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On motion of inhomogeneous incompressible fluid-like bodies with Navier's slip conditions*

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Abstract

An initial-boudary value problem for the system of equations governing the flow of inhomogeneous incompressible fluid-like bodies is studied. The boundary conditions assigned here are called the generalized Navier's slip conditions which represent the slip phenomena at the boundary. Rewriting this problem by Lagrangian coordinates, we prove its solvability and convergence results concerning slip-rate *etc.* in anisotropic Sobolev-Slobodetskiĭ spaces.

1 Introduction

In this study we are concerned with motion of inhomogeneous incompressible fluid-like bodies (IIFB). This model arises from the study of incompressible flows of granular materials. Granular materials are some sorts of materials which consist of grains. In certain situations granular matter behaves in fluid-like manner, for example, quicksand, avalanches, and so on. Even it flows, however, the profile of the flow is completely different from that of usual liquids.

Granular materials are substantially compressible due to existence of the interstices between the particles and are inhomogeneous since they are composed of a mixture of several types of particles. However, in some special conditions, the compressibility which influences the motion can be neglected. Here, we restrict the subject of our investigation to the granular bodies satisfying such incompressible conditions.

Málek and Rajagopal [9] derived the constitutive equations for inhomogeneous incompressible fluid-like bodies whose free energy depends on the density and the

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gradient of the density, taking into account the conservation law of energy, the second law of thermomechanics and the concept of maximization of the entropy production. We call the body under consideration is of Korteweg type, since such a material was firstly considered by Korteweg [6]. It is the consequece of the inhomogeneity of the body.

We should remark on the slip phenomena of the granular body at the boundary. Unlike the adhering behaviour of Newtonian fluids at the boundary, non-Newtonian fluids including granular materials may in general slip at the surface of solid wall in contact with the continuum. Moreover, this slip effect may cause the significant consequence for motion. Here, taking into account this slip phenomena, we analyse the motion of inhomogenous incompressible fluid-like bodies.

2 Mathematical Issues and Main Results

2.1 Initial-boundary value problem for IIFB models

In this study we are concerned with the following initial-boundary value problem for the motion of inhomogeneous incompressible fluid-like bodies:

$$\begin{cases} \dfrac{\mathrm{D}\varrho}{\mathrm{D}t} = 0, \quad \nabla \cdot \mathbf{v} = 0 \quad \text{in } Q_T \equiv \Omega \times (0,T), \\ \varrho \dfrac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} = \nabla \cdot \mathbb{T} + \varrho \mathbf{b} \quad \text{in } Q_T, \end{cases} \tag{2.1}$$

$$\mathbb{T} = -p\,\mathbb{I} + 2\nu(\varrho)\mathbb{D}(\mathbf{v}) - \beta\left(\nabla\varrho \otimes \nabla\varrho - \frac{1}{3}|\nabla\varrho|^2\,\mathbb{I}\right) \text{ in } Q_T, \tag{2.2}$$

$$\begin{cases} (\varrho, \mathbf{v})|_{t=0} = (\varrho_0, \mathbf{v}_0) \quad \text{in } \Omega, \\ \mathbf{v} \cdot \mathbf{n} = 0, \quad \mathbf{v} + K\Pi\mathbb{T}\mathbf{n} = \mathbf{0} \quad \text{on } G_T \equiv \Gamma \times (0,T), \end{cases} \tag{2.3}$$

where $\Omega(\subset \mathbb{R}^3)$ is a domain where a material occupies; Γ the boundary of Ω; ϱ the density of the body; $\mathbf{v}$ the velocity vector field; $\frac{\mathrm{D}}{\mathrm{D}t}$ the Lagrangian derivative; $\mathbf{b}$ the external body forces; $\mathbb{T}$ the Cauchy stress represented by the constitutive equations (2.2); p the pressure; $\mathbb{D}(\mathbf{v}) = \frac{1}{2}(\nabla\mathbf{v} + [\nabla\mathbf{v}]^{\mathrm{T}})$ the symmetric part of the velocity gradient; ν the viscosity; β a positive constant; $\mathbf{n}$ the unit outward normal vector on Γ; $K \geq 0$ the slip rate; $\Pi\mathbf{f} \equiv \mathbf{f} - (\mathbf{f}\cdot\mathbf{n})\mathbf{n}$ the projection to the tangential plane.

Here, we assign so-called the generalized Navier's slip boundary condition $(2.3)_3$ with slip rate K. If $K \equiv 0$, the condition immediately becomes the usual adherence condition $\mathbf{v} = 0$. When $K > 0$, the condition is refered to the slip at the boundary. Moreover, if $K \equiv \infty$ (of course, taking the limit after dividing the condition by K), then it becomes $\Pi\mathbb{T}\mathbf{n} = \mathbf{0}$ which represents the perfect-slip condition. Hence, the slip boundary condition $(2.3)_3$ connects the no-slip case to the perfect-slip case.

The condition $(2.3)_3$ is the generalized form of the slip condition which was first derived by Navier [13].

This problem arised from a study of some flows of granular materials. In certain situations granular matter behaves in fluid-like manner, however, the profile of the flow is completely different from that of usual liquids. Rajagopal and Massoudi [15] proposed the constitutive equations of granular materials as complex continua. In their work they paid attention to the quantity $\nabla\varrho \otimes \nabla\varrho$. Thereafter Málek and Rajagopal [9] derived the constitutive equation (2.2) for $\mathbb{T}$. $\nabla\varrho\otimes\nabla\varrho$ however cause some mathematical difficulties. In the conservation law of linear momentum, for example, a non-linear term $\operatorname{div}(\nabla\varrho \otimes \nabla\varrho)$ appears and it is definitely one of the principal terms of the system, which may degenerate. Thus we need to remove such difficulties to investigate the problem.

The initial-boundary value problem (2.1)-(2.3) is represented in the Eulerian coordinates X. Now, we rewrite it in Lagrangian coordinates x. Let $\mathbf{u}(x,t)$ and $q(x,t)$ be the velocity vector field and pressure, respectively, expressed as functions of the Lagrangian coordinates. The relationship between Lagrangian and Eulerian coordinates is given by

$$X = x + \int_0^t \mathbf{u}(x,\tau)\,\mathrm{d}\tau \equiv X_{\mathbf{u}}(x,t), \quad \mathbf{u}(x,t) = \mathbf{v}(X_{\mathbf{u}}(x,t),t).$$

From $(2.1)_1$ it is easy to derive

$$\frac{\partial}{\partial t}\varrho_{\mathbf{u}}(x,t) = 0$$

for $\varrho_{\mathbf{u}}(x,t) := \varrho(X_{\mathbf{u}}(x,t),t)$, thus we have $\varrho_{\mathbf{u}}(x,t) = \varrho_0(x)$. This means the density function of isochoric motion expressed in Lagrangian coordinates does not vary in time. Moreover, we denote the Jacobian matrix of the transformation $X_{\mathbf{u}}$ by $\mathbb{A} = (a_{ij})_{i,j=1,2,3}$ with elements $a_{ij}(x,t) = \delta_{ij} + \int_0^t \frac{\partial u_i}{\partial x_j}(x,\tau)\,\mathrm{d}\tau$ and its adjugate matrix by $\mathcal{A} = (A_{ij})_{i,j=1,2,3} = \det\mathbb{A}\cdot\mathbb{A}^{-1}$. $J_{\mathbf{u}}(x,t) = \det\mathbb{A}(x,t)$ satisfies the equality

$$\begin{aligned}\frac{\partial J_{\mathbf{u}}(x,t)}{\partial t} &= \sum_{i,j=1}^{3}\frac{\partial a_{ij}}{\partial t}A_{ji} = \sum_{i,j=1}^{3}A_{ji}\frac{\partial u_i}{\partial x_j} = \sum_{i,j=1}^{3}A_{ji}\sum_{k=1}^{3}\frac{\partial v_i}{\partial X_k}(X_{\mathbf{u}}(x,t),t)a_{kj}\\ &= J_{\mathbf{u}}(x,t)(\nabla\cdot\mathbf{v})(X_{\mathbf{u}}(x,t),t) = 0\end{aligned}$$

according to $(2.1)_2$. Since $J_{\mathbf{u}}(x,0) = 1$, we have $J_{\mathbf{u}}(x,t) \equiv 1$, namely $\mathbb{A}^{-1} = \mathcal{A}$. Using this $\mathcal{A}$, we have

$$\nabla_X F(X,t) = \mathbb{A}^{-\mathrm{T}}\nabla_x F_{\mathbf{u}}(x,t) = \mathcal{A}^{\mathrm{T}}\nabla_x F_{\mathbf{u}}(x,t) \equiv \nabla_{\mathbf{u}}F_{\mathbf{u}}(x,t)$$

for $F_{\mathbf{u}}(x,t) := F(X_{\mathbf{u}}(x,t),t)$.

Thus the problem (2.1)-(2.3) becomes

$$\begin{cases} \varrho_0 \dfrac{\partial \mathbf{u}}{\partial t} = \nabla_{\mathbf{u}} \cdot \mathbb{T}_{\mathbf{u}} + \varrho_0 \mathbf{b}_{\mathbf{u}}, \quad \nabla_{\mathbf{u}} \cdot \mathbf{u} = 0 \quad \text{in } Q_T, \\ \mathbf{u}|_{t=0} = \mathbf{v}_0 \quad \text{in } \Omega, \\ \mathbf{u} \cdot \mathbf{n}_{\mathbf{u}} = 0, \quad \mathbf{u} + K_{\mathbf{u}} \Pi_{\mathbf{u}} \mathbb{T}_{\mathbf{u}} \mathbf{n}_{\mathbf{u}} = \mathbf{0} \quad \text{on } G_T. \end{cases} \tag{2.4}$$

Here, $\quad \mathbb{T}_{\mathbf{u}} = -q\,\mathbb{I} + 2\nu(\varrho_0)\mathbb{D}_{\mathbf{u}}(\mathbf{u}) - \beta\Big(\nabla_{\mathbf{u}}\varrho_0 \otimes \nabla_{\mathbf{u}}\varrho_0 - \frac{1}{3}|\nabla_{\mathbf{u}}\varrho_0|^2\,\mathbb{I}\Big),$

$$\mathbb{D}_{\mathbf{u}}(\mathbf{w}) = \tfrac{1}{2}(\nabla_{\mathbf{u}}\mathbf{w} + [\nabla_{\mathbf{u}}\mathbf{w}]^{\mathrm{T}}),\ \mathbf{b}_{\mathbf{u}}(x,t) = \mathbf{b}(X_{\mathbf{u}}(x,t),t),\ \mathbf{n}_{\mathbf{u}}(x,t) = \mathbf{n}(X_{\mathbf{u}}(x,t)),$$
$$K_{\mathbf{u}}(x,t) = K(X_{\mathbf{u}}(x,t),t),\ \Pi_{\mathbf{u}}\mathbf{f} = \mathbf{f} - (\mathbf{f}\cdot\mathbf{n}_{\mathbf{u}})\mathbf{n}_{\mathbf{u}},$$
$$\Pi_{\mathbf{u}}\mathbb{T}_{\mathbf{u}}\mathbf{n}_{\mathbf{u}} = 2\nu(\varrho_0)\Pi_{\mathbf{u}}\mathbb{D}_{\mathbf{u}}(\mathbf{u})\mathbf{n}_{\mathbf{u}} - \beta\Pi_{\mathbf{u}}(\nabla_{\mathbf{u}}\varrho_0 \otimes \nabla_{\mathbf{u}}\varrho_0)\mathbf{n}_{\mathbf{u}}.$$

In this study we proved the theorem on the time-local solvability for the quasi-linear problem (2.4) in Sobolev-Slobodetskiĭ spaces.

