数理解析研究所講究録
第 1691 巻 2010 年 1-7

A Note on Tatami Tilings

Artiom ALHAZOV[1,2]*, Kenichi MORITA[1], Chuzo IWAMOTO[1]

[1] IEC, Department of Information Engineering
Graduate School of Engineering, Hiroshima University
Higashi-Hiroshima 739-8527 Japan
E-mail: {morita@iec.,chuzo@}hiroshima-u.ac.jp

[2] Institute of Mathematics and Computer Science
Academy of Sciences of Moldova
Academiei 5, Chişinău MD-2028 Moldova
E-mail: artiom@math.md

Abstract Inspired by the rules of Japanese tatami layouts, we present in these paper some characterization results of tiling rectangles with 1x2 and 2x1 blocks in case a single 1x1 block is allowed. The tatami condition is that four blocks cannot meet in the same point.

1 Introduction

In Japan, the size of a room is typically measured by the number of tatami mats (jō). There are various rules concerning the number and layout of tatami mats. In this paper we focus on just one rule: in any layout there is never a point where the corners of three or four mats touch. It is equivalent to the requirement that four mats never meet in one point.

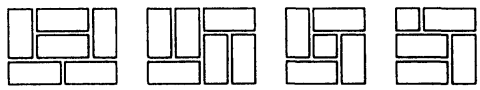

Figure 1: 6-jō and 4.5 -jō layouts

We formalize this concept in Section 2 and we call tilings satisfying this property the tatami tilings. The case when only dominoes are used has been characterized in [1] and [2]; one explains why the number of tilings is unexpectedly small when the ratio of dimensions is not big. In Subsection 3.1 we recall the known results for tatami tilings. In Subsection 3.2 we characterize tilings of odd area, using dominoes and a single monomino. In Section 4 we prove this characterization.

2 Definitions

We formalize a rectangular array as a set $A = \{(i,j) \mid 1 \leq i \leq m,\ 1 \leq j \leq n\}$, and call m and n its height and width, respectively. We recall a classic algebra fact

*Artiom Alhazov gratefully acknowleges the support of the Japan Society for the Promotion of Science and the Grant-in-Aid for Scientific Research, project 20·08364. He also acknowledges the support by the Science and Technology Center in Ukraine, project 4032.

that an equivalence relation defined on a set induces a partition on it, and vice versa. A domino tile (if there is no confusion, we may omit the word domino) is a subset of A of type $\{(i,j),(i+1,j)\}$ or $\{(i,j),(i,j+1)\}$; we call it horizontal in the first case and vertical in the second case. A monomino is a single element of A. A tiling is a partition of a rectangular grid in tiles.

A tiling T is called a tatami tiling if for every c,d, $1 \le c \le m-1$, $1 \le d \le n-1$, elements (c,d), $(c+1,d)$, $(c,d+1)$, $(c+1,d+1)$ belong to at most 3 tiles of T.

Figure 1 graphically represents tatami tilings with height 3 and widths 4 and 3.

3 Building blocks

3.1 Even area tilings

In this subsection we speak of tatami tilings that only use domino tiles. Naturally, the area of the corresponding rectangles must be even (whereas if a single monomino is used, then the area is odd). Without restricting generality, assume that $m \le n$. It has been shown in [1] (see also [2]) that **all** tatami tilings are horizontal compositions of specific building blocks. For even m, tilings consist of H_m and V_m with blocks L_m between them and, possibly, on the left and right edges, see Figure 2.

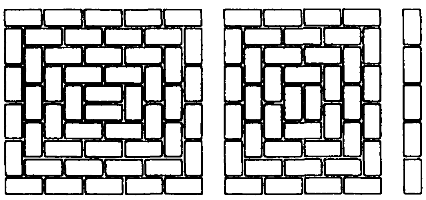

Figure 2: Tilings H_{10}, V_{10} and L_{10}

For odd m, tilings consist of H_m, V_m and their vertical reflections, see Figure 3.

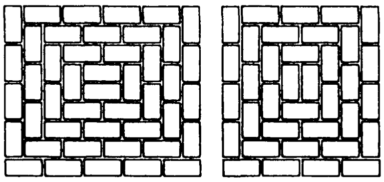

Figure 3: Tilings H_9 and V_9

In [1] one defines the function $t(m,n)$ of the number of tatami tilings m by n and describes how to compute it. In [2] one also gives the corresponding generating functions.

