数理解析研究所講究録
第 1690 巻 2010 年 105-116

On the Stokes and Navier-Stokes equations with Robin boundary condition in a perturbed half space

Waseda university Yuka Naito

1 Introduction

We consider Navier-Stokes equation with Robin boundary condition in a perturbed half space. We show that Navier-Stokes equation has a unique strong solution $u(t)$ on $(0, \infty)$ with a small data. Navier-Stokes equation is given by the following:

$$(1.1) \qquad \begin{cases} u_t - \Delta u + u \cdot \nabla u + \nabla p = 0 & \text{in } \Omega \times (0, \infty), \\ \nabla \cdot u = 0 & \text{in } \Omega \times (0, \infty), \\ u \cdot \nu = 0, \ B_{\alpha,\beta}(u,p) = 0 & \text{on } \partial\Omega \times (0, \infty), \\ u(x, 0) = a(x) & \text{in } \Omega. \end{cases}$$

where $u = (u_1, \cdots, u_n)$ is velocity, p is pressure, ν is a unit outer normal vector of $\partial\Omega$, a is a initial value. And we assume $\Omega \subset \mathbb{R}^3$ is a perturbed half space with smooth boundary $\partial\Omega$. Here a perturbed half space is a domain that satisfies the following condition:
there exists $R > 0$ such that $\Omega \setminus B_R = \mathbb{R}^n_+ \setminus B_R$, where $B_R = \{x \in \mathbb{R}^n \mid |x| < R\}$. And Robin boundary condition is given by

$$u \cdot \nu = 0, \quad B_{\alpha,\beta}(u,p) = \alpha u + \beta\{T(u,p)\nu - (T(u,p)\nu, \nu)\nu\} = 0 \quad (\alpha + \beta = 1).$$

where $T(u,p) = D(u) - pI$ denotes stress tensor of the Stokes flow, $D(u)_{jk} = \partial_k u_j + \partial_j u_k$ is strain tensor, where $\partial_k u_j = \frac{\partial u_j}{\partial x_k}$. We know easily that $B_{\alpha,\beta}(u,p)$ is independent of u:

$$B_{\alpha,\beta}(u,p) = B_{\alpha,\beta}(u).$$

Especially when $\alpha = 0$, Robin boundary condition becomes Navier's slip condition ($\partial_n u = 0$). And when $\beta = 0$, it becomes non-slip condition ($u = 0$). With non-slip condition there are many papers. R. Farwig and H. Sohr showed that Stokes operator A generates an analytic semigroup $T(t)$ on $J_p(\Omega)$

in a half space and a perturbed half space [1]. Kubo-Shibata tracted the non-slip condition case in a perturbed half space in [2]. In this paper we would like to extend their results to the case of Robin boundary condition. When parametrix is constructed with non-slip condition, Bogovskĭ lemma is very useful. But with Robin boundary condition we can use the lemma, and so we can not do by same way. And in Navier's slip condition case, we need the different way, and we assume $\alpha > 0$, $\beta > 0$ in this paper. The following theorem is our main result.

Theorem 1.1. *Let $n \geq 3$.*
There is a constant $\epsilon = \epsilon(\Omega, n) > 0$ such that if $a \in J^n(\Omega)$ satisfies

$$\|a\|_{L^n(\Omega)} \leq \epsilon,$$

Navier-Stokes equation admits a unique strong solution $u(t)$ on $(0, \infty)$. Moreover as $t \longrightarrow \infty$,

$$\|u(t)\|_{L^p(\Omega)} = o(t^{-\frac{1}{2}+\frac{n}{2p}}) \quad \text{for} \quad n \leq p \leq \infty,$$
$$\|\nabla u(t)\|_{L^n(\Omega)} = o(t^{-\frac{1}{2}}).$$

To get this main theorem, we consider Stokes equation which is given by the following:

$$(1.2) \qquad \begin{cases} u_t - \Delta u + \nabla p = 0 & \text{in } \Omega \times (0, \infty), \\ \nabla \cdot u = 0 & \text{in } \Omega \times (0, \infty), \\ u \cdot \nu = 0, \ B_{\alpha,\beta}(u, p) = 0 & \text{on } \partial\Omega \times (0, \infty), \\ u(x, 0) = a(x) & \text{in } \Omega. \end{cases}$$

Using semigroup argument, we define a operator the following. We consider the solenoidal space : $J^p(\Omega)$ which is given by

$$J^p(\Omega) = \{u \in L^p(\Omega)^n \mid \nabla \cdot u = 0 \text{ in } \Omega, \ u \cdot \nu = 0 \text{ on } \partial\Omega\}.$$

And we define Stokes operator by the following:

$$Au = -P\Delta u \ \text{ for } u \in D(A)$$

$$D(A) = J^p \cap \{u \in W^{2,p}(\Omega) \mid B_{\alpha,\beta}(u, p) = 0 \text{ on } \partial\Omega\},$$

here P is a continuous projection from $L^p(\Omega)^n$ onto $J^p(\Omega)$. According to Kato's argument [4], to get the main theorem, it suffices to show the following two results about Stokes equation:

1. Stokes operator A generates an anlytic semigroup $\{T(t)\}_{t \geq 0}$,

2. $L^n - L^q$ decay estimates of the Stokes semigroup $\{T(t)\}_{t\geq 0}$.