2.2 Function spaces

In this subsection we introduce the function spaces used in this paper. Let $\mathcal{G}$ be a domain in $\mathbb{R}^n$ $(n = 1, 2, 3, \ldots)$ and γ a non-negative number. By $W_2^\gamma(\mathcal{G})$ we denote the space of functions equipped with the standard norm

$$\|u\|^2_{W_2^\gamma(\mathcal{G})} = \sum_{|\alpha|<\gamma} \|D^\alpha u\|^2_{L_2(\mathcal{G})} + \|u\|^2_{\dot{W}_2^\gamma(\mathcal{G})},$$

where

$$\|u\|^2_{\dot{W}_2^\gamma(\mathcal{G})} = \sum_{|\alpha|=\gamma} \|D^\alpha u\|^2_{L_2(\mathcal{G})} \qquad \text{if } \gamma \text{ is an integer,}$$

$$\|u\|^2_{\dot{W}_2^\gamma(\mathcal{G})} = \sum_{|\alpha|=[\gamma]} \int_{\mathcal{G}}\int_{\mathcal{G}} \frac{|D^\alpha u(x) - D^\alpha u(y)|^2}{|x-y|^{n+2\{\gamma\}}}\,\mathrm{d}x\,\mathrm{d}y \quad \text{if } \gamma \text{ is not an integer.}$$

Here $[\gamma]$ and $\{\gamma\}$ are the integral and the fractional parts of γ, respectively. $\|f\|_{L_p(\mathcal{G})} = (\int_{\mathcal{G}} |f(x)|^p \mathrm{d}x)^{1/p}$ and $\|f\|_{L_\infty(\mathcal{G})} = \operatorname{esssup}_{x\in\mathcal{G}}|f(x)|$ are the norms in $L_p(\mathcal{G})$ for $1 \le p < +\infty$ and $L_\infty(\mathcal{G})$, respectively. $D^\alpha f = \partial^{|\alpha|} f / \partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_n^{\alpha_n}$ is the generalized derivative of the function f in the distribution sense of order $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n$ with $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{Z}_+^n$ being a multi-index.

Similarly, the norm in $W_2^{\gamma/2}(0,T)$ is defined by

$$\|u\|^2_{W_2^{\gamma/2}(0,T)} \sum_{j=0}^{\gamma/2} = \left\|\frac{\mathrm{d}^j u}{\mathrm{d}\,t^j}\right\|^2_{L_2(0,T)} \qquad \text{for integral } \gamma/2,$$

$$\|u\|^2_{W_2^{\gamma/2}(0,T)} = \sum_{j=0}^{[\gamma/2]} \left\|\frac{\mathrm{d}^j u}{\mathrm{d}\,t^j}\right\|^2_{L_2(0,T)}$$
$$+ \int_0^T \mathrm{d}t \int_0^t \left|\frac{\mathrm{d}^{[\gamma/2]}u(t)}{\mathrm{d}\,t^{[\gamma/2]}} - \frac{\mathrm{d}^{[\gamma/2]}u(t-\tau)}{\mathrm{d}\,t^{[\gamma/2]}}\right|^2 \frac{\mathrm{d}\tau}{\tau^{1+2\{\gamma/2\}}} \quad \text{for non-integral } \gamma/2.$$

The anisotropic space $W_2^{\gamma,\gamma/2}(\mathfrak{G}_T)$ on a cylindrical domain $\mathfrak{G}_T = \mathcal{G} \times (0,T)$ is defined by $L_2(0,T; W_2^\gamma(\mathcal{G})) \cap L_2(\mathcal{G}; W_2^{\gamma/2}(0,T))$, whose norm is introduced by the formula

$$\begin{aligned}\|u\|^2_{W_2^{\gamma,\gamma/2}(\mathfrak{G}_T)} &= \int_0^T \|u\|^2_{W_2^\gamma(\mathcal{G})}\,\mathrm{d}t + \int_{\mathcal{G}} \|u\|^2_{W_2^{\gamma/2}(0,T)}\,\mathrm{d}x \\ &\equiv \|u\|^2_{W_2^{\gamma,0}(\mathfrak{G}_T)} + \|u\|^2_{W_2^{0,\gamma/2}(\mathfrak{G}_T)},\end{aligned}$$

where $W_2^{\gamma,0}(\mathfrak{G}_T) = L_2(0,T; W_2^\gamma(\mathcal{G}))$ and $W_2^{0,\gamma/2}(\mathfrak{G}_T) = L_2(\mathcal{G}; W_2^{\gamma/2}(0,T))$. Other equivalent norms in these spaces can be introduced. For any $l \in (0,1)$ and $T \in (0,+\infty)$ we set

$$\|u\|^{(l,l/2)}_{\mathfrak{G}_T} = \left\{\|u\|^2_{W_2^{l,l/2}(\mathfrak{G}_T)} + \frac{1}{T^l}\|u\|^2_{L_2(\mathfrak{G}_T)}\right\}^{1/2},$$

and

$$\begin{aligned}\|u\|^{(2+l,1+l/2)}_{\mathfrak{G}_T} = \Bigg\{&\|u\|^2_{W_2^{2+l,1+l/2}(\mathfrak{G}_T)} + \left(\|u_t\|^{(l,l/2)}_{\mathfrak{G}_T}\right)^2 \\ &+ \sum_{|\alpha|=2}\left(\|D_x^\alpha u\|^{(l,l/2)}_{\mathfrak{G}_T}\right)^2 + \sup_{t\in(0,T)}\|u\|^2_{W_2^{1+l}(\mathcal{G})}\Bigg\}^{1/2},\end{aligned}$$

which are equivalent to the norms in the spaces $W_2^{l,l/2}(\mathfrak{G}_T)$ and $W_2^{2+l,1+l/2}\,(\mathfrak{G}_T)$, respectively. Also let

$$\|u\|^{(0,l/2)}_{\mathfrak{G}_T} = \left\{\|u\|^2_{W_2^{0,l/2}(\mathfrak{G}_T)} + \frac{1}{T^l}\|u\|^2_{L_2(\mathfrak{G}_T)}\right\}^{1/2}.$$

Finally, we denote by $H_h^{\gamma,\gamma/2}(\mathfrak{G}_T)$, $h > 0$ the space of functions $u(x,t)$ with a finite form

$$\|u\|^2_{H_h^{\gamma,\gamma/2}(\mathfrak{G}_T)} = \|u\|^2_{H_h^{\gamma,0}(\mathfrak{G}_T)} + \|u\|^2_{H_h^{0,\gamma/2}(\mathfrak{G}_T)},$$

$$\|u\|^2_{H_h^{\gamma,0}(\mathfrak{G}_T)} = \int_0^T e^{-2ht}\|u\|^2_{\dot{W}_2^\gamma(\mathcal{G})}\,\mathrm{d}t,$$

$$\begin{aligned}\|u\|^2_{H_h^{0,\gamma/2}(\mathfrak{G}_T)} &= h^\gamma \int_0^T e^{-2ht}\|u\|^2_{L_2(\mathcal{G})}\,\mathrm{d}t \\ &+ \int_0^T e^{-2ht}\mathrm{d}t \int_0^\infty \left\|\frac{\partial^{[\gamma/2]}u_0(\cdot,t)}{\partial\, t^{[\gamma/2]}} - \frac{\partial^{[\gamma/2]}u_0(\cdot,t-\tau)}{\partial\, t^{[\gamma/2]}}\right\|^2_{L_2(\mathcal{G})}\frac{\mathrm{d}\tau}{\tau^{1+2\{\gamma/2\}}},\end{aligned}$$

if $\gamma/2$ is not a integer. Here, $u_0(x,t) = u(x,t)$ for $t > 0$, $u_0(x,t) = 0$ for $t < 0$.

Remark 2.1 For $T < \infty$, the space $H_h^{\gamma,\gamma/2}(\mathfrak{G}_T)$ can be identified with the subspace of $W_2^{\gamma,\gamma/2}(\mathfrak{G}_T)$ consisting of functions $u(x,t)$ that can be extended by zero into the domain $\{t < 0\}$ without loss of regularity. In the case $\gamma > 1$ this implies that

$$\left.\frac{\partial^i u}{\partial t^i}\right|_{t=0} = 0, \quad i = 0, 1, \ldots, \left[\frac{\gamma - 1}{2}\right].$$

If $\mathcal{G}$ is a smooth manifold (in this paper the boundary of a domain in $\mathbb{R}^3$ may play this role), then the norm in $W_2^\gamma(\mathcal{G})$ is defined by means of local charts, each of which is mapped into a domain of Euclidean space where the norms of W_2^γ are defined by formula above. After this the spaces $W_2^{\gamma,\gamma/2}(\mathfrak{G}_T)$ are introduced as indicated above.

The same symbols $W_2^\gamma(\mathcal{G})$, $W_2^{\gamma,\gamma/2}(\mathfrak{G}_T)$ are used for the spaces of vector fields. Their norms are introduced in standard form; for example, for $\mathbf{f} = (f_1, f_2, \ldots, f_n)$

$$\|\mathbf{f}\|^2_{W_2^\gamma(\mathcal{G})} = \sum_{i=1}^n \|f_i\|^2_{W_2^\gamma(\mathcal{G})}.$$

2.3 Main results

Let us now describe the results obtained in this study.

Theorem 2.1 *Let Ω be a bounded domain in $\mathbb{R}^3$, $\Gamma \in W_2^{7/2+l}$, $l \in (1/2, 1)$, $\mathbf{v}_0 \in W_2^{1+l}(\Omega)$, $\varrho_0 \in W_2^{2+l}(\Omega)$, $\varrho_0(x) \geq R_0 > 0$, $\nu \in C^2(\overline{\mathbb{R}}_+)$, $\nu > 0$, $0 < T < +\infty$, $\mathbf{b} \in W_2^{l,l/2}(Q_T)$. Assume that $\mathbf{b}(x,t)$ has continuous derivatives with respect to x and $\mathbf{b}$, $\mathbf{b}_{x_k}$ satisfy the Lipschitz condition in x and the Hölder condition with exponent $1/2$ in t, that $K(X,t)$ has continuous derivatives up to order 2 with respect to X and $D_X^\alpha K$ ($|\alpha| \leq 2$) satisfy the Hölder condition with exponent $1/2$ in x and $1/4$ in t, and suppose either condition for K such as*

$$\begin{cases} (i) & K(X,t) \equiv k \geq 0: \ \textit{constant}, \\ & \textit{or} \\ (ii) & \inf K(X,t) > 0. \end{cases} \tag{2.5}$$

In addition, asuume the following compatibility conditions

$$\nabla \cdot \mathbf{v}_0 = 0 \ \textit{in } \Omega, \quad \mathbf{v}_0 \cdot \mathbf{n} = 0 \ \textit{on } \Gamma,$$

$$\mathbf{v}_0 + K(\cdot, 0)\Pi\left\{2\nu(\varrho_0)\mathbb{D}(\mathbf{v}_0)\mathbf{n} - \beta(\nabla\varrho_0 \otimes \nabla\varrho_0)\mathbf{n}\right\} = \mathbf{0} \ \textit{on } \Gamma.$$

Then problem (2.4) has a unique solution $(\mathbf{u}, \nabla q) \in W_2^{2+l,1+l/2}(Q_{T'}) \times W_2^{l,l/2}(Q_{T'})$ on some interval $(0, T')$ $(0 < T' \leq T)$, whose magnitude T' depends on the data. Moreover, when $K(X,t) \equiv k$ constant, T' can be taken uniformly in k.

Investigating the proof in detail again, we can prove that the dependence of the solution on the slip-rate. We point out the following theorem.

Theorem 2.2 *Let Ω, Γ, l, ϱ_0, $\mathbf{v}_0$, ν, β, T, T', $\mathbf{b}$ be the same as in Theorem 2.1, and assume that $K(x,t) \equiv k \geq 0$: constant. We denote the solution of problem (2.4) with $K(x,t) \equiv k$ by $(\mathbf{u}^{(k)}, \nabla q^{(k)})$. Then the sequence of the solutions of Navier's slip problem $\{(\mathbf{u}^{(k)}, \nabla q^{(k)})\}_{k>0}$ converges to the solution of the adherence problem $(\mathbf{u}^{(0)}, \nabla q^{(0)})$ as $k \to 0$.*

According to this result, not only the system of slip problems converges to that of the no-slip problem formally, but also the solutions of slip problems also converge to that of the no-slip problem (in strong topology). Thus the generalized Navier's slip conditions are regular and meaningful boundary conditions. We also remark that the time-local existence of $(\mathbf{u}^{(k)}, \nabla q^{(k)})$ is already obtained by Nakano and Tani [11, 12] for each k. But we need to prove the uniform estimates in k, therefore we shall show the proof of convergence result in this paper. Theorem 2.2 is proved in §4.

The bodies under consideration in this study are so-called fluid-like bodies. If $\beta = 0$ in the Cauchy stress $\mathbb{T}$, the governing equation becomes completely same as that of incompressible Navier-Stokes fluids. The terms related to β are originally derived from the Helmholtz free energy of the body. In the case $\beta = 0$, the free energy of the body under consideration doe not depend on $\nabla\varrho$. Thus β represents the magnitude of the influence of material inhomogeneity on the motion. We can assure the relation between fluid-like bodies and Navier-Stokes fluids by the following theorem.

