数理解析研究所講究録
第 1690 巻 2010 年 139-146

ON A STABILITY OF THE BURGERS VORTEX WITH RESPECT TO THREE DIMENSIONAL PERTURBATIONS

Dedicated to Professor Kenji Nishihara on his sixtieth birthday

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1. Introduction

The Burgers vortices are exact stationary solutions to the three dimensional Navier-Stokes equations, and they represent a balance between two basic mechanisms in fluid dynamics - vorticity stretching effect and diffusion effect. The Burgers vortices are also known as a simple model of vortex tubes which are coherent structures observed in turbulent flows [23, 12]. For this reason they has been widely studied physically [13, 18, 21, 3, 22] and mathematically [11, 2, 10, 7, 8, 9, 15, 16, 17, 5]. In this report we discuss three dimensional stability of the Burgers vortex and introduce a recent result obtained by [5] on this problem.

2. Formulation of the problem

We consider the Navier-Stokes equations for viscous incompressible flows in $\mathbb{R}^3$,

$$\partial_t V - \nu\Delta V + (V,\nabla)V + \frac{1}{\rho}\nabla P = 0, \qquad \nabla\cdot V = 0. \tag{2.1}$$

Here $V(x,t) = (V_1(x,t), V_2(x,t), V_3(x,t))^\top$ and $P(x,t)$ denote the velocity field and the pressure field, respectively, and the parameters in (2.1) are the kinematic viscosity $\nu > 0$ and the density $\rho > 0$. We assume that the velocity V has the form

$$V = V^s + U. \tag{2.2}$$

Here V^s is a given background straining flow defined by (2.3) below,

$$V^s(x) = \gamma(-\frac{x_1}{2}, -\frac{x_2}{2}, x_3)^\top = \gamma M x, \tag{2.3}$$

and U is the unknown perturbation velocity field. The parameter $\gamma > 0$ describes the magnitude of the straining flow, and M is a matrix of the form

$$M = \begin{pmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

By performing the scaling transformation

$$\tilde{x} = (\frac{\gamma}{\nu})^{\frac{1}{2}} x, \qquad \tilde{t} = \gamma t, \qquad \tilde{V} = \frac{V}{(\gamma\nu)^{\frac{1}{2}}}, \qquad \tilde{P} = \frac{P}{\rho\gamma\nu},$$

we may assume that $\gamma = \nu = \rho = 1$. For simplicity of notations we use x and t for $\tilde{x}$ and $\tilde{t}$. From the assumption of $V = V^s + U$, the equation for the vorticity field $\Omega = \nabla \times V = \nabla \times U$ is given by

$$\partial_t \Omega - L\Omega + (U, \nabla)\Omega - (\Omega, \nabla)U = 0, \qquad \nabla \cdot \Omega = 0. \tag{2.4}$$

Here L is a partial differential operator defined by

$$L\Omega = \Delta\Omega - (Mx, \nabla)\Omega + M\Omega. \tag{2.5}$$

The velocity field U is formally recovered from the vorticity field Ω via the Biot-Savart law

$$U(x,t) = (K_{3D} * \Omega)(x,t) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{(x-y) \times \Omega(y,t)}{|x-y|^3} dy. \tag{2.6}$$

If U is two dimensional, that is, if $U(x,t) = (U_1(x_h,t), U_2(x_h,t), 0)^\top$, $x_h = (x_1, x_2)^\top \in \mathbb{R}^2$, then the associated Ω becomes $\Omega(x,t) = (0,0,\Omega_3(x_h,t))^\top$, and (2.6) is replaced by the two dimensional Biot-Savart law

$$U_h(x_h,t) = (K_{2D} * \Omega_3)(x_h,t) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{(x_h - y_h)^\perp}{|x_h - y_h|^2} \Omega_3(y_h,t) dy_h. \tag{2.7}$$

Here $U_h = (U_1, U_2)^\top$ and $x_h^\perp = (-x_2, x_1)^\top$.
We set the vorticity field G as

$$G(x) = (0, 0, g(x_h))^\top, \qquad g(x_h) = \frac{1}{4\pi} e^{-|x_h|^2/4}. \tag{2.8}$$

Then by direct calculations we can check that $\{\alpha G\}_{\alpha \in \mathbb{R}}$ gives a family of stationary solutions to (2.4). The velocity field associated with G is

$$(2.9)\qquad U^G(x) = u^g(|x_h|^2)(-x_2, x_1, 0)^\top, \qquad u^g(r) = \frac{1}{2\pi r}(1 - e^{-\frac{r}{4}}).$$

The vorticity field αG is called the (axisymmetric) Burgers vortex. The parameter α is the circulation number which represents the magnitude of the vorticity field. To consider the asymptotic stability of the Burgers vortex, we first note the following lemma.