To consider the two theorems about Stokes equation, I consider the resolvent problem:

$$\begin{cases} \lambda u - \Delta u + \nabla p = f & \text{in } \Omega, \\ \nabla \cdot u = 0 & \text{in } \Omega, \\ u \cdot \nu = 0, \ B_{\alpha,\beta}(u) = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

About this resolvent problem we introduce the known result ([5]). It is the theorem about resolvent estimate with large λ.

Theorem 1.2. *For all $\epsilon > 0$ there exists λ_0 and C_ϵ such that satisfies the following:*

$$|\lambda| \ \|u\|_{L^q(\Omega)} + |\lambda|^{\frac{1}{2}} \ \|\nabla u\|_{L^q(\Omega)} + \|\nabla^2 u\|_{L^q(\Omega)} + \|\nabla p\|_{L^q(\Omega)} \leq C\|f\|_{L^q(\Omega)}$$

for $\lambda \in \sum_\epsilon, \quad |\lambda| > \lambda_0$ *where* $\sum_\epsilon = \{\lambda \in \mathbb{C} \setminus \{0\} \mid |\arg \lambda| \leq \pi - \epsilon\}$.

This theorem implies 1, that is the generation of an anlytic semigroup $\{T(t)\}_{t\geq 0}$. But we can not know 2, that is $L^n - L^q$ decay estimates of the Stokes semigroup $\{T(t)\}_{t\geq 0}$. Therefore our aim of this paper is to show the $L^n - L^q$ decay estimates. To do so, we have to analyze the resolvent problem with small λ. Therefore first we show the resolvent expansion with small λ in section 2. In section 3 we show $L^p - L^\infty$ estimates in a half space. Concretely we show the resolvent estimates in a half space. In section 4 we get $L^n - L^q$ decay estimates, using the estimates with small λ .

2 The resolvent expansion with small λ

Instead of (1.3) we consider generalized resolvent problem in a perturbed half space:

$$\begin{cases} \lambda u - \Delta u + \nabla p = f & \text{in } \Omega, \\ \nabla \cdot u = g & \text{in } \Omega, \\ u \cdot \nu = 0, \ B_{\alpha,\beta}(u) = h & \text{on } \partial\Omega. \end{cases}$$

Generalized means that the right members : g, h are not zero. Let us define the solution operator $U(\lambda)$ and $\Pi(\lambda)$ by the formula:

$$U(\lambda)F = u, \Pi(\lambda)F = p,$$

where we set $F = {}^t(f, g, h)$. Then we know

$$U(\lambda) : L^p_{R+3}(\Omega)^n \times W^{1,p}_{R+3,0}(\Omega) \times W^{1,p}_{R+3}(\Omega)^n \longrightarrow W^{2,p}_{loc}(\Omega),$$

$$F = {}^t(f,g,h) \longmapsto u,$$
$$\Pi(\lambda) : L^p_{R+3}(\Omega)^n \times W^{1,p}_{R+3,0}(\Omega) \times W^{1,p}_{R+3}(\Omega)^n \longrightarrow W^{1,p}_{loc}(\Omega),$$
$$F = {}^t(f,g,h) \longmapsto p,$$

where we have set the function spaces:

$$L^p_{R+3}(\Omega) = \{f \in L^p(\Omega) \mid \text{supp } f \subset B_{R+3}\},$$
$$W^{1,p}_{R+3,0}(\Omega) = \left\{f \in W^{1,p}(\Omega) \mid \text{supp } f \subset B_{R+3}, \int_\Omega f dx = 0\right\},$$
$$W^{1,p}_{R+3}(\Omega) = \left\{f \in W^{1,p}(\Omega) \mid \text{supp } f \subset B_{R+3}\right\}.$$

About the solution operator $U(\lambda)$ and $\Pi(\lambda)$ we can get the following:

Theorem 2.1. *Let $n \geq 3$ and $1 < p < \infty$.*

$$G_\Omega = \mathcal{L}(L^p_{R+3}(\Omega)^n \times W^{1,p}_{R+3,0}(\Omega) \times W^{1,p}_{R+3}(\Omega)^n, W^{2,p}_{loc}(\Omega) \times W^{1,p}_{loc}(\Omega))$$

Then solution operators $(U(\lambda), \Pi(\lambda)) \in G_\Omega$ for $\lambda \in U_{\frac{\alpha^2}{(1+\sqrt{2})^2\beta^2}}$, moreover they have the following expansion formula

$$(U(\lambda)F, \Pi(\lambda)F)$$
$$= \lambda^{\frac{n-1}{2}} H_1(\lambda)F + \lambda^{\frac{n-2}{2}} H_2(\lambda)F + (\lambda \log \lambda) H_3(\lambda)F + H_4(\lambda)F,$$

where $\mathcal{L}(X,Y)$ is the Banach space of all bounded linear operators from X to Y, H_j $(j = 1,2,3,4)$ are G_Ω-valued holomorphic functions in $U_{\frac{\alpha^2}{(1+\sqrt{2})^2\beta^2}}$, $F = {}^t(f,g,h)$, $U_\lambda = \{\lambda \in \mathbb{C} \mid |\lambda| < r\}$.