Theorem 2.3 *Let Ω, Γ, l, ϱ_0, $\mathbf{v}_0$, ν, β, T, T', $\mathbf{b}$, K be the same as in Theorem 2.1. We denote the solution of problem (2.4) with β by $(\mathbf{u}_{(\beta)}, \nabla q_{(\beta)})$.*

Then the sequence of the solutions $\{(\mathbf{u}_{(\beta)}, \nabla q_{(\beta)})\}_{\beta>0}$ converges to the solution of the Navier-Stokes equation ($\beta = 0$ in problem (2.4)) $(\mathbf{U}, \nabla Q)$ as $\beta \to 0$.

The time-local solvability of the Navier-Stokes equation with Navier's slip condition is already obtained by Tani *et al.* [22]. Theorem 2.3 can be proved easily if one precisely investigate the proof of the existence of the solution of (2.4) [11, 12], thus we omit the proof in this paper.

3 Linearized problem

3.1 Key lemmata

In this section we consider the linearized problems of (2.4) such as

$$\begin{cases} \varrho_0 \dfrac{\partial \mathbf{u}^{(k)}}{\partial t} = -\nabla q^{(k)} + \nu_1(x)\Delta \mathbf{u}^{(k)} + \varrho_0 \mathbf{f}, \quad \nabla \cdot \mathbf{u}^{(k)} = g \quad \text{in } Q_T, \\ \mathbf{u}^{(k)}|_{t=0} = \mathbf{v}_0 \quad \text{in } \Omega, \\ \mathbf{u}^{(k)} + 2\nu_1(x) k \Pi \mathbb{D}(\mathbf{u}^{(k)})\mathbf{n} = \mathbf{b} + k\mathbf{d} \quad \text{on } G_T. \end{cases} \tag{3.1}$$

where $\nu_1(x)$ is a given positive function defined in Ω, $\mathbf{f}$ and g given functions defined in Q_T, and $\mathbf{b}$ and $\mathbf{d}$ given functions defined on G_T satisfying $\mathbf{d}\cdot\mathbf{n}=0$. We shall show the convergence result for the solutions of the problems, thus we should express the dependence of the solution on slip constant k, namely $(\mathbf{u}^{(k)}, q^{(k)})$. For this problem we have the following key lemmata.

Lemma 3.1 Let Ω be a bounded domain in $\mathbb{R}^3$ with a boundary $\Gamma \in W_2^{5/2+l}$, $l \in (1/2,1)$, $0<T<+\infty$, $\mathbf{v}_0 \equiv \mathbf{0}$, $\varrho_0 \in W_2^{2+l}(\Omega)$, $\varrho_0(x) \geq R_0 > 0$, $\nu_1 \in W_2^{2+l}(\Omega)$, $\inf \nu_1(x) > 0$ and k is a non-negative constant. For arbitrary $\mathbf{f} \in H_h^{l,l/2}(Q_T)$, $g \in H_h^{1+l,1/2+l/2}(Q_T)$, $g = \nabla\cdot\mathbf{G}$, $\mathbf{G} \in H_h^{2+l,1+l/2}(Q_T)$, $\mathbf{b} \in H_h^{3/2+l,3/4+l}(G_T)$, $\mathbf{b} = \mathbf{G}|_\Gamma$, $\mathbf{d} \in H_h^{1/2+l,1/4+l}$ (G_T), and $\mathbf{d}\cdot\mathbf{n}=0$, problem (3.1) has a unique solution $\mathbf{u}^{(k)} \in H_h^{2+l,1+l/2}(Q_T)$, $\nabla q^{(k)} \in H_h^{l,l/2}(Q_T)$, provided h is sufficiently large. And this solution satisfies the following estimate:

$$\|\mathbf{u}^{(k)}\|_{H_h^{2+l,1+l/2}(Q_T)} + \|\nabla q^{(k)}\|_{H_h^{l,l/2}(Q_T)} \leq c\Big(\|\mathbf{f}\|_{H_h^{l,l/2}(Q_T)} + \|g\|_{H_h^{1+l,1/2+l/2}(Q_T)}$$
$$+\|\mathbf{G}\|_{H_h^{0,1+l/2}(Q_T)} + \|\mathbf{b}\|_{H_h^{3/2+l,3/4+l/2}(G_T)} + \|\mathbf{d}\|_{H_h^{1/2+l,1/4+l/2}(G_T)}\Big), \tag{3.2}$$

where c is independent of k. Moreover, it also holds that

$$(\mathbf{u}^{(k)}, \nabla q^{(k)}) \longrightarrow (\mathbf{u}^{(0)}, \nabla q^{(0)}) \quad \text{as } k\downarrow 0 \quad \text{in } H_h^{2+l,1+l/2}(Q_T) \times H_h^{l,l/2}(Q_T). \tag{3.3}$$

For a non-zero initial data $\mathbf{v}_0$ we obtain the similar result to Lemma 3.1.

Lemma 3.2 Let Ω, Γ, T, l, ϱ_0, R_0, ν_1 and k be the same as in Lemma 3.1. For arbitrary $\mathbf{v}_0 \in W_2^{1+l}(\Omega)$, $\mathbf{f} \in W_2^{l,l/2}(Q_T)$, $g \in W_2^{1+l,1/2+l/2}(Q_T)$, $g = \nabla\cdot\mathbf{G}$, $\mathbf{G} \in W_2^{2+l,1+l/2}(Q_T)$, $\mathbf{d} \in W_2^{3/2+l,3/4+l/2}$ and $\mathbf{d} \in W_2^{1/2+l,1/4+l/2}$ (G_T) satisfying the compatibility conditions

$$\nabla\cdot\mathbf{v}_0 = \nabla\cdot\mathbf{G}(\,\cdot\,,0)\text{in } \Omega, \quad \mathbf{b}=\mathbf{G}|_\Gamma, \quad \mathbf{d}\cdot\mathbf{n}=0,$$
$$\mathbf{v}_0 + k\Pi\mathbb{D}(\mathbf{v}_0)\mathbf{n} = \mathbf{b}(\cdot,0) + k\mathbf{d}(\cdot,0) \text{ on } \Gamma,$$

problem (3.1) has a unique solution $(\mathbf{u},\nabla q)$ in $W_2^{2+l,1+l/2}(Q_T) \times W_2^{l,l/2}(Q_T)$ and

$$\|\mathbf{u}\|_{Q_T}^{(2+l,1+l/2)} + \|\nabla q\|_{Q_T}^{(l,l/2)} \leq c(T)\Big(\|\mathbf{f}\|_{Q_T}^{(l,l/2)} + \|g\|_{W_2^{1+l,1/2+l/2}(Q_T)} + \|\mathbf{v}_0\|_{W_2^{1+l}(\Omega)}$$
$$+\|\mathbf{G}\|_{Q_T}^{(0,1+l/2)} + \|\mathbf{b}\|_{W_2^{3/2+l,3/4+l/2}(G_T)} + \|\mathbf{d}\|_{W_2^{1/2+l,1/4+l/2}(G_T)}\Big), \tag{3.4}$$

where $c(T)$ is a non-decreasing function of T independent of k. Moreover, it also holds that

$$(\mathbf{u}^{(k)}, \nabla q^{(k)}) \longrightarrow (\mathbf{u}^{(0)}, \nabla q^{(0)}) \quad \text{as } k\downarrow 0 \quad \text{in } W_2^{2+l,1+l/2}(Q_T) \times W_2^{l,l/2}(Q_T). \tag{3.5}$$

3.2 Half space problem for homogeneous systems

In order to prove Lemma 3.1 we first consider the half space problem with constant coefficients.

$$\begin{cases} \dfrac{\partial \mathbf{u}^{(k)}}{\partial t} - \nu_0 \Delta \mathbf{u}^{(k)} + \nabla q^{(k)} = \mathbf{0}, \quad \nabla \cdot \mathbf{u}^{(k)} = 0 \quad \text{in } D_{+T} \equiv \mathbb{R}^3_+ \times (0,T), \\ \mathbf{u}^{(k)}|_{t=0} = \mathbf{0} \quad \text{in } \mathbb{R}^3_+, \quad u_3^{(k)}|_{x_3=0} = 0 \quad \text{on } D_T \equiv \mathbb{R}^2 \times (0,T), \\ u_j^{(k)} - \nu_0 k \left(\dfrac{\partial u_j^{(k)}}{\partial x_3} + \dfrac{\partial u_3^{(k)}}{\partial x_j} \right)\Bigg|_{x_3=0} = b_j - k d_j \quad \text{on } D_T \ (j=1,2), \end{cases} \tag{3.6}$$

where ν_0 is a positive constant, k non-negative constant, $b_j \in H_h^{3/2+l,3/4+l/2}(D_T)$ and $d_j \in H_h^{1/2+l,1/4+l/2}(D_T)$ $(j=1,2)$ with $l \in (1/2,1)$.

Before considering problem (3.6), we extend b_j and d_j from D_T to D_∞ such that $b_j \in H_h^{3/2+l,3/4+l/2}(D_\infty)$ and $d_j \in H_h^{1/2+l,1/4+l/2}(D_\infty)$ (denoted by the same symbol) and

$$\|b_j\|_{H_h^{3/2+l,3/4+l/2}(D_\infty)} \le c \|b_j\|_{H_h^{3/2+l,3/4+l/2}(D_T)}, \tag{3.7}$$

$$\|d_j\|_{H_h^{1/2+l,1/4+l/2}(D_\infty)} \le c \|d_j\|_{H_h^{1/2+l,1/4+l/2}(D_T)}, \tag{3.8}$$

where c is independent of h and T (see [19], §2).

Next, we extend $\mathbf{u}^{(k)} = (u_1^{(k)}, u_2^{(k)}, u_3^{(k)})$, $q^{(k)}$, $\mathbf{b}' = (b_1, b_2)$ and $\mathbf{d}' = (d_1, d_2)$ to the half-space $t < 0$ by 0 and make the Fourier transformation with respect to $x' = (x_1, x_2)$ and the Laplace transformation with respect to t:

$$\hat{f}(\xi', x_3, s) = \int_0^\infty e^{-st}\, \mathrm{d}t \int_{\mathbb{R}^2} e^{-ix' \cdot \xi'} f(x', x_3, t)\, \mathrm{d}x'.$$

Then we have the following system of ordinary differential equations:

$$\begin{cases} \nu_0 \left(r^2 - \dfrac{\mathrm{d}^2}{\mathrm{d}x_3^2} \right) \hat{u}_j^{(k)} + i\xi_j \hat{q}^{(k)} = 0 \quad (j=1,2), \\ \nu_0 \left(r^2 - \dfrac{\mathrm{d}^2}{\mathrm{d}x_3^2} \right) \hat{u}_3^{(k)} + \dfrac{\mathrm{d}\hat{q}^{(k)}}{\mathrm{d}x_3} = 0, \quad i\xi_1 \hat{u}_1^{(k)} + i\xi_2 \hat{u}_2^{(k)} + \dfrac{\mathrm{d}\hat{u}_3^{(k)}}{\mathrm{d}x_3} = 0, \\ \hat{u}_3^{(k)}|_{x_3=0} = 0, \quad \hat{u}_j^{(k)} - \nu_0 k \left(\dfrac{\mathrm{d}\hat{u}_j^{(k)}}{\mathrm{d}x_3} + i\xi_j \hat{u}_3^{(k)} \right)\Bigg|_{x_3=0} = \hat{b}_j - k\hat{d}_j, \\ (\hat{\mathbf{u}}^{(k)}, \hat{q}^{(k)}) \longrightarrow (\mathbf{0}, 0) \ (x_3 \to +\infty), \end{cases} \tag{3.9}$$

where

$$r^2 = \frac{s}{\nu_0} + |\xi'|^2, \quad |\xi'|^2 = \xi_1^2 + \xi_2^2, \quad \arg r \in \left(-\frac{\pi}{4}, \frac{\pi}{4} \right).$$

This problem is easily solved by the same way as in [11, 19], whose solution is given explicitly by

$$\left\{\begin{aligned} \hat{u}_j^{(k)} &= \frac{\hat{b}_j - k\hat{d}_j}{1+\nu_0 kr} e_0(x_3) + \frac{i\xi_j \nu_0 k \sum_{m=1}^2 i\xi_m(\hat{b}_m - k\hat{d}_m)}{|\xi'|(1+\nu_0 kr)\{\nu_0 k(r+|\xi'|)+1\}} e_0(x_3) \\ &\quad + \frac{-i\xi_j \sum_{m=1}^2 i\xi_m(\hat{b}_m - k\hat{d}_m)}{|\xi'|\{\nu_0 k(r+|\xi'|)+1\}} e_1(x_3) \quad (j=1,2), \\ \hat{u}_3^{(k)} &= \frac{\sum_{m=1}^2 i\xi_m(\hat{b}_m - k\hat{d}_m)}{\nu_0 k(r+|\xi'|)+1} e_1(x_3), \\ \hat{q}^{(k)} &= \frac{-\nu_0(r+|\xi'|)\sum_{m=1}^2 i\xi_m(\hat{b}_m - k\hat{d}_m)}{|\xi'|\{\nu_0 k(r+|\xi'|)+1\}} e_2(x_3), \end{aligned}\right. \tag{3.10}$$

where

$$e_0(x_3) = e^{-rx_3}, \qquad e_1(x_3) = \frac{e^{-rx_3} - e^{-|\xi'|x_3}}{r - |\xi'|}, \qquad e_2(x_3) = e^{-|\xi'|x_3}.$$