3.2 Odd area tilings

Without restricting generality we assume $m \le n$. We claim that in case a single monomino is used, **all** tatami tilings are horizontal compositions of the building blocks (and reflections of some of them) H_m, V_m, M_m, CH_m, CV_m, L_m, with the following conditions: only one of the latter four blocks is used in a tiling, and exactly once; block CH_m or CV_m or its horizontal reflection can only be used in the left (only in the right if it is also vertically reflected). The blocks are illustrated in Figure 4, and the proofs follow after that.

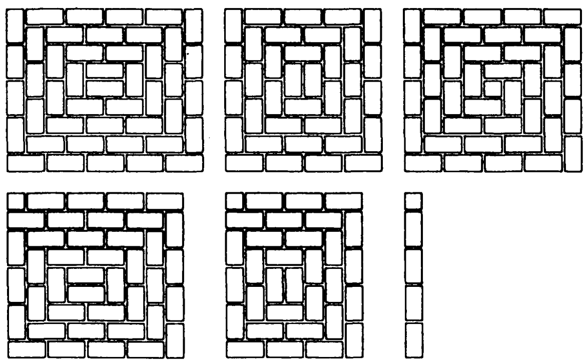

Figure 4: Tilings H_9, V_9, M_9, CH_9, CV_9 and L_9

4 Auxiliary results for proofs

Lemma 1 *Let $A = \{(i,j) \mid 1 \leq i \leq m,\ 1 \leq j \leq n\}$ and $B \subseteq A$ is a region of a rectangle. Let $B_1 = \{(i,j) \in B \mid i+j \text{ is even}\}$ and $B_2 = \{(i,j) \in B \mid i+j \text{ is odd}\}$. Then B can only be tiled if $|B_1| - |B_2|$ is either -1, 0 or 1, and the monomino is in B_2, absent or in B_1, respectively.*

Proof. B_1 and B_2 represent a "checkerboard" partition of B, and any domino consists of one element of B_1 and one element of B_2. □

Corollary 1 *If the area of a rectangle is odd, then one monomino (i,j) is used where $i+j$ is even.*

Proof. By Lemma 1, taking $B = A$. □

Corollary 2 *A triangular corner (see Figure 5) cannot be tiled, except if its area is 1 or 3.*

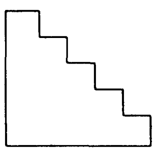

Figure 5: Triangle

Proof. If the height of a triangle B is m by m, then its area is $m(m+1)/2$. Partition it as in Lemma 1, then $|\,|B_1| - |B_2|\,| = \lfloor (m+1)/2 \rfloor$. □

Corollary 3 *A right trapezoid region of a rectangle (see Figure 6) cannot be tiled by dominoes only, unless its height is 1.*

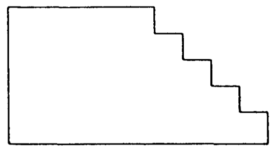

Figure 6: Right trapezoid

Proof. If the height of the trapezoid is m, then partitioning it as in Lemma 1 yields $|\,|B_1| - |B_2|\,|$ equal to either $\lfloor (m+1)/2 \rfloor$ or (if it is a union of a triangle with height m and an odd-area rectangle) $\lfloor (m-1)/2 \rfloor$. The statement follows from Lemma 1. □

For the next result we join two right trapezoids with equal long bases, see Figure 7, left.

Lemma 2 *A "pentagon" region (see Figure 7, left) cannot be tiled by dominoes only, unless one of two leftmost elements is in the middle row).*

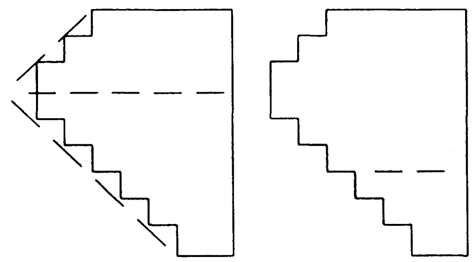

Figure 7: Pentagon region from Lemma 2

Proof. Such a pentagon B consists of a symmetrical region and a smaller right trapezoid T, see Figure 7, right. Notice that in the symmetrical region the number of elements the sum of whose coordinates is odd is equal to the number of elements the sum of whose coordinates is even, so the value of $|\,|B_1| - |B_2|\,|$ in the partitioning defined in Lemma 1 is the same as the corresponding value for the trapezoid T. By Corollary 3, this value cannot be 0 unless the height of T is 1, and the statement of this lemma follows from Lemma 1. □