Lemma 2.1. *If $\Omega \in (L^1_{loc}(\mathbb{R}; L^1(\mathbb{R}^2)))^3$ satisfies $\nabla \cdot \Omega = 0$ in the sense of distributions, then there exists $\alpha \in \mathbb{R}$ such that $\int_{\mathbb{R}^2} \Omega_3(x_h, x_3)dx_h = \alpha$ for a.e. $x_3 \in \mathbb{R}$.*

Although it is easily proved by the integration by parts, Lemma 2.1 is useful since the quantity $\int_{\mathbb{R}^2} \Omega_3(x_h, x_3, t)dx_h$ is conserved under the equation (2.4). Especially, if the solution Ω to (2.4) converges to αG at time infinity, then the value α is determined in terms of the initial data, i.e., we must have $\alpha = \int_{\mathbb{R}^2} \Omega_3(x_h, x_3, 0)dx_h$.

3. Main results

To state our main results, we introduce function spaces. Since the Burgers vortex is essentially a two-dimensional flow, it is natural to choose a function space which allows for perturbations in the same class. Following [8], we assume that the perturbations are localized in the horizontal variables, but merely bounded in the vertical direction. For each $m > 1$ we set ρ_m by

$$(3.1)\qquad \rho_m(r) = (1 + \frac{r}{4m})^m, \qquad r \geq 0.$$

Then we introduce the weighted L^2 space

$$(3.2)\qquad L^2(m) = \{f \in L^2(\mathbb{R}^2) \mid \int_{\mathbb{R}^2} |f(x_h)|^2 \rho_m(|x_h|^2)dx_h < \infty\}$$

$$(3.3)\qquad L^2_0(m) = \{f \in L^2(m) \mid \int_{\mathbb{R}^2} f(x_h)dx_h = 0\}.$$

Next, we define the three-dimensional space $X(m)$ as the set of all $\phi : \mathbb{R}^3 \to \mathbb{R}$ for which the map $x_h \mapsto \phi(x_h, x_3)$ belongs to $L^2(m)$ for any $x_3 \in \mathbb{R}$, and is a bounded and continuous function of x_3. In other words, we set

$$(3.4)\qquad X(m) = BC(\mathbb{R}; L^2(m)), \qquad X_0(m) = BC(\mathbb{R}; L^2_0(m)),$$

which are equipped with the norm

$$\|\phi\|_{X(m)} = \sup_{x_3 \in \mathbb{R}} \|\phi(\cdot, x_3)\|_{L^2(m)}.$$

Then our main result is stated as follows.

Theorem 3.1. *Let $m > 2$. Assume that $\Omega_0 = (\Omega_{0,1}, \Omega_{0,2}, \Omega_{0,3})^\top$ belongs to $X(m)^3$ and satisfies $\nabla \cdot \Omega_0 = 0$. Set $\alpha = \int_{\mathbb{R}^2} \Omega_{0,3}(x_h, x_3)dx_h$. Then there exist δ and C such that if $\|\Omega_0 - \alpha G\|_{X(m)^3} \leq \delta$, then Eq. (2.4) has a unique solution $\Omega \in L^\infty(0, \infty; X(m)^3)$ with initial data Ω_0. Moreover, it satisfies*

$$\|\Omega(t) - \alpha G\|_{X(m)^3} \leq Ce^{-\frac{t}{2}} \|\Omega_0 - \alpha G\|_{X(m)^3}, \qquad t \geq 0. \tag{3.5}$$

Here δ and C depend only on α and m.

Remark 3.1. Theorem 3.1 was firstly proved by [8] under the assumption of $|\alpha| \ll 1$. The smallness of $|\alpha|$ is removed by [5].