In estimating this solution, it is convinient to introduce the new norms

$$\|f\|^2_{\gamma,h,D_\infty} \equiv \int_{\mathbb{R}^2} \mathrm{d}\xi' \int_{-\infty}^{+\infty} |\hat{f}(\xi', h+i\xi_0)|^2 |r|^{2\gamma}\, \mathrm{d}\xi_0$$

and

$$\begin{aligned} \|f\|^2_{\gamma,h,D_{+\infty}} \equiv &\sum_{j<\gamma} \int_{\mathbb{R}^2} \mathrm{d}\xi' \int_{-\infty}^{+\infty} \left\| \left(\frac{\mathrm{d}}{\mathrm{d}x_3}\right)^j \hat{f}(\xi', \cdot, h+i\xi_0) \right\|^2_{L_2(\mathbb{R}_+)} |r|^{2(\gamma-j)}\, \mathrm{d}\xi_0 \\ &+ \int_{\mathbb{R}^2} \mathrm{d}\xi' \int_{-\infty}^{+\infty} \left\| \hat{f}(\xi', \cdot, h+i\xi_0) \right\|^2_{\dot{W}_2^\gamma(\mathbb{R}_+)} \mathrm{d}\xi_0 \end{aligned}$$

for $\gamma \geq 0$, which are equivalent to the norms in $H_h^{\gamma,\gamma/2}(D_\infty)$ and $H_h^{\gamma,\gamma/2}(D_{+\infty})$, respectively (see [19]). Moreover, for the functions $e_j(x_3)$, $j = 0,1,2$, we have

Lemma 3.3 ([19]) Let $s = h + i\xi_0$, $h > 0$, j be a non-negative integer and $\alpha \in (0,1)$. Then there exists a positve constant c independent of r and $|\xi'|$ such that

$$(i) \quad \int_0^{+\infty} \left| \left(\frac{\mathrm{d}}{\mathrm{d}x_3}\right)^j e_0(x_3) \right|^2 \mathrm{d}x_3 \leq c|r|^{2j-1},$$

$$(ii) \quad \int_0^{+\infty}\int_0^{+\infty} \left| \left(\frac{\mathrm{d}}{\mathrm{d}x_3}\right)^j e_0(x_3+z) - \left(\frac{\mathrm{d}}{\mathrm{d}x_3}\right)^j e_0(x_3) \right|^2 \frac{\mathrm{d}x_3\,\mathrm{d}z}{z^{1+2\alpha}} \leq c|r|^{2(j+\alpha)-1},$$

$$(iii) \quad \int_0^{+\infty} \left| \left(\frac{\mathrm{d}}{\mathrm{d}x_3}\right)^j e_1(x_3) \right|^2 \mathrm{d}x_3 \leq c\frac{|r|^{2j-1} + |\xi'|^{2j-1}}{|r|^2},$$

$$(iv)\quad \int_0^{+\infty}\int_0^{+\infty}\left|\left(\frac{\mathrm{d}}{\mathrm{d}x_3}\right)^j e_1(x_3+z)-\left(\frac{\mathrm{d}}{\mathrm{d}x_3}\right)^j e_1(x_3)\right|^2\frac{\mathrm{d}x_3\,\mathrm{d}z}{z^{1+2\alpha}}$$
$$\le c\frac{|r|^{2(j+\alpha)-1}+|\xi'|^{2(j+\alpha)-1}}{|r|^2}$$

for all $\xi'\in\mathbb{R}^2$.

The formula (3.10) and Lemma 3.3 yield that for $h>0$ the solution $(\mathbf{u}^{(k)},q^{(k)})$ of the problem (3.6) with $T=\infty$ satisfies the estimate

$$\|\mathbf{u}^{(k)}\|^2_{2+l,h,D_{+\infty}}+\|\nabla q^{(k)}\|^2_{l,h,D_{+\infty}}\le c\left(\|\mathbf{b}'\|^2_{3/2+l,h,D_\infty}+\sum_{j=1}^2\left(\langle\langle d_j\rangle\rangle^{(k)}_{1/2+l,h,D_\infty}\right)^2\right), \tag{3.11}$$

where c is a constant independent of h and k, and

$$\langle\langle f\rangle\rangle^{(k)}_{\gamma,h,D_\infty}\equiv\left(\int_{\mathbb{R}^2}\mathrm{d}\xi'\int_{-\infty}^{+\infty}\left|\frac{\nu_0kr}{1+\nu_0kr}\right|^2|\hat f(\xi',h+i\xi_0)|^2|r|^{2\gamma}\mathrm{d}\xi_0\right)^{1/2}.$$

If $f\in H_h^{\gamma,\gamma/2}(D_T)$, it holds

$$\left|\frac{\nu_0kr}{1+\nu_0kr}\right|^2|\hat f(\xi',h+i\xi_0)|^2|r|^{2\gamma}\le|\hat f(\xi',h+i\xi_0)|^2|r|^{2\gamma}\in L_1(\mathbb{R}^2_{\xi'}\times\mathbb{R}_{\xi_0}).$$

According to Lebesgue's dominant convergence theorem, for any $k_0\ge 0$ it holds

$$\lim_{k\to k_0}\langle\langle f\rangle\rangle^{(k)}_{\gamma,h,D_\infty}=\langle\langle f\rangle\rangle^{(k_0)}_{\gamma,h,D_\infty}. \tag{3.12}$$

Moreover, $\langle\langle f\rangle\rangle^{(k)}_{\gamma,h,D_\infty}$ is monotonically increasing in k, namely for $k\ge 0$

$$0=\langle\langle f\rangle\rangle^{(0)}_{\gamma,h,D_\infty}\le\langle\langle f\rangle\rangle^{(k)}_{\gamma,h,D_\infty}\le\langle\langle f\rangle\rangle^{(\infty)}_{\gamma,h,D_\infty}=\|f\|_{\gamma,h,D_\infty}. \tag{3.13}$$

From (3.11) and (3.13) we obtain the uniform estimate in k as follows:

$$\|\mathbf{u}^{(k)}\|^2_{2+l,h,D_{+\infty}}+\|\nabla q^{(k)}\|^2_{l,h,D_{+\infty}}\le\begin{cases}c\left(\|\mathbf{b}'\|^2_{3/2+l,h,D_\infty}+\|\mathbf{d}'\|^2_{1/2+l,h,D_\infty}\right) & \text{if } k>0,\\ c\,\|\mathbf{b}'\|^2_{3/2+l,h,D_\infty} & \text{if } k=0.\end{cases} \tag{3.14}$$

Consequently, taking into account (3.7), (3.8), (3.14) and the equivalence of the norms and restricting the domain of $\mathbf{u}$ and q, we have

Lemma 3.4 Let $h > 0$ and $l \in (1/2, 1)$. Then the solution $(\mathbf{u}, q)$ of the problem (3.6) satisfies the estimate

$$\begin{aligned}
&\|\mathbf{u}^{(k)}\|_{H_h^{2+l,1+l/2}(D_{+T})} + \|\nabla q^{(k)}\|_{H_h^{l,l/2}(D_{+T})} \\
&\leq c\left(\|\mathbf{b}'\|_{H_h^{3/2+l,3/4+l/2}(D_T)} + \langle\langle \bar{\mathbf{d}}' \rangle\rangle^{(k)}_{1/2+l,h,D_\infty}\right) \qquad (3.15)\\
&\leq \begin{cases} c\left(\|\mathbf{b}'\|_{H_h^{3/2+l,3/4+l/2}(D_T)} + \|\mathbf{d}'\|_{H_h^{1/2+l,1/4+l/2}(D_T)}\right) & \text{if } k > 0, \\ c\,\|\mathbf{b}'\|_{H_h^{3/2+l,3/4+l/2}(D_T)} & \text{if } k = 0, \end{cases} \qquad (3.16)
\end{aligned}$$

where c is a constant independent of h and k, and $\bar{\mathbf{d}}'$ is the expansion of $\mathbf{d}'$ into D_∞.

Moreover, we can prove the convergence theorem for problem (3.6). Let $\mathbf{U}^{(k)} = \mathbf{u}^{(k)} - \mathbf{u}^{(0)}$ and $Q^{(k)} = q^{(k)} - q^{(0)}$. Then $(\mathbf{U}^{(k)}, Q^{(k)})$ $(k > 0)$ satisfies the following relation:

$$\begin{cases} \dfrac{\partial \mathbf{U}^{(k)}}{\partial t} - \nu_0 \Delta \mathbf{U}^{(k)} + \nabla Q^{(k)} = \mathbf{0}, \quad \nabla \cdot \mathbf{U}^{(k)} = 0 \quad \text{in } D_{+T}, \\ \mathbf{U}^{(k)}|_{t=0} = \mathbf{0} \quad \text{in } \mathbb{R}^3_+, \quad U_3^{(k)}|_{x_3=0} = 0 \quad \text{on } D_T, \\ U_j^{(k)} - \nu_0 k \left(\dfrac{\partial U_j^{(k)}}{\partial x_3} + \dfrac{\partial U_3^{(k)}}{\partial x_j} \right)\Bigg|_{x_3=0} = -k d_j^* \quad \text{on } D_T \ (j = 1, 2), \end{cases} \qquad (3.17)$$

where

$$d_j^* = d_j - \nu_0 \left(\frac{\partial u_j^{(0)}}{\partial x_3} + \frac{\partial u_3^{(0)}}{\partial x_j} \right)\Bigg|_{x_3=0}.$$

We should remark that $d_j^* \in H_h^{1/2+l,1/4+l/2}(D_T)$ since $\mathbf{u}^{(0)} \in H_h^{2+l,1+l/2}(D_{+T})$, and d_j^* is also independent of k. We extend d_j^* from D_T to D_∞ again.

Applying (3.11) to (3.17), we obtain

$$\|\mathbf{U}^{(k)}\|^2_{2+l,h,D_{+\infty}} + \|\nabla Q^{(k)}\|^2_{l,h,D_{+\infty}} \leq c \sum_{j=1}^{2} \left(\langle\langle d_j^* \rangle\rangle^{(k)}_{1/2+l,h,D_\infty} \right)^2,$$

where c is a constant independent of h and k. Using (3.12) and (3.13), we therefore obtain

$$\begin{aligned}
&\lim_{k\downarrow 0} \left(\|\mathbf{U}^{(k)}\|^2_{H_h^{2+l,1+l/2}(D_{+T})} + \|\nabla Q^{(k)}\|^2_{H_h^{l,l/2}(D_{+T})} \right) \\
&\leq c \lim_{k\downarrow 0} \left(\|\mathbf{U}^{(k)}\|^2_{2+l,h,D_{+\infty}} + \|\nabla Q^{(k)}\|^2_{l,h,D_{+\infty}} \right) \leq c \sum_{j=1}^{2} \left(\lim_{k\downarrow 0} \langle\langle d_j^* \rangle\rangle^{(k)}_{1/2+l,h,D_\infty} \right)^2 \\
&= 0.
\end{aligned}$$

Thus we have the following result.

Lemma 3.5 Let $h > 0$ and $l \in (1/2, 1)$. Solutions of problem (3.6) $(\mathbf{u}^{(k)}, q^{(k)})$ $(k \geq 0)$ hold

$$(\mathbf{u}^{(k)}, \nabla q^{(k)}) \longrightarrow (\mathbf{u}^{(0)}, \nabla q^{(0)}) \quad \text{as } k \downarrow 0 \quad \text{in } H_h^{2+l,1+l/2}(D_{+T}) \times H_h^{l,l/2}(D_T).$$

3.3 Inhomogeneous systems in the half and whole space

Next we consider the non-homogeneous problem in the half space with constant coefficients.

$$\begin{cases} \dfrac{\partial \mathbf{u}^{(k)}}{\partial t} - \nu_0 \Delta \mathbf{u}^{(k)} + \nabla q^{(k)} = \mathbf{f}, \quad \nabla \cdot \mathbf{u}^{(k)} = g \quad \text{in } D_{+T}, \\ \mathbf{u}^{(k)}|_{t=0} = \mathbf{0} \quad \text{in } \mathbb{R}^3_+, \quad u_3^{(k)}|_{x_3=0} = b_3 \quad \text{on } D_T, \\ u_j^{(k)} - \nu_0 k \left(\dfrac{\partial u_j^{(k)}}{\partial x_3} + \dfrac{\partial u_3^{(k)}}{\partial x_j} \right)\Bigg|_{x_3=0} = b_j - k d_j \quad \text{on } D_T \ (j = 1, 2). \end{cases} \tag{3.18}$$

For (3.18) we have a uniform estimate and a convergence result similar to Lemmata 3.4 and 3.5.