Lemma 3 *If a monomino does not touch any side of a rectangle m by n, $m \leq n$, then it must be in row $(m+1)/2$.*

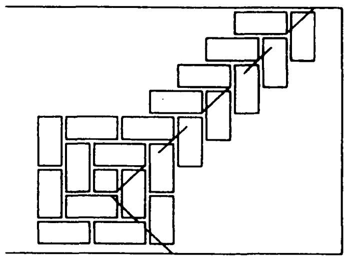

Figure 8: Tiling around a monomino

Proof. The claim trivially holds for $m = 1$, so from now on we suppose $m \geq 3$. There must be at least one domino touching the monomino with its long side, otherwise the tatami condition would not hold. To satisfy the tatami condition, there must be another domino touching both the monomino and the domino with its long side. Similarly, a third domino must touch the monomino and the second domino with its long side, and a fourth domino must touch the monomino and the third domino with its long side. This creates a 3 by 3 block. While the block does not touch any side of the rectangle, by similar logic it is clear that it is a part of a larger square block with odd dimensions.

Multiple tiles are forced, see Figure 8. Disregarding the vertical tiles in the right part, it is clear that a pentagon region (see dashed lines) is isolated. The lemma statement follows from Lemma 2. □

Lemma 4 *A monomino cannot touch only the left side of an odd-area rectangle m by n, $m \leq n$.*

Proof. (idea) The claim follows by a proof technique similar to the one from Theorem 2.2 of [2], by proceeding as if the monomino was a part of horizontal domino extending beyond the left edge of the rectangle. The only special case is row 2 or row $m-1$ (excluded by Corollary 1). □

Lemma 5 *A horizontal domino cannot touch the left side of an odd-area rectangle m by n, $m \leq n$, unless it is in rows $1, 2, m-1, m$.*

Proof. Suppose a horizontal domino touches the left side of the rectangle only. The width is odd, so consider the leftmost non-horizontal tatami t in this row. If t is a monomino and it touches the right side of the rectangle, then the claim is reduced to Lemma 4. If t is a monomino and it does not touch the right side of the rectangle, then two horizontal tiles are forced above and below the left edge of t, and any tatami to the right of t contradicts the tatami condition.

If t is a vertical domino, then the proof is analogous to the proof of case 2 in Theorem 2.2 of [2], with one difference in subcase 2-1, namely, if the top left triangle is small. The top left triangle of 1 block is excluded from the claim of the lemma, so the remaining case is when the top left triangle consists of 3 blocks. Notice that this triangle is isolated, so we can freely choose to put a monomino in the leftmost column. However, Lemma 4 shows that such tiling is impossible, concluding the proof. □

5 Proving the tiling characterization

Without restricting generality, we again assume $m \leq n$. It follows from Lemma 4 and

Lemma 5 that the left edge of the rectangle is touched by vertical dominoes only, except a monomino may be placed in one corner and horizontal dominoes may be placed in the top two rows and the bottom two rows. Hence, only the following cases are possible.

- No horizontal tiles touch the left edge. Since the height is odd, the monomino must be placed in the corner, so we obtain block L_m, see Figure 4.

- A horizontal tile touches the second top row or the second bottom row. Due to Lemma 5, the monomino must be placed in the corner next to it. Since the height is odd, another horizontal tile must be placed in the other left corner, and the rest of the left column is tiled by vertical dominoes. More vertical tiles are forced, and then multiple horizontal tiles are forced, see Figure 9, right. Then we have two choices how to tile the two leftmost empty elements. Using a vertical domino forces us to complete CV_m block, see Figure 4. Using two horizontal dominoes forces us to complete CH_m block, see Figure 4.