To prove this theorem, we need the results of a half space problem. Therefore we shall consider generalized resolvent problem in half space:

$$\text{(2.1)} \qquad \begin{cases} \lambda v_h - \Delta v_h + \nabla \theta_h = f & \text{in } \mathbb{R}^n_+, \\ \nabla \cdot v_h = g & \text{in } \mathbb{R}^n_+, \\ \alpha v_{hi} - \beta \partial_n v_{hi} = h_j \ (j = 1, \cdots n-1),\ v_{hn} = 0 & \text{on } \partial\mathbb{R}^n_+, \end{cases}$$

where the unit outer normal vector becomes $\nu = (0, \cdots, 0, -1)$ in a half space. We shall introduce theorems about this problem. They were proved in [6] by Y.Naito.

Theorem 2.2. *Let $n \geq 3$ and $1 < p < \infty$.*

$$G_{\mathbb{R}^n_+} = \mathcal{L}(L^p_{R+3}(\mathbb{R}^n_+)^n \times W^{1,p}_{R+3,0}(\mathbb{R}^n_+) \times W^{1,p}_{R+3}(\mathbb{R}^n_+)^n, W^{2,p}_{loc}(\mathbb{R}^n_+) \times W^{1,p}_{loc}(\mathbb{R}^n_+))$$

Then solution operators $(U_h(\lambda), \Pi_h(\lambda)) \in G_{\mathbb{R}^n_+}$ *for* $\lambda \in U_{\frac{\alpha^2}{(1+\sqrt{2})^2\beta^2}}$, *moreover they have the following expansion formula*

$$(U_h(\lambda)F, \Pi_h(\lambda)F)$$
$$= \lambda^{\frac{n-1}{2}} H_1(\lambda)F + \lambda^{\frac{n-2}{2}} H_2(\lambda)F + (\lambda \log \lambda)H_3(\lambda)F + H_4(\lambda)F,$$

where H_j $(j = 1, 2, 3, 4)$ *are* $G_{\mathbb{R}^n_+}$*-valued holomorphic functions in* $U_{\frac{\alpha^2}{(1+\sqrt{2})^2\beta^2}}$, $F = {}^t(f, g, h)$, $U_\lambda = \{\lambda \in \mathbb{C} \mid |\lambda| < r\}$, *where we have set* $U_h(\lambda)F = v_h, \Pi_h(\lambda) = \theta_h$.

Theorem 2.3. *Let* $1 < p < \infty$, $n \geq 3$. *Let* $(U_h(\lambda), \Pi_h(\lambda))$ *be the solution operator to* (2.1) *for* $\lambda \in \mathbb{C}\backslash(\infty, 0]$. *Then there exists operator*

$$(U_h(0), \Pi_h(0)) : (L^p_{R+3}(\mathbb{R}^n_+)^n \times W^{1,p}_{R+3,0}(\mathbb{R}^n_+) \times W^{1,p}_{R+3}(\mathbb{R}^n_+)^n, W^{2,p}_{loc}(\mathbb{R}^n_+) \times W^{1,p}_{loc}(\mathbb{R}^n_+))$$

which have the following properties:
(1)If we set $U_h(0)F = v_h$ *and* $\Pi_h(0)F = \theta_h$, *then then* (v_h, θ_h) *satisfies the equation:*

$$\begin{cases} -\Delta v_h + \nabla\theta_h = f & \text{in } \mathbb{R}^n_+, \\ \nabla \cdot v_h = g & \text{in } \mathbb{R}^n_+, \\ \alpha v_{hi} - \beta\partial_n v_{hi} = h_j \ (j = 1, \cdots n-1), \ v_{hn} = 0 & \text{on } \partial\mathbb{R}^n_+, \end{cases}$$

(2)(v_h, θ_h) *satisfies the estimates:*

$$\|v_h\|_{W^{2,p}(B^+_L)} + \|\theta_h\|_{W^{1,p}(B^+_L)} \leq C_{R,L}\|F\|_{A(\mathbb{R}^n_+)},$$
$$\sup_{|x|\geq 1, x\in\mathbb{R}^n_+} \{|x|^{n-1}|v_h(x)| + |x|^{n-1}|\nabla v_h(x)| + |x|^{n-1}|\theta_h(x)|\} \leq C_{R,L}\|F\|_{A(\mathbb{R}^n_+)}$$
$$\|U_h(\lambda)F - U_h(0)F\|_{W^{1,p}(B^+_R)} + \|\Pi_h(\lambda)F - \Pi_h(0)F\|_{L^p(B^+_R)}$$
$$\leq C(|\lambda|^{\frac{n-2}{2}} + |\lambda|^{\frac{n-1}{2}} \log \lambda)\|F\|_{A(\mathbb{R}^n_+)}$$

where $\|F\|_{A(\mathbb{R}^n_+)} = \|f\|_{L^p(\mathbb{R}^n_+)} + \|g\|_{W^{1,p}(\mathbb{R}^n_+)} + \|h\|_{W^{1,p}(\mathbb{R}^n_+)}$.

