-1)/2}p^{k-1}z)\varphi^-(p^{-(2k-1)/4}p^{k}z)\varphi^-(p^{-(2k-1)/4}p^{k+1}z)\cdots\varphi^-(p^{-(2k-1)/4}p^{m-2}z):\\&=\exp\Big(\sum_{n>0}\frac{1-t^{-n}}{n}p^{n(2m-k-3)/2}a_{-n}z^n\Big)\exp\Big(-\sum_{n>0}\frac{1-t^{n}}{n}p^{-n(k-1)/2}a_{n}z^{-n}\Big):\end{aligned} \tag{3.35}$$

If $k>i$, then by Lemma 2.6 the component is

$$\begin{aligned}&:\eta(p^{-(k-1)/2}p^{k-2}z)\varphi^-(p^{-(2k-1)/4}p^{k-1}z)\varphi^-(p^{-(2k-1)/4}p^{k}z)\cdots\varphi^-(p^{-(2k-1)/4}p^{m-2}z):\\&=\exp\Big(\sum_{n>0}\frac{1-t^{-n}}{n}p^{n(2m-k-3)/2}a_{-n}z^n\Big)\exp\Big(-\sum_{n>0}\frac{1-t^{n}}{n}p^{-n(k-3)/2}a_{n}z^{-n}\Big):\end{aligned} \tag{3.36}$$

Then the i-th tensor component of (3.32) is the product of (3.33),(3.30) and (3.34). After a short calculation, one finds that it is $\xi(p^{(i-2)/2}z)$, which is the i-th component of $\widetilde{\Lambda}_i^*(p^{(m-2)/2}z)$.

If $k<i$, then the k-th tensor component of (3.32) is the product of (3.33),(3.30) and (3.35). It is 1, that is, the k-th component of $\widetilde{\Lambda}_i^*(p^{(m-2)/2}z)$.

Finally, for $k>i$, the k-th tensor component of (3.32) is the product of (3.33),(3.30) and (3.36). It turns out to be $\varphi^-(p^{(2j-5)/4}z)$, which is the k-th component of $\widetilde{\Lambda}_i^*(p^{(m-2)/2}z)$.

3.3.4. *Proof of (4).* From the known identities (2.20) and (2.22), it is not difficult to calculate $\Big[\prod_{k,l=1}^{m-1}f_{1.m}(p^{-k+l}w/z)\Big]\Lambda_i^*(z)\Lambda_j^*(w)$ in terms of Λ_k's.

First we consider the case $i=j$. From the operator product (2.20), we have

$$\begin{aligned}&\Big[\prod_{k,l=1}^{m-1}f_{1,m}(p^{-k+l}\tfrac{w}{z})\Big]\Lambda_i^*(z)\Lambda_i^*(w)\\&=\Big[\prod_{k=1}^{m-2}\prod_{l=k+1}^{m-1}\gamma_+(p^{-k+l}\tfrac{w}{z})\Big]\Big[\prod_{k=2}^{m-1}\prod_{l=1}^{k-1}\gamma_-(p^{-k+l}\tfrac{w}{z})\Big]:\Lambda_i^*(z)\Lambda_i^*(w):\\&=\exp\Big(\sum_{n>0}\frac{1}{n}(1-q^n)(1-t^{-n})\frac{1-p^{-n(m-2)}}{1-p^{-n}}\frac{1-p^{n(m-1)}}{1-p^n}(\frac{w}{z})^n\Big):\Lambda_i^*(z)\Lambda_i^*(w):.\end{aligned}$$

Here we used the abbreviation $\gamma_\pm(\frac{w}{z}):=\gamma_\pm(z,w;q,p)$. Then we have

$$\begin{aligned}&\Big[\prod_{k,l=1}^{m-1}f_{1,m}(p^{-k+l}\tfrac{w}{z})\Big]\times\exp\Big(-\sum_{n>0}\frac{1}{n}(1-q^n)(1-t^{-n})\frac{1-p^{-n(m-2)}}{1-p^{-n}}\frac{1-p^{n(m-1)}}{1-p^n}(\frac{w}{z})^n\Big)\\&=\exp\Big(-\sum_{n>0}\frac{1}{n}\frac{(1-q^n)(1-t^{-n})(1-p^{(m-1)n}}{1-p^{mn}}(\frac{w}{z})^n\Big)=f_{1,m}(\tfrac{w}{z}).\end{aligned}$$

Thus the desired equation $f_{1,m}(\frac{w}{z})\Lambda_i^*(z)\Lambda_i^*(w) =: \Lambda_i^*(z)\Lambda_i^*(w):$ is proved.

Next, note that the calculation of the case $i \neq j$ reduces to that of $k = i$. If $i < j$, then

$$f_{1,m}(\tfrac{w}{z})\Lambda_i^*(z)\Lambda_j^*(w) =: \Lambda_i^*(z)\Lambda_j^*(w) : \frac{\gamma_-(\frac{w}{z})^{j-i}}{\gamma_+(p\frac{w}{z})^{j-i-1}} =: \Lambda_i^*(z)\Lambda_j^*(w) : \gamma_-(\tfrac{w}{z}).$$

At the last line we used the formula $\gamma_-(z)/\gamma_+(pz) = 1$. For the final case $i > j$, we have

$$f_{1,m}(\tfrac{w}{z})\Lambda_i^*(z)\Lambda_j^*(w) =: \Lambda_i^*(z)\Lambda_j^*(w) : \frac{\gamma_+(\frac{w}{z})^{i-j}}{\gamma_-(p^{-1}\frac{w}{z})^{i-j-1}} =: \Lambda_i^*(z)\Lambda_j^*(w) : \gamma_+(\tfrac{w}{z}).$$

Thus all the cases are proved.

3.3.5. *Proof of (5).* This is similary shown as (2) and (3), so we omit the detail.

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