Lemma 3.6 Let ν_0, k, l, b_j $(j = 1, 2)$ and d_j $(j = 1, 2)$ are the same as in (3.6). In addition suppose $\mathbf{f} \in H_h^{l,l/2}(D_{+T})$, $g \in H_h^{1+l,1/2+l/2}(D_{+T})$, $g = \nabla \cdot \mathbf{G}$, $\mathbf{G} \in H_h^{0,1+l/2}(D_{+T})$, $b_3 \in H_h^{3/2+l,3/4+l/2}(D_T)$ and $\int_{\mathbb{R}^2} G_3 \mathrm{d}x' = \int_{\mathbb{R}^2} b_3 \mathrm{d}x'$.

Then problem (3.18) has a unique solution $\mathbf{u}^{(k)} \in H_h^{2+l,1+l/2}(D_{+T})$, $\nabla q^{(k)} \in H_h^{l,l/2}(D_{+T})$ for $k \geq 0$ satisfying a uniform estimate

$$\begin{aligned} &\|\mathbf{u}^{(k)}\|_{H_h^{2+l,1+l/2}(D_{+T})} + \|\nabla q^{(k)}\|_{H_h^{l,l/2}(D_{+T})} \\ &\leq c \Big(\|\mathbf{f}\|_{H_h^{l,l/2}(D_{+T})} + \|g\|_{H_h^{1+l,1/2+l/2}(D_{+T})} + \|\mathbf{G}\|_{H_h^{0,1+l/2}(D_{+T})} \\ &\qquad + \|\mathbf{b}\|_{H_h^{3/2+l,3/4+l/2}(D_T)} + \langle\langle \mathbf{d}' \rangle\rangle^{(k)}_{1/2+l,h,D_\infty} \Big) \\ &\leq \begin{cases} c \Big(\|\mathbf{f}\|_{H_h^{l,l/2}(D_{+T})} + \|g\|_{H_h^{1+l,1/2+l/2}(D_{+T})} + \|\mathbf{G}\|_{H_h^{0,1+l/2}(D_{+T})} \\ \qquad + \|\mathbf{b}\|_{H_h^{3/2+l,3/4+l/2}(D_T)} + \|\mathbf{d}'\|_{H_h^{1/2+l,1/4+l/2}(D_T)} \Big) & \text{if } k > 0, \\ c \Big(\|\mathbf{f}\|_{H_h^{l,l/2}(D_{+T})} + \|g\|_{H_h^{1+l,1/2+l/2}(D_{+T})} + \|\mathbf{G}\|_{H_h^{0,1+l/2}(D_{+T})} \\ \qquad + \|\mathbf{b}\|_{H_h^{3/2+l,3/4+l/2}(D_T)} \Big) & \text{if } k = 0, \end{cases} \end{aligned} \tag{3.19}$$

where c is a positive constant independent of h and k. Moreover, it also holds

$$(\mathbf{u}^{(k)}, \nabla q^{(k)}) \longrightarrow (\mathbf{u}^{(0)}, \nabla q^{(0)}) \quad \text{as } k \downarrow 0 \quad \text{in } H_h^{2+l,1+l/2}(D_{+T}) \times H_h^{l,l/2}(D_T). \tag{3.20}$$

Proof. According to [22], the solution of (3.18) can be expressed in the form

$$(\mathbf{u}^{(k)}, q^{(k)}) = (\mathbf{w} + \nabla\phi + \mathbf{W}^{(k)},\ \pi^{(k)} - \phi_t + \nu_0 g').$$

Here $\mathbf{w}$ is a solution of the Dirichlet problem for the heat equation:

$$\begin{cases} \dfrac{\partial \mathbf{w}}{\partial t} - \nu_0 \Delta \mathbf{w} = \mathbf{f} \quad \text{in } D_{+T}, \\ \mathbf{w}|_{t=0} = \mathbf{0} \quad \text{in } \mathbb{R}^3_+, \qquad \mathbf{w}|_{x_3=0} = \mathbf{0} \quad \text{on } D_T. \end{cases} \tag{3.21}$$

While ϕ is a solution of the Neumann problem:

$$\Delta\phi = g - \nabla\cdot\mathbf{w}^{(1)} \equiv g' \quad \text{in } \mathbb{R}^3_+, \qquad \frac{\partial\phi}{\partial x_3}\Big|_{x_3=0} = b_3 \quad \text{on } \mathbb{R}^2. \tag{3.22}$$

And then $(\mathbf{W}^{(k)}, \pi^{(k)})$ is a solution of the problem similar to (3.6):

$$\begin{cases} \dfrac{\partial \mathbf{W}^{(k)}}{\partial t} - \nu_0 \Delta \mathbf{W}^{(k)} + \nabla\pi^{(k)} = \mathbf{0}, \quad \nabla\cdot\mathbf{W}^{(k)} = 0 \quad \text{in } D_{+T}, \\ \mathbf{W}^{(k)}|_{t=0} = \mathbf{0} \quad \text{in } \mathbb{R}^3_+, \qquad W_3^{(k)}|_{x_3=0} = b_3 \quad \text{on } D_T, \\ W_j^{(k)} - \nu_0 k \left(\dfrac{\partial W_j^{(k)}}{\partial x_3} + \dfrac{\partial W_3^{(k)}}{\partial x_j} \right)\Bigg|_{x_3=0} = \tilde{b}_j - k\tilde{d}_j \quad \text{on } D_T\ (j = 1, 2), \end{cases} \tag{3.23}$$

where $\tilde{b}_j = b_j - \frac{\partial\phi}{\partial x_j}$ $(j = 1, 2)$ and $\tilde{d}_j = d_j - \nu_0 \left(\frac{\partial w_j}{\partial x_3} + \frac{\partial w_3}{\partial x_j} + 2\frac{\partial^2\phi}{\partial x_j \partial x_3} \right)\Big|_{x_3=0}$ $(j = 1, 2)$.

Obviously, problems (3.21) and (3.22) are independent of k, thus we have the following uniform estimates [22]

$$\|\mathbf{w}\|_{H_h^{2+l,1+l/2}(D_{+T})} \leq c\|\mathbf{f}\|_{H_h^{l,l/2}(D_{+T})}, \tag{3.24}$$

$$\|\nabla\phi\|_{H_h^{2+l,1+l/2}(D_{+T})} \leq c \left(\|g\|_{H_h^{1+l,1/2+l/2}(D_{+T})} + \|\mathbf{G}\|_{H_h^{0,1+l/2}(D_{+T})} \right. \\ \left. + \|b_3\|_{H_h^{3/2+l,3/4+l/2}(D_T)} \right), \tag{3.25}$$

where c is independent of h and k.

Because of $\mathbf{w}$, $\nabla\phi \in H_h^{2+l,1+l/2}(D_{+T})$, it follows $\tilde{b}_j \in H_h^{3/2+l,3/4+l/2}(D_T)$ and $\tilde{d}_j \in H_h^{1/2+l,1/4+l/2}(D_T)$ $(j = 1, 2)$. Hence, applying Lemma 3.4 to (3.23), it holds

$$\|\mathbf{W}^{(k)}\|_{H_h^{2+l,1+l/2}(D_{+T})} + \|\nabla\pi^{(k)}\|_{H_h^{l,l/2}(D_{+T})} \\ \leq c \left(\|\mathbf{b}'\|_{H_h^{3/2+l,3/4+l/2}(D_T)} + \langle\langle \bar{\mathbf{d}}' \rangle\rangle^{(k)}_{1/2+l,h,D_\infty} \right. \\ \left. + \|\mathbf{w}\|_{H_h^{2+l,1+l/2}(D_{+T})} + \|\nabla\phi\|_{H_h^{2+l,1+l/2}(D_{+T})} \right) \tag{3.26}$$

where c is independent of h and k. Consequently, the estimate (3.19) follows from (3.13), (3.24), (3.25) and (3.26).

Again let $\mathbf{U}^{(k)} = \mathbf{u}^{(k)} - \mathbf{u}^{(0)}$ and $Q^{(k)} = q^{(k)} - q^{(0)}$. Then $(\mathbf{U}^{(k)}, Q^{(k)})$ $(k > 0)$ satisfies the exactly same relation as (3.17):

$$\begin{cases} \dfrac{\partial \mathbf{U}^{(k)}}{\partial t} - \nu_0 \Delta \mathbf{U}^{(k)} + \nabla Q^{(k)} = \mathbf{0}, \quad \nabla \cdot \mathbf{U}^{(k)} = 0 \quad \text{in } D_{+T}, \\ \mathbf{U}^{(k)}|_{t=0} = \mathbf{0} \quad \text{on } \mathbb{R}^3_+, \quad U_3^{(k)}|_{x_3=0} = 0 \quad \text{in } D_T, \\ U_j^{(k)} - \nu_0 k \left(\dfrac{\partial U_j^{(k)}}{\partial x_3} + \dfrac{\partial U_3^{(k)}}{\partial x_j} \right) \Bigg|_{x_3=0} = -k d_j^* \quad \text{on } D_T \ (j = 1, 2), \end{cases}$$

thus (3.20) immediately follows from Lemma 3.5.

Furthermore, we also obtain a result for the whole-space problem.

$$\begin{cases} \dfrac{\partial \mathbf{w}}{\partial t} - \nu_0 \Delta \mathbf{w} + \nabla \pi = \mathbf{f}, \quad \nabla \cdot \mathbf{w} = g \quad \text{in } \mathbb{R}^3_T \equiv \mathbb{R}^3 \times (0, T), \\ \mathbf{w}|_{t=0} = \mathbf{0} \quad \text{in } \mathbb{R}^3. \end{cases} \tag{3.27}$$

For this problem we have the following result.

Lemma 3.7 Let ν_0 and l are the same as in (3.6). Suppose $\mathbf{f} \in H_h^{l,l/2}(\mathbb{R}^3_T)$, $g \in H_h^{1+l,1/2+l/2}(\mathbb{R}^3_T)$, $g = \nabla \cdot \mathbf{G}$ and $\mathbf{G} \in H_h^{0,1+l/2}(\mathbb{R}^3_T)$. Then problem (3.27) has a unique solution $\mathbf{w} \in H_h^{2+l,1+l/2}(\mathbb{R}^3_T)$, $\nabla \pi \in H_h^{l,l/2}(\mathbb{R}^3_T)$ satisfying

$$\begin{aligned} &\|\mathbf{w}\|_{H_h^{2+l,1+l/2}(\mathbb{R}^3_T)} + \|\nabla \pi\|_{H_h^{l,l/2}(\mathbb{R}^3_T)} \\ &\quad \le c \left(\|\mathbf{f}\|_{H_h^{l,l/2}(\mathbb{R}^3_T)} + \|g\|_{H_h^{1+l,1/2+l/2}(\mathbb{R}^3_T)} + \|\mathbf{G}\|_{H_h^{0,1+l/2}(\mathbb{R}^3_T)} \right), \end{aligned} \tag{3.28}$$

where c is a positive constant independent of h.

3.4 Proof of Lemma 3.1

We present some preliminaries. Because of the condition of Ω and Γ, in the neighbourhood of an arbitrary point $\xi \in \Gamma$, the surface Γ is represented by the equation

$$y_3 = \varphi(y'), \quad y' = (y_1, y_2) \in K_d \quad (K_d = \{y' : |y'| < d\})$$

in a Cartesian local coordinate system (y_1, y_2, y_3) with the origin at ξ and with y_3-axis directed along $-\mathbf{n}(\xi)$, $\mathbf{n}(\xi)$ being the unit outward normal vector to Γ at ξ. The function φ may be considered to be defined on $\mathbb{R}^2$ such that its support is included in a disc K_{2d} and $\varphi(0) = 0$, $\nabla' \varphi(0) = 0$ (∇' is the gradient with respect to y') and $\|\varphi\|_{W_2^{5/2+l}(\mathbb{R}^2)} \le M$ $(M > 0)$ hold. It is to be noted that the constants d and M are taken indepenently of ξ. Furthermore, φ can be extended into $\mathbb{R}^3_+$ (see [19, 21]) so that it belongs to $W_2^{3+l}(\mathbb{R}^3_+)$, and $\varphi(0) = 0$, $\nabla \varphi(0) = 0$ and $\sup_{|y| \le \lambda}(|\varphi(y)| + |\nabla \varphi(y)|) \le cM\lambda$. Then the transformation $y = Y(z)$:

$$y_1 = z_1, \ y_2 = z_2, \ y_3 = z_3 + \varphi(z) \tag{3.29}$$

is invertible if $|\varphi_{z_3}| < 1$ and maps $\mathbb{R}^3_+$ onto the domain $\{y_3 > \varphi(y')\}$.