Moreover we use the follwoing lemma:

Lemma 2.4. *Let* $1 < p < \infty$, $n \geq 3$.
We assume that $u \in W^{2,p}_{loc}$, $p \in W^{1,p}_{loc}$ *and they sastify the following condition:*

$$\begin{cases} -Div\, T(u, p) = 0 & \text{in } \Omega, \\ \nabla \cdot u = 0 & \text{in } \Omega, \\ u \cdot \nu = 0, \ B_{\alpha,\beta}(u, p) = 0 & \text{on } \partial\Omega, \end{cases}$$

$$\sup_{x\in\mathbb{R}_{R+3}} \{|x|^{n-1}|u(x)| + |x|^{n-1}|\nabla u(x)| + |x|^{n-1}|p(x)|\} < \infty.$$

Then $u = p = 0$.

Proof. Let $\psi(x) \in C_0^\infty(\mathbb{R}^n)$ be a cutoff function such that

$$\psi(x) = \begin{cases} 1 & \text{for } |x| \geq R+1 \\ 0 & \text{for } |x| \leq R \end{cases} .$$

We set $\psi_l(x) = \psi(\frac{x}{l}) \in C_0^\infty(\mathbb{R}^n)$.

$$\begin{aligned}
0 = &- (\mathrm{Div}\, T(u,p), \psi_l u)_\Omega = \\
= &- (T(u,p)\nu, \psi_l u)_\Gamma + (T(u,p), \nabla(\psi_l u))_\Omega \\
= &(T(u,p)\nu - (T(u,p)\nu, \nu)\nu, \psi_l u)_\Gamma - ((T(u,p)\nu, \nu)\nu, \psi_l u)_\Gamma \\
&+ (T(u,p), \nabla(\psi_l)u)_\Omega + (T(u,p), \psi_l D(u))_\Omega \\
= &\frac{\alpha}{\beta}(u, \psi_l u)_\Gamma - (T(u,p)\nu, \nu)\,(\nu, \psi_l u)_\Gamma \\
&+ (T(u,p), \nabla(\psi_l)u)_\Omega + (T(u,p), \psi_l D(u))_\Omega
\end{aligned}$$

As $l \longrightarrow \infty$, we can get $\frac{\alpha}{\beta}\|u\|^2_{\partial\Omega} + \|D(u)\|_\Omega = 0$. Therefore we know $\|D(u)\|_\Omega = 0$ which implies $u = 0$ by the boundary condition. By the equation we can get $p = 0$. □

Under these preparations we prove Theorem 2.1

Proof. We consider a zero extension which is given by

$$f^*(x) = \begin{cases} f(x) & \text{for } |x| > R \\ 0 & \text{for } |x| \leq R \end{cases} .$$

Let $R_h(\lambda), \Pi_h(\lambda)$ be a solution operator to a half space problem. And we set $v_h = R_h(\lambda)F^*$, $\theta_h = \Pi_h(\lambda)F^*$ where $F^* = {}^t(f^*, g^*, h^*)$. That is v_h, θ_h sastisfy the following problem:

$$\begin{cases} \lambda v_h - \Delta v_h + \nabla\theta_h = f^* & \text{in } \mathbb{R}^n_+, \\ \nabla \cdot v_h = g^* & \text{in } \mathbb{R}^n_+, \\ \alpha v_{hi} - \beta\partial_n v_{hi} = h_j^* \ (j = 1, \cdots n-1), \ v_{hn} = 0 & \text{on } \partial\mathbb{R}^n_+, \end{cases}$$

Moreover we consider the following problem:

$$\begin{cases} -\Delta w + \nabla\theta = f, \ \ \nabla \cdot w = g & \text{in } E_R \\ w \cdot \nu = 0, \ B_{\alpha,\beta}(w, \theta) = h & \text{on } \partial E_R. \end{cases}$$

Knowing the existence of the solution of this problem, we set $AF = w$, $BF = \theta$. We use a cut off function : $\psi_R^\infty(x) \in C^\infty$ which is given by

$$\psi_R^\infty(x) = \begin{cases} 1 & \text{for } |x| \geq R+1 \\ 0 & \text{for } |x| \leq R \end{cases} \tag{2.2}$$

And we set

$$\begin{cases} U(\lambda)F = \psi^\infty_{R+1} R_h(\lambda)F^* + (1-\psi^\infty_{R+1})AF \\ \Theta(\lambda)F = \psi^\infty_{R+1}\pi_h(\lambda)F^* + (1-\psi^\infty_{R+1})BF. \end{cases}$$

We know $U(\lambda)F,\ \Theta(\lambda)F$ satisfy the following:

$$\begin{cases} (\lambda-\Delta)U(\lambda)F + \nabla\Theta(\lambda)F = f + S^1_\lambda F & \text{in } \Omega, \\ \nabla\cdot U(\lambda)F = g + S^2_\lambda F & \text{in } \Omega, \\ U(\lambda)F\cdot\nu = 0,\ B_{\alpha,\beta}(U(\lambda)F) = h + S^3_\lambda F & \text{on } \partial\Omega. \end{cases}$$