- Neither of the previous cases hold. Since the height is odd, exactly one horizontal tile must be placed in one of the left corners, and the rest of the left column is tiled by vertical dominoes. More vertical tiles are forced, and then we have choices.

  - We do not use the monomino in the top or bottom row. We recall that in this case the monomino cannot be used except in the middle row. Hence, multiple horizontal tiles are forced, see Figure 9, left. Then we have three choices how to tile the two leftmost empty elements. Using the monomino in the middle row forces us to complete M_m block, see Figure 4. Using a vertical domino forces us to complete V_m block, see Figure 4. Using two horizontal dominoes forces us to complete H_m block, see Figure 4.

  - The monomino is in the top or bottom row. The top row is excluded by Corollary 1. The monomino in the bottom row gives us the following cases: 1)we complete CV_m or CH_m rotated by 180 degrees, see Figure 4, 2)a contradiction caused by some horizontal domino t and two other dominoes over and under the right part of t with common corners with t (see Figure 10), so the tiling cannot continue to the right, while another horizontal domino is forced to the right, excluding the possibility that t touches the right side of the rectangle.

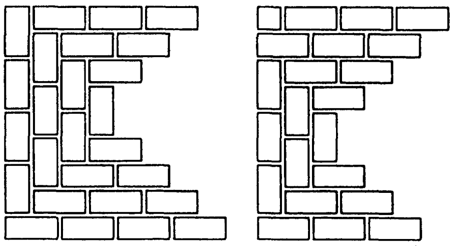

Figure 9: Tiling left-to right

Figure 10: Tiling cannot continue to the right

In each case, one of the 6 mentioned blocks is completed, and then we proceed with the remaining rectangle. Since the monomino has to be used exactly once, any tiling will have exactly one occurrence of L_m, M_m, CH_m or CV_m. It is also easy to see that if a monomino is used in a corner of a block, then this must be a corner of entire rectangle: any possible left neighbor of the horizontal domino under the monomino would violate the tatami condition.

6 Enumerating the tilings

We recall that the function $t(m,n)$ was defined in [1], see also [2]. Similarly, we define the function $t_1(m,n)$ of tilings where at most one monomino is used. Note that reflections and rotations of the same tiling are counted multiple times, unless that tiling has corresponding symmetries. Since $t_1(m,n) = t(m,n)$ if mn is even, it suffices to define t_1 for odd values of m and n ($t(m,n)$ was 0 there). Clearly, $t_1(m,n) = t_1(n,m)$, so it suffices to consider the case $m \leq n$.

It is very easy to see that in case of only one row and n columns, the tiling is defined by the position of the monomino between $(n-1)/2$ dominoes, so

$$t_1(1,n) = (n+1)/2.$$

Finally, if $m \geq 3$, then we split the function as follows:

$$t_1(m,n) = t_2(m,n) + t_3(m,n),$$

where t_2 enumerates the tilings with CH_m/CV_m and t_3 enumerates all other tilings. We also need the function $c(m,n)$ of the number of representations of n as an ordered sum of numbers $m+1$ and $m-1$ (we only need it for odd values of $m \geq 3$):

$$c(m,n) = \begin{cases} 0 & , \quad n < 0 \\ 1 & , \quad n = 0, \\ c(m,n-m+1) & \\ +c(m,n-m-1) & , \quad n > 0. \end{cases}$$

If m,n are odd and $m \geq 5$, then

$$t_2(m,n) = 4c(m,n-m) + 4c(m,n-m-2).$$

Indeed, every tiling with CH_m/CV_m consists of a tiling by dominoes only, obtained by removing that block. Multiplier 4 is used to account for the reflections. Note that the block CV_3 degenerates into L_3, and the corresponding tilings are counted in t_3. Therefore, $t_2(3,n) = 4c(m,n-m)$.

We also assume $t_3(m,n) = 0$ if $n \leq 0$. If m,n are odd and $m \geq 3$, then

$$t_3(m,n) = 2c(m,n-1) + 2c(m,n-m) + t_3(m,n-m-1) + t_3(m,n-m+1).$$

Indeed, the above formula sums up the cases depending on whether the leftmost block is L_m, M_m, H_m or V_m, respectively. Multiplier 2 is used to account for vertical reflections.