Considering the neighbourhood of $\xi \in \Gamma$, we assume for the sake of simplicity that $\xi = 0$ and that the coordinates $\{y_j\}$ coincide with $\{x_j\}$. Let $\zeta_\lambda(x) = \zeta(x/\lambda)$ where $\zeta \in C_0^\infty(\mathbb{R}^3)$, $\zeta(x) = 1$ for $|x| \le 1/2$, $\zeta(x) = 0$ for $|x| \ge 1$. Then $(\mathbf{u}_\lambda^{(k)}, q_\lambda^{(k)}) = (\zeta_\lambda \mathbf{u}^{(k)}, \zeta_\lambda q^{(k)})$ satisfies the following relation

$$\begin{cases} \dfrac{\partial \mathbf{u}_\lambda^{(k)}}{\partial t} - \dfrac{\nu_1(x)}{\varrho_0(x)} \Delta \mathbf{u}_\lambda^{(k)} + \dfrac{1}{\varrho_0(x)} \nabla q_\lambda^{(k)} = \zeta_\lambda \mathbf{f} - \mathbf{F}_1, \\ \nabla \cdot \mathbf{u}_\lambda^{(k)} = \zeta_\lambda g - F_2 \quad \text{in } Q_T, \qquad \mathbf{u}_\lambda^{(k)}|_{t=0} = \mathbf{0} \quad \text{in } \Omega, \\ \mathbf{u}_\lambda^{(k)} + 2\nu_1(x) k \Pi \mathbb{D}(\mathbf{u}_\lambda^{(k)}) \mathbf{n} = \zeta_\lambda \mathbf{b} + k(\zeta_\lambda \mathbf{d} - \mathbf{F}_3) \quad \text{on } G_T, \end{cases} \tag{3.30}$$

where

$$\begin{aligned} \mathbf{F}_1 &= -\zeta_\lambda \Delta \mathbf{u}^{(k)} + \Delta(\mathbf{u}_\lambda^{(k)}) + \zeta_\lambda \nabla q^{(k)} - \nabla q_\lambda^{(k)}, \\ F_2 &= \zeta_\lambda \nabla \cdot \mathbf{u}^{(k)} - \nabla \cdot \mathbf{u}_\lambda^{(k)}, \\ \mathbf{F}_3 &= 2\nu_1(x) k \Pi \left(\zeta_\lambda \mathbb{D}(\mathbf{u}^{(k)}) - \mathbb{D}(\mathbf{u}_\lambda^{(k)}) \right) \mathbf{n}. \end{aligned}$$

We consider (3.30) in local coordinates $\{z\}: \ z = Y^{-1}(x)$, then we have

$$\begin{cases} \dfrac{\partial \bar{\mathbf{u}}_\lambda^{(k)}}{\partial t}(z,t) - \dfrac{\nu_1(0)}{\varrho_0(0)} \Delta_z \bar{\mathbf{u}}_\lambda^{(k)}(z,t) + \dfrac{1}{\varrho_0(0)} \nabla_z \bar{q}_\lambda^{(k)}(z,t) = \bar{\mathbf{F}}_1(z,t) \quad \text{in } D_{\lambda,+T}, \\ \nabla_z \cdot \bar{\mathbf{u}}_\lambda^{(k)}(z,t) = \bar{F}_2(z,t) \quad \text{in } D_{\lambda,+T}, \\ \bar{\mathbf{u}}_\lambda^{(k)}(z,t)|_{t=0} = \mathbf{0} \quad \text{in } Y^{-1}(\Omega_\lambda), \\ \bar{\mathbf{u}}_\lambda^{(k)}(z,t) + 2\nu_1(0) k \Pi_0 \mathbb{D}_z(\bar{\mathbf{u}}_\lambda^{(k)}(z,t)) \mathbf{n}_0|_{z_3=0} \\ \qquad = (\zeta_\lambda \mathbf{b})(Y(z),t) + k\bar{\mathbf{F}}_3(z,t)|_{z_3=0} \quad \text{on } D_{\lambda,T}, \end{cases} \tag{3.31}$$

where $\bar{\mathbf{u}}_\lambda^{(k)}(z,t) = \mathbf{u}_\lambda^{(k)}(Y(z),t)$, $\bar{q}_\lambda^{(k)}(z,t) = q_\lambda^{(k)}(Y(z),t)$, $D_{\lambda,+T} = Y^{-1}(\Omega_\lambda) \times (0,T)$, $D_{\lambda,T} = Y^{-1}(\Gamma_\lambda) \times (0,T)$, $\Omega_\lambda = \Omega \cap \{|x| \le \lambda\}$, $\Gamma_\lambda = \Gamma \cap \{|x| \le \lambda\}$, $\mathbf{n}_0 = (0,0,-1)^{\mathrm{T}}$, $\Pi_0 \mathbf{f} = (f_1, f_2, 0)^{\mathrm{T}}$, $\bar{\nabla} = (\frac{\partial x}{\partial z})^{-\mathrm{T}} \nabla_z$, $\bar{\Delta} = \bar{\nabla} \cdot \bar{\nabla}$,

$$\begin{aligned} \bar{\mathbf{F}}_1 &= (\zeta_\lambda \mathbf{f})(Y(z),t) - \mathbf{F}_1(Y(z),t) - \frac{\nu_1(0)}{\varrho_0(0)} \Delta_z \bar{\mathbf{u}}_\lambda^{(k)}(z,t) + \frac{\nu_1(Y(z))}{\varrho_0(Y(z))} \bar{\Delta} \bar{\mathbf{u}}_\lambda^{(k)}(z,t) \\ &\quad - \frac{1}{\varrho_0(Y(z))} \bar{\nabla} \bar{q}_\lambda^{(k)}(z,t) + \frac{1}{\varrho_0(0)} \nabla_z \bar{q}_\lambda^{(k)}(z,t), \\ \bar{F}_2 &= (\zeta_\lambda g)(Y(z),t) - F_2(Y(z),t) - \bar{\nabla} \cdot \bar{\mathbf{u}}_\lambda^{(k)}(z,t) + \nabla_z \cdot \bar{\mathbf{u}}_\lambda^{(k)}(z,t), \\ \bar{\mathbf{F}}_3 &= (\zeta_\lambda \mathbf{d})(Y(z),t) - \mathbf{F}_3(Y(z),t) \\ &\quad - 2\nu_1(Y(z)) \bar{\Pi} \bar{\mathbb{D}}(\bar{\mathbf{u}}_\lambda^{(k)}) \bar{\mathbf{n}} + 2\nu_1(0) \Pi_0 \mathbb{D}_z(\bar{\mathbf{u}}_\lambda^{(k)}(z,t)) \mathbf{n}_0, \end{aligned}$$

$\bar{\mathbf{n}}(z) = \mathbf{n}(Y(z))$, $\bar{\Pi} \mathbf{f} = \mathbf{f} - (\mathbf{f} \cdot \bar{\mathbf{n}}) \mathbf{n}$ and $\bar{\mathbb{D}}(\mathbf{f}) = \frac{1}{2}(\bar{\nabla} \mathbf{f} + [\bar{\nabla} \mathbf{f}]^{\mathrm{T}})$.

Since $\operatorname{supp} \bar{\mathbf{u}}_\lambda^{(k)}$, $\operatorname{supp} \bar{q}_\lambda^{(k)}$, $\operatorname{supp} \bar{\mathbf{F}}_1$, $\operatorname{supp} \bar{F}_2 \subset Y^{-1}(\Omega_\lambda)$ and $\operatorname{supp} (\zeta_\lambda \mathbf{b})(Y(\cdot),t)$, $\operatorname{supp} \bar{\mathbf{F}}_3 \subset Y^{-1}(\Gamma_\lambda)$, we can extend these functions by zero into the outside of their supports. Extending ϱ_0 and ν_1 into $\mathbb{R}^3$, we can consider (3.31) as the initial-boundary value problem in $\mathbb{R}^3_+$. Applying (3.19) to (3.31), we obtain

$$\begin{aligned}
&\|\bar{\mathbf{u}}_\lambda^{(k)}\|_{H_h^{2+l,1+l/2}(D_{+T})} + \|\bar{\nabla} q_\lambda^{(k)}\|_{H_h^{l,l/2}(D_{+T})} \\
&\le c\Big(\|\bar{\mathbf{F}}_1\|_{H_h^{l,l/2}(D_{+T})} + \|\bar{F}_2\|_{H_h^{1+l,1/2+l/2}(D_{+T})} + \|\bar{\mathbf{F}}_4\|_{H_h^{0,1+l/2}(D_{+T})} \\
&\quad + \|(\zeta_\lambda \mathbf{b})(Y(\cdot),\cdot)\|_{H_h^{3/2+l,3/4+l/2}(D_T)} + \langle\langle \bar{\mathbf{F}}_3^*\rangle\rangle^{(k)}_{1/2+l,h,D_\infty}\Big), \qquad (3.32)
\end{aligned}$$

where $\bar{\mathbf{F}}_4$ is the gradient of the Newtonian potential of $\bar{F}_2$, namely

$$\bar{\mathbf{F}}_4 = \frac{-1}{4\pi}\nabla\int_{\mathbb{R}^3_+}\frac{\bar{F}_2(\omega,t)}{|z-\omega|}\mathrm{d}\omega,$$

$\bar{\mathbf{F}}_3^*$ the expansion of $\bar{\mathbf{F}}_3$ into D_∞ and c is a positive constant independent of h and k. In the same way as in [21] we can prove that

$$\begin{aligned}
&\|\bar{\mathbf{u}}_\lambda^{(k)}\|_{H_h^{2+l,1+l/2}(D_{+T})} + \|\bar{\nabla} q_\lambda^{(k)}\|_{H_h^{l,l/2}(D_{+T})} \\
&\le c\Big\{\|(\zeta_\lambda \mathbf{f})(Y(\cdot),\cdot)\|_{H_h^{l,l/2}(D_{2\lambda,+T})} + \|(\zeta_\lambda g)(Y(\cdot),\cdot)\|_{H_h^{1+l,1/2+l/2}(D_{2\lambda,+T})} \\
&\quad + \|(\zeta_\lambda \mathbf{G})(Y(\cdot),\cdot)\|_{H_h^{0,1+l/2}(D_{2\lambda,+T})} + \|\mathbf{G}(Y(\cdot),\cdot)\|_{H_h^{0,l/2}(D_{2\lambda,+T})} \\
&\quad + \|(\zeta_\lambda \mathbf{b})(Y(\cdot),\cdot)\|_{H_h^{3/2+l,3/4+l/2}(D_{2\lambda,T})} + \langle\langle \bar{\mathbf{F}}_3^*\rangle\rangle^{(k)}_{1/2+l,h,D_\infty} \\
&\quad + (\lambda^{1/2} + h^{-l/2})\Big(\|\mathbf{u}_\lambda^{(k)}\|_{H_h^{2+l,1+l/2}(Q_{2\lambda,+T})} + \|\nabla q_\lambda^{(k)}\|_{H_h^{l,l/2}(Q_{2\lambda,+T})}\Big)\Big\}, \qquad (3.33) \\
&\le c\Big\{\|(\zeta_\lambda \mathbf{f})(Y(\cdot),\cdot)\|_{H_h^{l,l/2}(D_{2\lambda,+T})} + \|(\zeta_\lambda g)(Y(\cdot),\cdot)\|_{H_h^{1+l,1/2+l/2}(D_{2\lambda,+T})} \\
&\quad + \|(\zeta_\lambda \mathbf{G})(Y(\cdot),\cdot)\|_{H_h^{0,1+l/2}(D_{2\lambda,+T})} + \|\mathbf{G}(Y(\cdot),\cdot)\|_{H_h^{0,l/2}(D_{2\lambda,+T})} \\
&\quad + \|(\zeta_\lambda \mathbf{b})(Y(\cdot),\cdot)\|_{H_h^{3/2+l,3/4+l/2}(D_{2\lambda,T})} + \|(\zeta_\lambda \mathbf{d})(Y(\cdot),\cdot)\|_{H_h^{1/2+l,1/4+l/2}(D_{2\lambda,T})} \\
&\quad + (\lambda^{1/2} + h^{-l/2})\Big(\|\mathbf{u}_\lambda^{(k)}\|_{H_h^{2+l,1+l/2}(Q_{2\lambda,+T})} + \|\nabla q_\lambda^{(k)}\|_{H_h^{l,l/2}(Q_{2\lambda,+T})}\Big)\Big\}, \qquad (3.34)
\end{aligned}$$

We remark that the similar inequalities hold in neighbourhoods of any point on Γ or in the interior of Ω. In the latter case $\mathbf{b}$ and $\mathbf{d}'$ do not enter into the estimates.