Set

$$\mathbb{X}(m) = X(m) \times X(m) \times X_0(m), \tag{3.6}$$

which is invariant under (2.4). Theorem 3.1 shows that the Burgers vortex αG is asymptotically stable with respect to perturbations in $\mathbb{X}(m)$, for any value of the circulation $\alpha \in \mathbb{R}$. However, the constants δ and C in Theorem 3.1 depend on α in such a way that $\delta(\alpha, m) \to 0$ and $C(\alpha, m) \to \infty$ as $|\alpha| \to \infty$.

To prove Theorem 3.1 it is useful to consider the equation for $\omega = \Omega - \alpha G$ in $\mathbb{X}(m)$,

(3.7)
$$\begin{cases} \partial_t \omega - (L - \alpha\Lambda)\omega &= -(K_{3D} * \omega, \nabla)\omega + (\omega, \nabla)K_{3D} * \omega, \quad x \in \mathbb{R}^3, \quad t > 0, \\ \nabla \cdot \omega(t) &= 0, \quad x \in \mathbb{R}^3, \quad t > 0, \\ \omega|_{t=0} &= \omega_0, \quad x \in \mathbb{R}^3. \end{cases}$$

Here Λ is a linear operator defined by

$$\Lambda\omega = (K_{3D}*G, \nabla)\omega - (\omega, \nabla)K_{3D}*G + (K_{3D}*\omega, \nabla)G - (G, \nabla)K_{3D}*\omega. \tag{3.8}$$

The key step to prove Theorem 3.1 is to analyze the linearized problem

$$\begin{cases} \partial_t \omega - (L - \alpha\Lambda)\omega &= 0, \quad x \in \mathbb{R}^3, \ t > 0 \\ \omega|_{t=0} &= \omega_0 \quad x \in \mathbb{R}^3. \end{cases} \tag{3.9}$$

Especially, $L - \alpha\Lambda$ has a uniform spectral gap for all $\alpha \in \mathbb{R}$, which leads to a uniform decay $e^{-\frac{t}{2}}$ in (3.5). More precisely, we can show

Theorem 3.2. *Let $m > 2$. Assume that $\omega_0 = (\omega_{0,1}, \omega_{0,2}, \omega_{0,3})^\top \in \mathbb{X}(m)$ satisfies $\nabla \cdot \omega_0 = 0$. Then Eq. (3.9) has a unique solution $\omega \in L^\infty(0,\infty;\mathbb{X}(m))$ with initial data ω_0 and it satisfies*

$$\|\omega(t)\|_{\mathbb{X}(m)} \le Ce^{-\frac{t}{2}}\|\omega_0\|_{\mathbb{X}(m)}, \qquad t \ge 0. \tag{3.10}$$

Here C depends only on α and m.

3.1. **Key lemmas for the linearized operator $L - \alpha\Lambda$.** In this section we collect several properties of L and Λ which are keys to prove Theorem 3.2. We first consider the operator L. Setting

$$\mathcal{L}_h = \Delta_h + \frac{x_h}{2}\cdot\nabla_h + 1 = \sum_{j=1}^{2}\partial_{x_j}^2 + \sum_{j=1}^{2}\frac{x_j}{2}\partial_{x_j} + 1, \tag{3.11}$$

$$\mathcal{L}_3 = \partial_{x_3}^2 - x_3\partial_{x_3}, \tag{3.12}$$

we write $L\omega$ as

$$L\omega = \begin{pmatrix} L_h\omega_h \\ L_3\omega_3 \end{pmatrix} = \begin{pmatrix} (\mathcal{L}_h + \mathcal{L}_3 - \frac{3}{2})\omega_h \\ (\mathcal{L}_h + \mathcal{L}_3)\omega_3 \end{pmatrix}.$$