Here we have set $S^1_\lambda F, S^2_\lambda F, S^3_\lambda F$ by the following:

$$\begin{aligned} S^1_\lambda F =& 2\nabla\psi^\infty_{R+1}\cdot\nabla R_h(\lambda)F^* + (\Delta\psi^\infty_{R+1})R_h(\lambda)F^* + \lambda\psi^\infty_{R+1}AF - 2\nabla\psi^\infty_{R+1}\cdot\nabla AF \\ &- (\Delta\psi^\infty_{R+1})AF - 2(\nabla\psi^\infty_{R+1})\pi_h(\lambda)F^* + 2(\nabla\psi^\infty_{R+1})BF \\ S^2_\lambda F =& -\nabla\psi^\infty_{R+1}\cdot R_h(\lambda)F^* + \nabla\cdot AF \\ S^3_\lambda F =& \beta(\nabla\psi^\infty_{R+1}\cdot\nu)(-R_h(\lambda)F^* + AF) \end{aligned}$$

Here we have use

$$\begin{aligned} &\int_\Omega (-\nabla\psi^\infty_{R+1}\cdot R_h(\lambda)F^* + \nabla\psi^\infty_{R+1}\cdot AF)dx \\ &= \int_{D^+_{R+1}} (-\nabla\psi^\infty_{R+1}\cdot R_h(\lambda)F^* + \nabla\psi^\infty_{R+1}\cdot AF)dx \\ &= \int_{D^+_{R+1}} (-\nabla\cdot(\psi^\infty_{R+1}R_h(\lambda)F^*) \\ &\quad + \nabla\cdot(\psi^\infty_{R+1}AF + \psi^\infty_{R+1}\nabla\cdot R_h(\lambda)F^* - \psi^\infty_{R+1}\nabla\cdot AF)dx \\ &= \int_{\partial D^+_{R+1}} (-\psi^\infty_{R+1}R_h(\lambda)F^*\cdot\nu + -\psi^\infty_{R+1}AF\cdot\nu)d\sigma. \end{aligned}$$

Where we have set $D^+_{R+1} = \{x\in\mathbb{R}^n \mid R+1 < |x| < R+2\}$ and $d\sigma$ denotes surface. Since $\mathbb{S}_\lambda = {}^t(S^1_\lambda, S^2_\lambda, S^2_\lambda)$ is a compact operator on $\mathcal{L}(L^p_{R+3}(\Omega)^n \times W^{1,p}_{R+3,0}(\Omega)\times W^{1,p}_{R+3}(\Omega)^n)$, to show $I+\mathbb{S}_\lambda$ has a inverse operator, it is sufficient to show $I+\mathbb{S}_\lambda$ is injective. And so we shall show $(I+\mathbb{S}_\lambda)F = 0 \implies F = 0$. Setting $u = U(0)F, p = \Theta(0)F$, we know that u, p satisfy the problem:

$$\begin{cases} -\Delta u + \nabla p = 0 & \text{in } \Omega, \\ \nabla\cdot u = 0 & \text{in } \Omega, \\ u\cdot\nu = 0,\ B_{\alpha,\beta}(u) = 0 & \text{on } \partial\Omega. \end{cases}$$

By Lemma 2.4 we can get $u, p = 0$ which implies

$$\begin{cases} \psi^\infty_{R+1}R_h(0)F^* + (1-\psi^\infty_{R+1})AF = 0 & \text{in } \Omega \\ \psi^\infty_{R+1}\pi_h(0)F^* + (1-\psi^\infty_{R+1})BF = 0 & \text{in } \Omega. \end{cases}$$

Considering suppψ^{∞}_{R+1}, we know

$$\begin{array}{ll} R_h(0)F^* = \pi_h(0)F^* = 0 & |x| \geq R+2 \\ AF = BF = 0 & |x| \leq R+1 \end{array}$$

Setting

$$w = \begin{cases} AF & |x| \geq R,\ x \in ER \\ 0 & |x| < R \end{cases} \quad \theta = \begin{cases} BF & |x| \geq R,\ x \in ER \\ 0 & |x| < R, \end{cases}$$

we know that w, θ satisfy the following:

$$\begin{cases} -\Delta w + \nabla\theta = f^* & \text{in } \tilde{ER}, \\ \nabla \cdot w = g^* & \text{in } \tilde{ER}, \\ w \cdot \nu = 0,\ B_{\alpha,\beta}(w) = h^* & \text{on } \partial\tilde{ER}, \end{cases}$$

where we have set $\tilde{ER} = \{x \in ER \mid |x| \geq R\} \cup B^+_R$. On the other hand, we know

$$\begin{cases} -\Delta R_h(0)F^* + \nabla\pi_h(0)F^* = f^* & \text{in } \tilde{ER}, \\ \nabla \cdot R_h(0)F^* = g^* & \text{in } \tilde{ER}, \\ R_h(0)F^* \cdot \nu = 0,\ B_{\alpha,\beta}(R_h(0)F^*) = h^* & \text{on } \partial\tilde{ER}. \end{cases}$$