When we cover Ω by a finite number of such neighbourhoods and make the summation of (3.34) over all the neighbourhoods, we obtain

$$\begin{aligned}
&\|\mathbf{u}^{(k)}\|_{H_h^{2+l,1+l/2}(Q_T)} + \|\nabla q^{(k)}\|_{H_h^{l,l/2}(Q_T)} \\
&\le c\Big\{\|\mathbf{f}\|_{H_h^{l,l/2}(Q_T)} + \|g\|_{H_h^{1+l,1/2+l/2}(Q_T)} + \|\mathbf{G}\|_{H_h^{0,1+l/2}(Q_T)} + \|\mathbf{b}\|_{H_h^{3/2+l,3/4+l/2}(G_T)} \\
&\quad + \|\mathbf{d}\|_{H_h^{1/2+l,1/4+l/2}(G_T)} + (\lambda^{1/2} + h^{-l/2})\Big(\|\mathbf{u}^{(k)}\|_{H_h^{2+l,1+l/2}(Q_T)} + \|\nabla q^{(k)}\|_{H_h^{l,l/2}(Q_T)}\Big)\Big\},
\end{aligned} \qquad (3.35)$$

where c is independent of h and k. Taking sufficiently small λ and large h, we obtain the uniform estimate (3.2).

Moreover, let $\mathbf{U}^{(k)} = \mathbf{u}^{(k)} - \mathbf{u}^{(0)}$ and $Q^{(k)} = q^{(k)} - q^{(0)}$. Then $(\mathbf{U}^{(k)}, Q^{(k)})$ $(k > 0)$ satisfies the following relation similar to (3.1):

$$\begin{cases} \varrho_0 \dfrac{\partial \mathbf{U}^{(k)}}{\partial t} - \nu_1(x)\Delta \mathbf{U}^{(k)} + \nabla Q^{(k)} = \mathbf{0}, \quad \nabla \cdot \mathbf{U}^{(k)} = 0 \quad \text{in } Q_T, \\ \mathbf{u}|_{t=0} = \mathbf{0} \quad \text{in } \Omega, \\ \mathbf{U}^{(k)} + 2\nu_1(x) k \Pi \mathbb{D}(\mathbf{U}^{(k)})\mathbf{n} = k\tilde{\mathbf{d}} \quad \text{on } G_T. \end{cases} \tag{3.36}$$

Taking into account (3.33) in this case, in the neighbourhood of $0 \in \Gamma$ we have

$$\|\bar{\mathbf{U}}_\lambda^{(k)}\|_{H_h^{2+l,1+l/2}(D_{+T})} + \|\bar{\nabla} Q_\lambda^{(k)}\|_{H_h^{l,l/2}(D_{+T})} \le c\Big\{\langle\langle \bar{\mathbf{F}}_3^* \rangle\rangle^{(k)}_{1/2+l,h,D_\infty}$$
$$+ (\lambda^{1/2} + h^{-l/2})\left(\|\mathbf{U}_\lambda^{(k)}\|_{H_h^{2+l,1+l/2}(Q_{2\lambda,+T})} + \|\nabla Q_\lambda^{(k)}\|_{H_h^{l,l/2}(Q_{2\lambda,+T})}\right)\Big\}.$$

When we make the summation over all the converings again we have

$$\|\mathbf{u}^{(k)}\|_{H_h^{2+l,1+l/2}(Q_T)} + \|\nabla q^{(k)}\|_{H_h^{l,l/2}(Q_T)} \le c\Bigg\{\sum_j \langle\langle \bar{\mathbf{F}}_{j,3}^* \rangle\rangle^{(k)}_{1/2+l,h,D_\infty}$$
$$+ (\lambda^{1/2} + h^{-l/2})\left(\|\mathbf{u}^{(k)}\|_{H_h^{2+l,1+l/2}(Q_T)} + \|\nabla q^{(k)}\|_{H_h^{l,l/2}(Q_T)}\right)\Bigg\}, \tag{3.37}$$

where $\bar{\mathbf{F}}_{j,3}^*$ denotes the $\bar{\mathbf{F}}_3^*$ for the neighbourhood of $\xi_j \in \Gamma$ which is the center of the covering of Ω. Taking sufficiently small λ and large h, we immediately arrive at (3.3).

4 Proof of Theorem 2.2

Finally, we shall prove Theorem 2.2.

Because of Lemma 3.2 and the proof of the existence of the time-local solution of (2.4) [12], we can easily see that the magnitude of time interval T' where solution exists can be taken uniformly in k. Thus we omit the proof of Theorem 2.1.

We consider the following condition for $\mathbf{u}^{(k)}$ and $q^{(k)}$

$$T'^{1/2}(\|\mathbf{U}^{(k)}\|_{Q_{T'}}^{(2+l,1+l/2)} + \|\nabla Q^{(k)}\|_{Q_{T'}}^{(l,l/2)}) \le \delta.$$

Let $\mathbf{U}^{(k)} = \mathbf{u}^{(k)} - \mathbf{u}^{(0)}$ and $Q^{(k)} = q^{(k)} - q^{(0)}$. Then $(\mathbf{U}^{(k)}, Q^{(k)})$ $(k > 0)$ satisfies the following equation similar to (2.4):

$$\begin{cases} \varrho_0 \dfrac{\partial \mathbf{U}^{(k)}}{\partial t} - \nu(\varrho_0)\Delta \mathbf{U}^{(k)} + \nabla Q^{(k)} = \mathbf{l}_1^{(\mathbf{u}^{(k)})}(\mathbf{U}^{(k)}, Q^{(k)}) \\ \qquad + \left(\mathbf{l}_1^{(\mathbf{u}^{(k)})}(\mathbf{u}^{(0)}, q^{(0)}) - \mathbf{l}_1^{(\mathbf{u}^{(0)})}(\mathbf{u}^{(0)}, q^{(0)})\right) \\ \qquad + 2\nu'(\varrho_0)\left(\mathbb{D}_{\mathbf{u}^{(k)}}(\mathbf{u}^{(k)})\nabla_{\mathbf{u}^{(k)}}\varrho_0 - \mathbb{D}_{\mathbf{u}^{(0)}}(\mathbf{u}^{(0)})\nabla_{\mathbf{u}^{(0)}}\varrho_0\right) \\ \qquad - \dfrac{\beta}{3}\left\{((\nabla^{(i)}_{\mathbf{u}^{(k)}}\nabla^{(j)}_{\mathbf{u}^{(k)}}\varrho_0)\nabla_{\mathbf{u}^{(k)}}\varrho_0 - (\nabla^{(i)}_{\mathbf{u}^{(0)}}\nabla^{(j)}_{\mathbf{u}^{(0)}}\varrho_0)\nabla_{\mathbf{u}^{(0)}}\varrho_0\right\} \\ \qquad - \beta\left(\Delta_{\mathbf{u}^{(k)}}\varrho_0\nabla_{\mathbf{u}^{(k)}}\varrho_0 - \Delta_{\mathbf{u}^{(0)}}\varrho_0\nabla_{\mathbf{u}^{(0)}}\varrho_0\right) + \varrho_0(\mathbf{b}_{\mathbf{u}^{(k)}} - \mathbf{b}_{\mathbf{u}^{(0)}}) \text{ in } Q_{T'}, \\ \nabla \cdot \mathbf{U}^{(k)} = l_2^{(\mathbf{u}^{(k)})}(\mathbf{U}^{(k)}) + \left(l_2^{(\mathbf{u}^{(k)})}(\mathbf{u}^{(0)}) - l_2^{(\mathbf{u}^{(0)})}(\mathbf{u}^{(0)})\right) \text{ in } Q_{T'}, \ \mathbf{U}^{(k)}|_{t=0} = \mathbf{0} \text{ in } \Omega, \\ \mathbf{U}^{(k)} \cdot \mathbf{n}|_\Gamma = l_3^{(\mathbf{u}^{(k)})}(\mathbf{U}^{(k)}) + \left(l_3^{(\mathbf{u}^{(k)})}(\mathbf{u}^{(0)}) - l_3^{(\mathbf{u}^{(0)})}(\mathbf{u}^{(0)})\right)|_\Gamma \text{ on } G_{T'}, \\ \mathbf{U}^{(k)} + 2\nu(\varrho_0)k\Pi\mathbb{D}(\mathbf{U}^{(k)})\mathbf{n}|_\Gamma = k\mathbf{l}_4^{(\mathbf{u}^{(k)})}(\mathbf{U}^{(k)}) + k\left(\mathbf{l}_4^{(\mathbf{u}^{(k)})}(\mathbf{u}^{(0)}) - \mathbf{l}_4^{(\mathbf{u}^{(0)})}(\mathbf{u}^{(0)})\right) \\ + \beta k\left\{\Pi_{\mathbf{u}^{(k)}}(\nabla_{\mathbf{u}^{(k)}}\varrho_0 \otimes \nabla_{\mathbf{u}^{(k)}}\varrho_0)\mathbf{n}_{\mathbf{u}^{(k)}} - \Pi_{\mathbf{u}^{(0)}}(\nabla_{\mathbf{u}^{(0)}}\varrho_0 \otimes \nabla_{\mathbf{u}^{(0)}}\varrho_0)\mathbf{n}_{\mathbf{u}^{(0)}}\right\}|_\Gamma + k\mathbf{d} \text{ on } G_{T'}, \end{cases}$$

where

$$\begin{cases} \nabla_{\mathbf{u}} = (\nabla^{(1)}_{\mathbf{u}}, \nabla^{(2)}_{\mathbf{u}}, \nabla^{(3)}_{\mathbf{u}}), \quad \Delta_{\mathbf{u}} = \nabla_{\mathbf{u}} \cdot \nabla_{\mathbf{u}}, \\ \mathbf{l}_1^{(\mathbf{u})}(\mathbf{w}, s) = \nu(\varrho_0)(\Delta_{\mathbf{u}} - \Delta)\mathbf{w} - (\nabla_{\mathbf{u}} - \nabla)s, \\ l_2^{(\mathbf{u})}(\mathbf{w}) = (\nabla - \nabla_{\mathbf{u}}) \cdot \mathbf{w} = \nabla \cdot \mathcal{L}^{(\mathbf{u})}(\mathbf{w}), \quad l_3^{(\mathbf{u})}(\mathbf{w}) = \mathbf{w} \cdot (\mathbf{n} - \mathbf{n}_{\mathbf{u}}), \\ \mathbf{l}_4^{(\mathbf{u})}(\mathbf{w}) = 2\nu(\varrho_0)\left(\Pi\mathbb{D}(\mathbf{w})\mathbf{n} - \Pi_{\mathbf{u}}\mathbb{D}_{\mathbf{u}}(\mathbf{w})\mathbf{n}_{\mathbf{u}}\right), \end{cases}$$

and

$$\mathbf{d} = \Pi_{\mathbf{u}^{(0)}}(\nabla_{\mathbf{u}^{(0)}}\varrho_0 \otimes \nabla_{\mathbf{u}^{(0)}}\varrho_0)\mathbf{n}_{\mathbf{u}^{(0)}}. \tag{4.1}$$

Obviously, $\mathbf{d}$ is independent of k.