Since the semigroups associated with $\mathcal{L}_h$ and $\mathcal{L}_3$ are explicitly given by

$$e^{t\mathcal{L}_h}\phi = \frac{1}{4\pi a(t)}\int_{\mathbb{R}^2} e^{-\frac{|x_h - y_h e^{-\frac{t}{2}}|^2}{4a(t)}}\phi(y_h)dy_h, \qquad a(t) = 1 - e^{-t}, \tag{3.13}$$

$$e^{t\mathcal{L}_3}\phi = \frac{1}{\sqrt{2\pi b(t)}}\int_{\mathbb{R}} e^{-\frac{|x_3 e^{-t} - y_3|^2}{2b(t)}}\phi(y_3)dy_3, \qquad b(t) = 1 - e^{-2t}, \tag{3.14}$$

we have the representation of the semigroup for L_3 such as

$$e^{tL_3}\phi = \frac{1}{\sqrt{2\pi b(t)}}\int_{\mathbb{R}} e^{-\frac{|x_3 e^{-t} - y_3|^2}{2b(t)}}(e^{t\mathcal{L}_h}\phi(\cdot, y_3))(x_h)dy_3. \tag{3.15}$$

Hence the semigroup associated with L is given by

$$e^{tL}\omega_0 = (e^{-\frac{3}{2}t}e^{tL_3}\omega_{0,1},\ e^{-\frac{3}{2}t}e^{tL_3}\omega_{0,2},\ e^{tL_3}\omega_{0,3})^\top. \tag{3.16}$$

In [6] the following estimates for $e^{t\mathcal{L}_h}$ are obtained:

$$\|e^{t\mathcal{L}_h}f\|_{L^2(m)} \le C\|f\|_{L^2(m)}, \qquad f \in L^2(m),\ m > 1, \tag{3.17}$$

$$\|e^{t\mathcal{L}_h}f\|_{L^2(m)} \le Ce^{-\frac{t}{2}}\|f\|_{L^2(m)}, \qquad f \in L^2_0(m),\ m > 2. \tag{3.18}$$

Combining these with (3.13)-(3.16), we can show that

$$\|e^{tL}f\|_{\mathbb{X}(m)} \le Ce^{-\frac{t}{2}}\|f\|_{\mathbb{X}(m)}, \qquad f \in \mathbb{X}(m), \qquad m > 2. \tag{3.19}$$

In particular, when $|\alpha|$ is sufficiently small, we have a control of the spectrum of $L - \alpha\Lambda$ from the general perturbation theory for linear operators. However, when $|\alpha|$ is not small, we need to use additional special structures of L and Λ in order to estimate the spectrum of $L - \alpha\Lambda$.

To overcome the difficulty for the case of not small $|\alpha|$, we first observe that L and Λ have a simple dependence on x_3 variable such as

$$[\partial_{x_3}, L] = \partial_{x_3} L - L\partial_{x_3} = -\partial_{x_3}, \tag{3.20}$$

$$[\partial_{x_3}, \Lambda] = 0, \tag{3.21}$$

which gives a relation $\partial_{x_3}^k e^{t(L-\alpha\Lambda)} = e^{-kt} e^{t(L-\alpha\Lambda)} \partial_{x_3}^k$. Hence, as a first step, we easily get an exponential decay estimate for $\partial_{x_3}^{k_0} e^{t(L-\alpha)}$ at least for sufficiently large k_0. So the second step is to show that $\partial_{x_3}^{k-1} e^{t(L-\alpha\Lambda)}$ is essentially estimated by $\partial_{x_3}^k e^{t(L-\alpha\Lambda)}$, which enables us to get the estimate for $e^{t(L-\alpha\Lambda)}$ itself by the backward induction on k.

For the proof of the second step we decompose $L - \alpha\Lambda$ as follows. Set Λ_j, $j = 1, 2, 3, 4$, as

$$\begin{aligned} \Lambda\omega &= (U^G, \nabla)\omega - (\omega, \nabla)U^G + (K_{3D} * \omega, \nabla)G - (G, \nabla)(K_{3D} * \omega) \\ &= \Lambda_1\omega - \Lambda_2\omega + \Lambda_3\omega - \Lambda_4\omega, \end{aligned} \tag{3.22}$$

and also set $\tilde{\Lambda}_3$ as

$$\tilde{\Lambda}_3\omega = (K_{2D} * \omega_3, \nabla)G. \tag{3.23}$$

Using these notations, we define linear operators $L_{2D,\alpha}$ and N by

$$L_{2D,\alpha}\omega = \begin{pmatrix} (\mathcal{L}_h - \frac{3}{2} - \alpha\Lambda_1 + \alpha\Lambda_2)\omega_h \\ (\mathcal{L}_h - \alpha\Lambda_1 - \alpha\tilde{\Lambda}_3)\omega_3 \end{pmatrix},$$