Therefore we get $R_h(0)F^* = w$ in $\tilde{ER}$ by uniqueness. And we know $R_h(0)F^* = w = AF,\ \pi_h(0)F^* = \theta = BF$ in $\tilde{ER}$. For $|x| \geq R+1$,

$$\begin{aligned} 0 &= \psi^{\infty}_{R+1}R_h(0)F^* + (1-\psi^{\infty}_{R+1})AF \\ &= -(1-\psi^{\infty}_{R+1})(R_h(0)F^* - AF) + R_h(0)F^* \\ &= R_h(0)F^* \end{aligned}$$

By similar argment, we get $\Pi(0)F^* = 0$ for $|x| \geq R+1$. Therefore for $|x| \geq R+1$ and $x \in \Omega$

$$\begin{aligned} f &= -\Delta R(0)F^* + \nabla\Pi(0)F^* = 0, \\ g &= \nabla \cdot (R(0)F^*) = 0, \\ h &= B_{\alpha,\beta}(R(0)F^*, \Pi(0)F^*) = 0. \end{aligned}$$

For $|x| \leq R+1$ and $x \in \Omega$ we know

$$\begin{aligned} 0 &= -\Delta AF + \nabla BF = f, \\ 0 &= \nabla \cdot Af = g, \\ 0 &= B_{\alpha,\beta}(AF, BF) = h. \end{aligned}$$

And we know $f = g = h = 0$ for $x \in \Omega$. Therefore we sum up Theorem 2.3, and we get the following lemma:

Lemma 2.5. *There exists $\lambda_0 > 0$ such that the following holds:*

$$(I+\mathbb{S}_\lambda)^{-1} = {}^t(I+S^1_\lambda, I+S^2_\lambda, I+S^2_\lambda)^{-1} \in \mathcal{L}(L^p_{R+3}(\Omega)^n \times W^{1,p}_{R+3,0}(\Omega) \times W^{1,p}_{R+3}(\Omega)^n),$$
$$\|(I+\mathbb{S}_\lambda)^{-1}\|_{\mathcal{L}(A(\Omega))} \le C$$

□

By lemma 2.5, we can write (u,p) as follows:

$$\begin{aligned} u(x) &= U(\lambda)(I+\mathbb{S}_\lambda)^{-1}F, \\ &= \psi^\infty_{R+1}R_h(\lambda)(I+\mathbb{S}_\lambda)^{-1}F + (1-\psi^\infty_{R+1})A(I+\mathbb{S}_\lambda)^{-1}F \\ p(x) &= \Pi(\lambda)(I+\mathbb{S}_\lambda)^{-1}F, \\ &= \psi^\infty_{R+1}\pi_h(\lambda)(I+\mathbb{S}_\lambda)^{-1}F + (1-\psi^\infty_{R+1})B(I+\mathbb{S}_\lambda)^{-1}F. \end{aligned}$$

Summing up this and theorem 2.2, we can get theorem 2.1.

3 $L^p - L^\infty$ estimates in a half space

In this section we treat a half space problem (2.1) continuously. About this problem we show the following theorem.

Theorem 3.1. *Let be v_h a solution of* (2.1). *Then it satisfies the estimates.*

$$\|v_h\|_{L^\infty(\mathbb{R}^n_+)} \le C|\lambda|^{-\frac{1}{2}-\frac{n}{2p}}\|f\|_{L^p(\mathbb{R}^n_+)},$$
$$\|\nabla v_h\|_{L^\infty(\mathbb{R}^n_+)} \le C|\lambda|^{-\frac{1}{2}}\|f\|_{L^p(\mathbb{R}^n_+)},$$

for $\lambda \in \sum_\epsilon, \quad |\lambda| < \lambda_0, \quad p \neq n.$

To show this theorem, we introduce the Gagliardo-Nirenberg-Sobolev theorem.

Theorem 3.2. *The Gagliardo-Nirenberg-Sobolev theorem Let* $1 \le p \le \infty, \quad 1 \le q \le \infty.$
Let j, m be integers such that satisfy $0 \le j < m.$
We assume $m - j - \frac{n}{p} \neq 0, 1, 2, \cdots.$
For $0 \le a \le 1$ *set*

$$\frac{1}{r} = \frac{j}{n} + a\left(\frac{1}{p} - \frac{m}{n}\right) + (1-a)\frac{1}{q}.$$

Then the following estimate holds:

$$\|u\|_{L^r(\mathbb{R}^n_+)} \le C\|\nabla^m u\|^a_{L^p(\mathbb{R}^n_+)}\|u\|^{1-a}_{L^q(\mathbb{R}^n_+)}.$$

And we use the following theorem:

Theorem 3.3. *Let* $1 < p < \infty$.

$$|\lambda| \; \|v_h\|_{L^p(\mathbb{R}^n_+)} + |\lambda|^{\frac{1}{2}} \; \|\nabla v_h\|_{L^p(\mathbb{R}^n_+)} + \|\nabla^2 v_h\|_{L^p(\mathbb{R}^n_+)} + \|\nabla \theta_h\|_{L^p(\mathbb{R}^n_+)}$$
$$\leq C\{\|f\|_{L^p(\mathbb{R}^n_+)} + |\lambda| \|g\|_{W^{-1,p}(\mathbb{R}^n_+)} + |\lambda|^{\frac{1}{2}} \|(g,h)\|_{L^p(\mathbb{R}^n_+)} + \|(\nabla g, \nabla h)\|_{L^p(\mathbb{R}^n_+)}$$

for $\lambda \in \sum_\epsilon = \{\lambda \in \mathbb{C} \setminus \{0\} \mid |\arg \lambda| \leq \pi - \epsilon\}$.

