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STABILITY OF VISCOUS SHOCK WAVES

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To my friend, Professor Kenji Nishihara with admiration

ABSTRACT. The idea of Kenji Nishihara on the nonlinear stability of shock waves for isentropic Navier-Stokes equations have been followed up by many researchers. We describe some of the thinkings that are motivated by the Nishihara method.

1. INTRODUCTION

Consider the hyperbolic-parabolic conservation laws

$$u_t + f(u)_x = (B(u)u_x)_x. \tag{1}$$

The most basic example is the isentropic Navier-Stokes equations

$$\begin{cases} \rho_t + (\rho v)_x = 0, \\ (\rho v)_t + (\rho v^2 + p(\rho))_x = (\kappa v_x)_x. \end{cases} \tag{2}$$

The viscous shocks, the traveling waves

$$(\rho, \rho v)(x,t) = \Phi(x - st)$$

of the system have been constructed long time ago. But their stability turned out to be hard to prove, in spite of the intense interests over the years. Thus the shock wave community was stunned by the paper [6] establishing their stability under the zero total mass constraint of the perturbation

$$(\rho, \rho v)(x,0) = \Phi(x) + (\bar\rho, \bar\rho\bar v)(x,0),$$

$$\int_{-\infty}^{\infty} (\bar\rho, \bar\rho\bar v)(x,0)dx = 0. \tag{3}$$

There are surprises about the paper: The first being the necessity of the zero total mass assumption (3). Then there is the simplicity in conception of the energy method in the paper.

Besides drawing on the deep understanding of the energy method for the perturbation of constant states for general system (1) by the Kyoto School, [5], [3], the energy method in [6] makes explicit use of the compressibility of the shock wave. The compressibility is the direct consequence of the nonlinearity of the system and the second law of thermodynamics, expressed in the form of the positivity of the viscosity coefficient $\kappa > 0$. The zero total mass condition makes essential use of another fundamental fact, that the mass and momentum are the conserved quantities of the system. Thus the two most fundamental properties of the system (2) are used in the paper [6] in clean and effective way. The remaining task for the subsequent researchers is to find ways of removing the zero total mass condition and the resulting consequences.

2. ZERO TOTAL MASS ASSUMPTION

Shock waves, being traveling waves of the autonomous system (1), (2), are *orbitally stable.* Thus a perturbation in general changes the location of the shock as it evolves in time. The zero total mass assumption implies that the shock location does not change time-asymptotically. This allows for the concentration of the analysis on the essential stability mechanism, the compressibility of the shock. The compressibility condition is the consideration of the hyperbolic conservation laws

$$u_t + f(u)_x = 0. \tag{4}$$

The characteristic values $\lambda_i(u)$ for (4) are the eigenvalues of $f'(u)$. For isentropic Navier-Stokes equations (2), the characteristic values are

$$\lambda_1 = v - c,\ \lambda_2 = v + c,$$

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where c is the sound speed, $c^2 = p'(\rho)$. An i-shock is a consequence of the compressibility of λ_i. For gases including the polytropic gases

$$p(\rho) = a\rho^\gamma, \ \gamma > 0,$$

there is the genuinely nonlinear property, [8], when

$$\frac{d^2}{d\tau^2}p \neq 0, \ \tau \equiv 1/\rho.$$

This implies that an i-shock $\Phi(x - st)$ is strongly compressible in that

$$\frac{d}{dx}\lambda_i(\Phi(x)) < 0.$$

In particular

$$\lambda(\Phi(-\infty)) < s < \lambda_i(\Phi(\infty)).$$

This inequality has the geometric meaning in that the i-waves propagate toward the i-shock Φ which is the basic reason of the stability of the shock. One can thus use the weighted energy method to obtain the decay rate, [4]. The zero total mass assumption precludes many of the other wave phenomena, chief among them is the nonlinear coupling of waves pertaining to distinct characteristic families. On the other hand, around the shock wave, the compressibility property is the main mechanism for its stability. There is the need to locate the shock wave through *wave tracing*. The tracing is done by the local conservation laws in place of the zero total mass assumption. With such a wave tracing setting the weighted energy method is appropriate for estimates around the shock, [11], [15]. Thus the energy method in [6] has been generalized to general situations.

When the shock wave is not pertaining to a genuinely nonlinear characteristic field, the compressibility property is weakened, and the stability analysis is largely open. There is an interesting result for scalar conservation laws, [7].

The tracing requires at least some more local, pointwise estimates of the solutions, a topic to be described in the next section.

3. Green's function approach

Pointwise estimates aim at describing the coupling of waves pertaining to different characteristic field for general systems. The study of the Green's functions is essential for the quantitative study. For system (1) linearized around a constant state, there is now an explicit expression for the Green's function, starting with the isentropic Navier-Stokes equation by Yanni Zeng, [16], and extended to general systems with physical viscosity, [12]. The idea of inversing the Fourier transform using the complex analytic method in [16] is a key step in the construction of the Green's function. With Green's function explicitly constructed, one can then use the Duhamel's principle to study the quantitative properties of the solutions. The Green's function captures the basic two properties of the wave behavior: Because physical systems are not uniformly parabolic, but hyperbolic-parabolic, the δ-function at the initial data propagates and decays into the Green's function at later time. This is so also for the dissipative, non-parabolic, systems such as the Boltzmann equation, [10]. That the viscosity matrix is not a diagonal matrix induces rich wave coupling phenomena, [13].

The study of shock stability, initiated in [6] for the most basic physical system (2), is so far complete only for systems with artificial viscosity

$$u_t + f(u)_x = u_{xx},$$

[9], [14]. In [14], it is shown that the shock is nonlinearly stable when both the shock and the perturbation are weak, though their relative strengths are allowed to vary. The study for physical viscosity is being completed by the same authors. one notes in the passing that the important zero dissipative limit has been studied only for systems with artificial viscosity, [1].

In [2], Kenji Nishihara and his coauthor did a detailed and very useful study of the Green's function for the Burgers equation linearized around a rarefaction wave. This work reveals the strong hyperbolic linear dissipation for the rarefaction wave in addition to the usual sublinear parabolic dissipation. Shih-Hsien Yu and the author are following this line of research and completing a work on rarefaction waves for the system.

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