$$N\omega = (\Lambda_3 - \tilde{\Lambda}_3 - \Lambda_4)\omega.$$

Note that $L_{2D,\alpha}$ is a two dimensional operator in the sense that it does not depend on x_3 variable. Now we can write $L - \alpha\Lambda$ as

$$L - \alpha\Lambda = L_{2D,\alpha} + \mathcal{L}_3 - \alpha N, \tag{3.24}$$

and thus, $e^{t(L-\alpha\Lambda)}$ satisfies the integral equation

$$e^{t(L-\alpha\Lambda)} = e^{t(L_{2D,\alpha}+\mathcal{L}_3)} - \alpha \int_0^t e^{(t-s)(L_{2D,\alpha}+\mathcal{L}_3)} N e^{s(L-\alpha\Lambda)} ds. \tag{3.25}$$

The integral equation (3.25) is useful to get the desired estimates. We first consider the semigroup $e^{t(L_{2D,\alpha}+\mathcal{L}_3)} = (e_h^{t(L_{2D,\alpha}+\mathcal{L}_3)}, e_3^{t(L_{2D,\alpha}+\mathcal{L}_3)})^\top$.

Lemma 3.1. *Let $m > 2$. Then we have*

(3.26) $\partial_{x_3}^k e^{t(L_{2D,\alpha}+\mathcal{L}_3)} = e^{-kt} e^{t(L_{2D,\alpha}+\mathcal{L}_3)} \partial_{x_3}^k,$

(3.27) $\|e_h^{t(L_{2D,\alpha}+\mathcal{L}_3)} f_h\|_{\mathbb{X}(m)} \le Ce^{-t}\|f_h\|_{\mathbb{X}(m)}, \qquad f_h \in X(m)^2, \qquad t > 0,$

(3.28) $\|e_3^{t(L_{2D,\alpha}+\mathcal{L}_3)} f_3\|_{\mathbb{X}(m)} \le Ce^{-\frac{t}{2}}\|f_3\|_{\mathbb{X}(m)}, \qquad f_3 \in X_0(m), \qquad t > 0.$

The equality (3.26) follows from $[\partial_{x_3}, L_{2D,\alpha}] = 0$ and $[\partial_{x_3}, \mathcal{L}_3] = -\partial_{x_3}$. The details of the proof for (3.27) and (3.28) will be given in [5]. Next we need the estimate for N to control the second term in the right hand side of (3.25). We write

$$N\omega = (N_1\omega,\ N_2\omega,\ N_3\omega)^\top = (N_h\omega,\ N_3\omega)^\top.$$

Lemma 3.2. *Let $m > 2$. Then $\partial_{x_3}^k N = N\partial_{x_3}^k$ holds, and we have for any $f = (f_h, f_3)^\top \in \mathbb{X}(m)$,*

(3.29) $\qquad \|N_h f\|_{\mathbb{X}(m)} \le C\|\partial_{x_3} f\|_{\mathbb{X}(m)},$

(3.30) $\qquad \|N_3 f\|_{\mathbb{X}(m)} \le C(\|\partial_{x_3} f\|_{\mathbb{X}(m)} + \|f_h\|_{X(m)^2}).$

The important fact in Lemma 3.2 is that $\|f_3\|_{X(m)}$ does not appear in the right hand side of (3.30). Combining Lemma 3.1 and Lemma 3.2 with the integral equation (3.25), we finally obtain

(3.31)

$$\|\partial_{x_3}^k e^{t(L-\alpha\Lambda)} f\|_{\mathbb{X}(m)} \le Ce^{-(\frac{1}{2}+k)t}\|\partial_{x_3}^k f\|_{\mathbb{X}(m)} + C\int_0^t e^{-(\frac{1}{2}+k)(t-s)}\|\partial_{x_3}^{k+1} e^{s(L-\alpha\Lambda)} f\|_{\mathbb{X}(m)} ds.$$

We note that, from the parabolic regularity, we may assume that f is smooth and $\partial_{x_3}^k f \in \mathbb{X}(m)$ for each k in (3.31). Then Theorem 3.2 is proved by the backward induction on k for $\partial_{x_3}^k e^{t(L-\alpha\Lambda)}$. The details will be stated in [5] and we omit them here.

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