Using this thorem, we show Theorem 3.1.

Proof. First we assume $p \neq n$, and we use the Gagliardo-Nirenberg-Sobolev theorem. When $p = n$, I consider p_1, p_2 such that $p_1 < n < p_2$. And I interpolate $L^{p_1}(\Omega)$ and $L^{p_2}(\Omega)$, I can remove the restriction $p \neq n$. □

4 $L^n - L^q$ decay estimates

In this section we show $L^n - L^q$ decay estimates which is given by the following

Theorem 4.1. *Let* $1 < p < \infty$ *and* $n \geq 3$. *Then there exists a unique solution* u *of* (1.2) *satisfying the estimates:*

$$\|u\|_{L^q(\Omega)} \leq C_q t^{-\frac{n}{2}(\frac{1}{n}-\frac{1}{q})} \|f\|_{L^n(\Omega)}, \quad t > 0, \; f \in J^n(\Omega), \quad q \geq n$$
$$\|\nabla u\|_{L^n(\Omega)} \leq C t^{-\frac{1}{2}} \|f\|_{L^n(\Omega)}, \quad t > 0, \; f \in J^n(\Omega).$$

We show the only following theorem which implies Theorem 4.1.

Theorem 4.2. *Let* $1 < p < \infty$ *and* $n \geq 3$. *Let* u *be the solution of* (1.3). *There exists a positive constant* C *such that* u *satisfies the following:*

$$\|u\|_{L^q(\Omega)} \leq C|\lambda|^{-\frac{1}{2}-\frac{n}{2q}} \|f\|_{L^n(\Omega)},$$
$$\|\nabla u\|_{L^n(\Omega)} \leq C|\lambda|^{-\frac{1}{2}} \|f\|_{L^n(\Omega)},$$

for $\lambda \in \sum_\epsilon = \{\lambda \in \mathbb{C} \setminus \{0\} \mid |\arg \lambda| \leq \pi - \epsilon\}, \quad |\lambda| < \lambda_0$.

Proof. We use a cut off function which is given by (2.2).First I shall consider a bounded part $\Omega_R = \Omega \cap B_R$. We set $u = \psi_R^\infty v_h + w$, $p = \psi_R^\infty \theta_h + \pi$ where v_h, θ_h is the solution of (2.1) with $g = h = 0$. And w, π satisfy the following:

$$\begin{cases} (\lambda - \Delta) w + \nabla \pi = K_1 & \text{in } \Omega, \\ \nabla \cdot w = K_2 & \text{in } \Omega, \\ w \cdot \nu = 0, \; B_{\alpha,\beta}(w) = K_3 & \text{on } \partial\Omega. \end{cases}$$

Here the formula of K_1, K_2, K_3 are given by the following:

$$\begin{aligned} K_1 &= K_1\left(\nabla\psi_R^\infty \cdot v_h, (\Delta\psi_R^\infty)v_h, \nabla\psi_R^\infty\theta_h, (1-\psi_R^\infty)f\right), \\ K_2 &= -\nabla\psi_R^\infty \cdot v_h, \\ K_3 &= K_3\left(\nabla\psi_R^\infty v_h\right). \end{aligned}$$

Since supp $K_1, K_2, K_3 \subset B_R$, we can use Theorem 2.1:

$$\begin{aligned} (w,\pi) = \lambda^{\frac{n-1}{2}} G_1(\lambda)^t(K_1, K_2, K_3) + \lambda^{\frac{n-2}{2}} G_2(\lambda)^t(K_1, K_2, K_3) \\ + (\lambda\log\lambda)G_3(\lambda)^t(K_1, K_2, K_3) + G_4(\lambda)^t(K_1, K_2, K_3), \end{aligned}$$

where G_j $(j = 1, 2, 3, 4)$ are holomorphic functions with respect to λ. Here K_1, K_2, K_3 have the estimates:

$$\|(K_1, K_2, K_3)\|_{L^\infty(\Omega_R)} \le C\|(v_h, \nabla v_h, \theta_h)\|_{L^\infty(\Omega_R)} \le C|\lambda|^{-1+\frac{n}{2p}}\|f\|_{L^p(\Omega)}.$$