The lemmata in §4 of [12] yield

$$\left\|\mathbf{l}_1^{(\mathbf{u}^{(k)})}(\mathbf{U}^{(k)}, Q^{(k)})\right\|_{Q_{T'}}^{(l,l/2)} \le c\delta\left(\|\mathbf{U}^{(k)}\|_{Q_{T'}}^{(2+l,1+l/2)} + \|\nabla Q^{(k)}\|_{Q_{T'}}^{(l,l/2)}\right),$$

$$\left\|\mathbf{l}_1^{(\mathbf{u}^{(k)})}(\mathbf{u}^{(0)}, q^{(0)}) - \mathbf{l}_1^{(\mathbf{u}^{(0)})}(\mathbf{u}^{(0)}, q^{(0)})\right\|_{Q_{T'}}^{(l,l/2)}$$

$$\le c\delta\left(\|\mathbf{U}^{(k)}\|_{Q_{T'}}^{(2+l,1+l/2)} + \|\nabla Q^{(k)}\|_{Q_{T'}}^{(l,l/2)}\right),$$

$$\|\mathbb{D}_{\mathbf{u}^{(k)}}(\mathbf{u}^{(k)})\nabla_{\mathbf{u}^{(k)}}\varrho_0 - \mathbb{D}_{\mathbf{u}^{(0)}}(\mathbf{u}^{(0)})\nabla_{\mathbf{u}^{(0)}}\varrho_0\|_{Q_{T'}}^{(l,l/2)}$$

$$\le c\|\varrho_0\|_{W_2^{2+l}(\Omega)}\left(1 + T'^{1/2-l/2}\|\mathbf{v}_0\|_{W_2^l(\Omega)}\right)T^{1/2}\|\mathbf{U}^{(k)}\|_{Q_{T'}}^{(2+l,1+l/2)},$$

$$\left\|(\nabla^{(i)}_{\mathbf{u}^{(k)}}\nabla^{(j)}_{\mathbf{u}^{(k)}}\varrho_0)\nabla_{\mathbf{u}^{(k)}}\varrho_0 - (\nabla^{(i)}_{\mathbf{u}^{(0)}}\nabla^{(j)}_{\mathbf{u}^{(0)}}\varrho_0)\nabla_{\mathbf{u}^{(0)}}\varrho_0\right\|_{Q_{T'}}^{(l,l/2)}$$

$$\leq c\|\varrho_0\|^2_{W_2^{2+l}(\Omega)}(T'^{1/2}+T'^{1/2-l/2})T'^{1/2}\|\mathbf{U}^{(k)}\|^{(2+l,1+l/2)}_{Q_{T'}},$$

$$\|\Delta_{\mathbf{u}^{(k)}}\varrho_0\nabla_{\mathbf{u}^{(k)}}\varrho_0-\Delta_{\mathbf{u}^{(0)}}\varrho_0\nabla_{\mathbf{u}^{(0)}}\varrho_0\|^{(l,l/2)}_{Q_{T'}}$$
$$\leq c\|\varrho_0\|^2_{W_2^{2+l}(\Omega)}(T'^{1/2}+T'^{1/2-l/2})T'^{1/2}\|\mathbf{U}^{(k)}\|^{(2+l,1+l/2)}_{Q_{T'}},$$

$$\|\mathbf{b}_{\mathbf{u}^{(k)}}-\mathbf{b}_{\mathbf{u}^{(0)}}\|^{(l,l/2)}_{Q_{T'}}\leq c\,T'^{1/2}\|\mathbf{U}^{(k)}\|^{(2+l,1+l/2)}_{Q_{T'}},$$

$$\|l_2^{(\mathbf{u}^{(k)})}(\mathbf{U}^{(k)})\|_{W_2^{1+l,1/2+l/2}(Q_{T'})}\leq c\delta\|\mathbf{U}^{(k)}\|^{(2+l,1+l/2)}_{Q_{T'}},$$

$$\|l_2^{(\mathbf{u}^{(k)})}(\mathbf{u}^{(k)})-l_2^{(\mathbf{u}^{(0)})}(\mathbf{u}^{(0)})\|_{W_2^{1+l,1/2+l/2}(Q_{T'})}\leq c\delta\|\mathbf{U}^{(k)}\|^{(2+l,1+l/2)}_{Q_{T'}},$$

$$\left\|\frac{\partial}{\partial t}\mathcal{L}^{(\mathbf{u}^{(k)})}(\mathbf{U}^{(k)})\right\|^{(0,l/2)}_{Q_{T'}}+\left\|\frac{\partial}{\partial t}\left(\mathcal{L}^{(\mathbf{u}^{(k)})}(\mathbf{u}^{(k)})-\mathcal{L}^{(\mathbf{u}^{(0)})}(\mathbf{u}^{(0)})\right)\right\|^{(0,l/2)}_{Q_{T'}}$$
$$\leq c\delta\|\mathbf{U}^{(k)}\|^{(2+l,1+l/2)}_{Q_{T'}},$$

$$\|l_3^{(\mathbf{u}^{(k)})}(\mathbf{U}^{(k)})\|_{W_2^{3/2+l,3/4+l/2}(G_{T'})}\leq c\delta\|\mathbf{U}^{(k)}\|^{(2+l,1+l/2)}_{Q_{T'}},$$

$$\|l_3^{(\mathbf{u}^{(k)})}(\mathbf{u}^{(0)})-l_3^{(\mathbf{u}^{(0)})}(\mathbf{u}^{(0)})\|_{W_2^{3/2+l,3/4+l/2}(G_{T'})}\leq c\delta\|\mathbf{U}^{(k)}\|^{(2+l,1+l/2)}_{Q_{T'}},$$

$$\|\mathbf{l}_4^{(\mathbf{u}^{(k)})}(\mathbf{U}^{(k)})\|_{W_2^{1/2+l,1/4+l/2}(G_{T'})}\leq c\delta\|\mathbf{U}^{(k)}\|^{(2+l,1+l/2)}_{Q_{T'}},$$

$$\|\mathbf{l}_4^{(\mathbf{u}^{(k)})}(\mathbf{u}^{(0)})-\mathbf{l}_4^{(\mathbf{u}^{(0)})}(\mathbf{u}^{(0)})\|_{W_2^{1/2+l,1/4+l/2}(G_{T'})}\leq c\delta\|\mathbf{U}^{(k)}\|^{(2+l,1+l/2)}_{Q_{T'}},$$

$$\|\Pi_{\mathbf{u}^{(k)}}(\nabla_{\mathbf{u}^{(k)}}\varrho_0\otimes\nabla_{\mathbf{u}^{(k)}}\varrho_0)\mathbf{n}_{\mathbf{u}^{(k)}}-\Pi_{\mathbf{u}^{(0)}}(\nabla_{\mathbf{u}^{(0)}}\varrho_0\otimes\nabla_{\mathbf{u}^{(0)}}\varrho_0)\mathbf{n}_{\mathbf{u}^{(0)}}\|_{W_2^{1/2+l,1/4+l/2}(G_{T'})}$$
$$\leq c\,T^{1/2}\|\varrho_0\|^2_{W_2^{2+l}(\Omega)}T^{1/2}\|\mathbf{U}^{(k)}\|^{(2+l,1+l/2)}_{Q_{T'}},$$

Applying the estimate (3.4) and taking (3.37) into account, we obtain

$$\|\mathbf{U}^{(k)}\|^{(2+l,1+l/2)}_{Q_{T'}}+\|\nabla Q^{(k)}\|^{(l,l/2)}_{Q_{T'}}$$
$$\leq c_2(T)\left\{(\delta+T^{1/2})\|\mathbf{U}^{(k)}\|^{(2+l,1+l/2)}_{Q_{T'}}+\delta\|\nabla Q^{(k)}\|^{(l,l/2)}_{Q_{T'}}\right.$$
$$\left.+\sum_j\langle\langle\bar{\mathbf{F}}^*_{j,3}\rangle\rangle^{(k)}_{1/2+l,h,D_\infty}\right\},\tag{4.2}$$

where $c_2(T)$ is a non-decreasing function with respect to T independent of k, and $\bar{\mathbf{F}}^*_{j,3}$ is the same as (3.37) for $\mathbf{d}$ of (4.1). In the proof of the time-local existence we take δ and T' in such a way that $c_2(T)(\delta+T')<\frac{1}{2}$, we therefore obtain

$$\lim_{k\downarrow 0}(\|\mathbf{U}^{(k)}\|^{(2+l,1+l/2)}_{Q_{T'}}+\|\nabla Q^{(k)}\|^{(l,l/2)}_{Q_{T'}})=0.$$

This completes the proof of Theorem 2.2.

References

[1] S. Agmon, A. Douglis and L. Nirenberg: *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions I.* Comm. Pure Appl. Math. **12**, 623-727 (1959).

[2] O. V. Besov, V. P. Il'in and S. M. Nikol'skiĭ: Integral representations of functions and imbedding theorems. *"Nauka", Moscow*, 1975; English transl., Vols. 1, 2, Wiley, 1979.

[3] E. J. Boyle and M. Massoudi: *A theory for granular materials exhibiting normal stress effects based on Enskog's dense gas theory.* Int. J. Eng. Sci. **28**, 1261-1275 (1990).

[4] S. Itoh, N. Tanaka, A. Tani: *The initial value problem for the Navier-Stokes equations with general slip boundary condition in Hölder space.* J. Math. Fluid Mech. **5**, 275-301 (2003).

[5] S. Itoh, A. Tani: *The initial value problem for the non-homogeneous Navier-Stokes equations with general slip boundary condition.* Proc. Roy. Soc. Edinburgh Sect. A, **130**, 827-835 (2000).

[6] D. J. Kortweg: *Sur la forme que prenent les équations du mouvements des fluides si l'on tient compte des forces capilaires causées par des variations de densité considérables mains continues et sur la théorie de la capollarité dans l'hypothése d'une varation continue de la densité.* Arch. Néerl. Sci. Exactes Nat. Ser. II, **6**, 1-24 (1901).

[7] O. A. Ladyženskaja, V. A. Solonnikov and N. N. Ural'ceva: Linear and Quasi-linear Equations of Parabolic Type. *Amer. Math. Soc., Providence, R. I.*, 1968.

[8] J. Málek and K. R. Rajagopal: *Mathematical Issues Concerning the Navier-Stokes Equations and Some of Its Generalizations.* Handbook of Differential Equations: Evolutionary Equations. Vol. II, North-Holland, Amsterdam, 371-459 (2005).

[9] J. Málek and K. R. Rajagopal: *On the modeling of inhomogeneous incompressible fluid-like bodies.* Mech. Mat. **38**, 233-242 (2006).

[10] J. Málek and K. R. Rajagopal: *Incompressible rate type fluids with pressure and shear-rate dependent material moduli.* Nonlinear Analysis: Real World Applications **8**, 156-164 (2007).

[11] N. Nakano and A. Tani: *An initial-boundary value problem for motion of incompressible inhomogeneous fluid-like bodies.* Adv. Math. Sci. Appl., accepted.

[12] N. Nakano and A. Tani: *Navier's slip problem for motion of incompressible inhomogeneous fluid-like bodies.* J. Math. Fluid Mech., accepted.

[13] C. L. M. H. Navier, *Mémoir sur les lois du mouvement des fluides*, Mem. Acad. R. Sci. Paris **6**, (1823), 389-416.

[14] K. R. Rajagopal: *Multiple natural configurations in continuum mechanics.* Technical Report 6, Institute for Computational and Applied Mechanics, University of Pittsburgh, 1995.

[15] K. R. Rajagopal and M. Massoudi: *A method for measuring material moduli of granular materials: flow in an orthogonal rheomer.* Topical Report U.S. Department of Energy DOE/PETC/TR-90/3 (1990).

[16] V. A. Solonnikov: *A priori estimates for certain boundary value problems.* Dokl. Akad. Nauk SSSR **138**, 781-784 (1961); English transl. in Soviet Math. Dokl., **2** (1961).

[17] V. A. Solonnikov: *Estimates for solutions of nonstationary Navier-Stokes equations.* Zap. Nauchn. Sem. LOMI **38**, 153-231 (1973); English transl. in J. Soviet Math. **8** (1977)

[18] V. A. Solonnikov: *On general initial-boundary value problems for linear parabolic systems.* Proc. Steklov Math. Inst. **83**, 1-184 (1965).

[19] V. A. Solonnikov: *On an initial-boundary value problem for the Stokes system arising in the study of a problem with a free boundary.* Trudy Mat. Inst. Steklov. **188**, 150-188 (1990); English transl. in Proc. Steklov Inst. Math. **1991**, Issue 3, 191-239 (1991).

[20] V. A. Solonnikov: *Solvability of the problem of evolution of a viscous incompressible fluid bounded by a free surface on a finite time interval.* St. Petersburg Math. J. **3**, 189-220 (1992).

[21] V. A. Solonnikov and A. Tani: *Free boundary problem for a viscous compressible flow with a surface tension.* Carathéodory: An International Tribute, World Scientific Publ. Co., 1270-1303 (1991).

[22] A. Tani, S. Itoh and N. Tanaka: *The initial value problem for the Navier-Stokes equations with general slip boundary condtition.* Adv. Math. Sci. Appl. **4**, 51-69 (1994).