Counting only incongruent tilings could be done in a similar way, but we do not focus on this here. Below is a table of t_1 for small values of m and n.

t_1	1	3	5	7	9	11	13	15
1	1	2	3	4	5	6	7	8
3	2	10	18	38	72	136	250	454
5	3	18	10	8	18	24	32	52
7	4	38	8	10	4	4	18	20
9	5	72	18	4	10	4	0	4
11	6	136	24	4	4	10	4	0
13	7	250	32	18	0	4	10	4
15	8	454	52	20	4	0	4	10

7 Generating functions

An ordinary generating function of a sequence is a formal power series whose coefficients coincide with elements of that sequence. The generating functions for the number of domino tilings were given in [2]. We now focus on $t'(m,n) = t_1(m,n) - t(m,n)$, i.e., on the number of tilings using exactly one monomino. Assume m is fixed and consider the sequence $t'(m,m), t'(m,m+1), \cdots$. We construct the generating functions f_m where the coefficient of z^n equals to $t'(m,n)$ whenever $m \geq n$.

Case $m = 1$ and no monominoes: $B = 1 + z^2B$ (empty or starts with a horizontal tile). This yields $B = 1/(1-z^2)$.

Case $m = 1$ and one monomino: $A = zB + z^2A$ (starts with a monomino or a horizontal tile). Solving for A yields

$$f_1 = \frac{z}{(1-z^2)^2}.$$

In what follows we only consider $m \geq 3$, m odd.

Computing c: $A = 1 + z^{m-1}A + z^{m+1}A$ (from formula in the previous section). This yields $A = 1/(1 - z^{m-1} - z^{m+1})$.

Computing t_2: $B = 4z^{m-2}A + 4z^mA$ (from formula in the previous section). This yields $B = 4(z^{m-2} + z^m)/(1 - z^{m-1} - z^{m+1})$ for $m \geq 5$. Omitting $4z^{m-2}$ for $m = 3$, we get $B = 4z^m/(1 - z^{m-1} - z^{m+1})$.

Computing t_3: $C = z^{m-1}C + z^{m+1}C + 2zA + 2z^mA$ (from formula in the previous section). This yields $C = 2(z + z^m)/(1 - z^{m-1} - z^{m+1})^2$.

Finally, $B + C$ yields: for $m \geq 5$, m odd

$$f_m = \frac{1}{(1 - z^{m-1} - z^{m+1})^2} \cdot (2z + 4z^{m-2} + 6z^m - 4z^{2m-3} - 8z^{2m-1} - 4z^{2m+1}),$$

while for $m = 3$ the generating function is

$$f_3 = \frac{2z + 6z^m - 4z^{2m-1} - 4z^{2m+1}}{(1 - z^{m-1} - z^{m+1})^2}.$$

We recall that for $m = 1$ we had

$$f_1 = \frac{z}{(1-z^2)^2}.$$

8 Multiple monominoes

We have given the characterizations of tiling rectangles by dominoes and a single monomino. Note that it would require at least two monominoes to tile a rectangle 7 by 10. It would also require at least three monominoes to tile a rectangle 9 by 13. However, the variety of tilings with arbitrary number of monominoes is quite "wild" in sense that such tilings cannot be easily decomposed, see Figure 11; therefore, most results presented here do not generalize to arbitrary number of monominoes, the techniques used here are not applicable, and it is expected that any characterization or enumeration of them would be much more complicated.

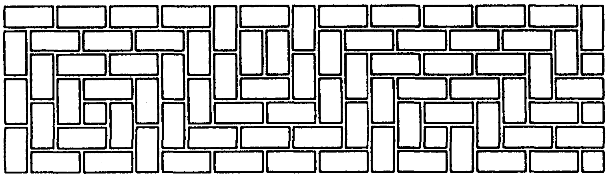

Figure 11: Tiling 7 by 23 with 5 monominoes

References

[1] D. Hickerson: Filling rectangular rooms with tatami mats, 2002. `http://www.research.att.com/~njas/sequences/a068920.txt`

[2] F. Ruskey, J. Woodcock: Counting Fixed-Height Tatami Tilings. *The Electronic Journal of Combinatorics* 16(1), 2009, R126, 20 pp., `http://www.combinatorics.org/`