Therefore I can get the following:

$$\begin{aligned} \|w\|_{L^\infty(\Omega_R)} &\le C|\lambda|^{\frac{n-2}{2}}\|(K_1, K_2, K_3)\|_{L^\infty(\Omega_R)} \\ &\le C|\lambda|^{\frac{n-2}{2}}\left(|\lambda|^{-1+\frac{n}{2p}}\|f\|_{L^p(\Omega)}\right) \\ &\le C|\lambda|^{-2+\frac{n}{2}+\frac{n}{2p}}\|f\|_{L^p(\Omega)} \end{aligned}$$

for $n \ge 3$ $p \le n$.
Next I shall consider (w, π) in $\Omega \setminus \Omega_R$. Setting $\psi_{R-2}^\infty w = z$, $\psi_{R-2}^\infty \pi = \theta$, we know that z, θ satisfy this problem:

$$\begin{cases} (\lambda - \Delta)z + \nabla\theta = H_1 & \text{in } \mathbb{R}_+^n, \\ \nabla \cdot z = H_2 & \text{in } \mathbb{R}_+^n, \\ \alpha z_i - \beta\partial_n z_i = H_{3j}\ (j = 1, \cdots n-1),\ z_n = 0 & \text{on } \partial\mathbb{R}_+^n, \end{cases}$$

where H_1, H_2, H_3 are given by these:

$$\begin{aligned} H_1 &= H_1\left(\nabla\psi_{R-2}^\infty \cdot w, (\Delta\psi_{R-2}^\infty)w, \nabla\psi_{R-2}^\infty\pi, \psi_{R-2}^\infty K_1\right), \\ H_2 &= -\nabla\psi_{R-2}^\infty \cdot w + \psi_{R-2}^\infty K_2, \\ H_3 &= H_3\left(\nabla\psi_{R-2}^\infty w, \psi_{R-2}^\infty K_3\right). \end{aligned}$$

Therefore we can get the following:

$$\begin{aligned} |\lambda|\ \|z\|_{L^p(\mathbb{R}_+^n)} + |\lambda|^{\frac{1}{2}}\ \|\nabla z\|_{L^p(\mathbb{R}_+^n)} + \|\nabla^2 z\|_{L^p(\mathbb{R}_+^n)} + \|\nabla\theta\|_{L^p(\mathbb{R}_+^n)} \\ \le C\Big\{\|H_1\|_{L^p(\mathbb{R}_+^n)} + |\lambda|\|H_2\|_{W^{-1,p}(\mathbb{R}_+^n)} \\ + |\lambda|^{\frac{1}{2}}\|(H_2, H_3)\|_{L^p(\mathbb{R}_+^n)} + \|(\nabla H_2, \nabla H_3)\|_{L^p(\mathbb{R}_+^n)}\Big\} \end{aligned}$$

for $\lambda \in \sum_\epsilon$.

$$
\begin{aligned}
\|z\|_{L^p(\mathbb{R}^n_+)} &\leq C\|\nabla^2 z\|_{L^q(\mathbb{R}^n_+)}^{\frac{n}{2}(\frac{1}{q}-\frac{1}{p})} \|z\|_{L^q(\mathbb{R}^n_+)}^{1-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})} \\
&\leq \|(H_1, H_2, H_3)\|_{L^q(\mathbb{R}^n_+)}^{\frac{n}{2}(\frac{1}{q}-\frac{1}{p})} \\
&\times \Big\{ |\lambda|^{-1}\|H_1\|_{L^n(\mathbb{R}^n_+)} + \|H_2\|_{W^{-1,q}(\mathbb{R}^n_+)} + |\lambda|^{-\frac{1}{2}}\|(H_2, H_3)\|_{L^q(\mathbb{R}^n_+)} \\
&\qquad + |\lambda|^{-1}\|(\nabla H_2, \nabla H_3)\|_{L^q(\mathbb{R}^n_+)} \Big\}^{1-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})} \\
&\leq C|\lambda|^{1-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}\|(H_1, H_2, H_3)\|_{L^q(\Omega_R)} \\
&\leq C|\lambda|^{-1+\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}\left(|\lambda|^{-2+\frac{n}{2}+\frac{n}{2q}}\|f\|_{L^q(\Omega)}\right) \\
&\leq C|\lambda|^{-3+\frac{n}{2}+\frac{n}{q}-\frac{n}{2p}}\|f\|_{L^q(\Omega)} \\
&\leq C|\lambda|^{-1+\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}\|f\|_{L^q(\Omega)} \qquad \text{for } q \leq p.
\end{aligned}
$$

About ∇u I can use the resolvent estimates.
Therefore I can get the theorem: □

References

[1] R. Farwig and H. Sohr, *Generalized resolvent estimates for the Stokes system in bounded and unbounded domains*, J. Math. Soc. Japan, **46** (1994), 607.643.

[2] T. Kubo and Y. Shibata, *On the Stokes and Navier-Stokes equations in a perturbed half space*, Adv. Differential Equations **10** No. 6 (2005), 695–720.

[3] J. Saal, *Robin Boundary Conditions and Bounded H1-calculus for the Stokes operater*, Doctor thesis, TU Darmstadt, Logos Verlag Berlin, (2003).

[4] T. Kato, *Strong L^p-solutions of the Navier-Stokes equation in $\mathbb{R}^n$ with applications to weak solutions*, Math.Z., **189** (1984) 471–480.

[5] R. Shimada and Y. Shibata, *On a generalized resolvent estimate for the Stokes system with Robin boundary condition* J. Math. Soc. Japan **59** No. 2 (2007) 69–519

[6] Y. Naito, *On the properties of solution to the Stokes equation with Robin boundary condition in a half space*, (preprint).